On Reachability of Positive Linear Discrete-Time Systems with Scalar Controls

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Abstract. Positive systems are defined as systems in which the state trajectory is always positive (or at least non-negative) whenever the initial state is positive (non-negative). Positive linear systems are defined on cones and not on linear spaces and that is why the reachability and controllability tests for linear systems prove to be false. In this paper necessary and sufficient condition for reachability of discrete-time positive linear systems with scalar input is proved. Criteria for recognising the reachability property of such systems are presented and complete characterisations of the generic structure of reachable non-negative pair (A, b) in both algebraic and graph-theoretic forms are developed. The paper gives a new general treatment of reachability properties of scalar-input positive linear systems.

1 Introduction

Consider the single-input positive linear discrete-time system (PLDS), see, for example, [5]

$$x(t+1) = Ax(t) + bu(t), \qquad t = 0, 1, 2,... \tag{1}$$

$$A \in R^{+}_{nxn}, \ , \ b \in R^{+}_{n} \tag{2}$$

$$u(t) \in R^{+}_{1} \ , \tag{3}$$

where $x(t)$ is the system state, $u(t)$ is the control, $R^{+}_{n \times n}$ is the space of all $n \times n$ real matrices with non-negative entries a_{ij}, and R^{+}_{n} is the set of all non-negative vectors with dimension n. Note that A and b being non-negative is the necessary and sufficient conditions for a discrete-time linear system with non-negative control to have a non-negative state trajectory for any non-negative initial state. Positive linear systems are defined on cones and not on linear spaces and that is why many well-known properties of linear systems prove to be false. Aside from the intrinsic system theoretic interest, it should be noted that a variety of models having positive linear system behaviour can be found in engineering, management science, economics, social sciences, compartmental analysis in biology and medicine, and other areas (see [2, 5, 6, 9, 10] and references cited there).

The system (1)-(3) (and the pair (A, b)) is said to be reachable [6] if for any state $x \in R^{+}_{n}$, $x \neq 0$, and some finite t there exists a non-negative control sequence $\{u(s), s=0,1,2,...,t-1\}$that transfers the system from the origin into the state $x(t) = x$. It is said to be null-controllable if for any state $x \in R^{+}_{n}$ and for some finite t, there exists a non-negative control sequence that transfers the system from x into the origin. The system (1)-(3) is controllable if it is reachable and null-controllable [6]. The non-negative pair $(A, b) \geq 0$ is reachable [2, 6] if and only if reachability matrix

$$R_t = [b \ \ A^2b \ \ldots \ A^3b \ \ A^{n-1}b \ \ldots A^{t-1}b] \geq 0 \tag{4}$$

contains a monomial submatrix for some $t \geq n$ (the product of a permutation matrix and a non-singular diagonal matrix is called a monomial matrix). Note that the timing t in the reachability criterion given above is not fixed. Timing, indeed, seems to be much more critical in controlling positive systems than unconstrained linear systems where the set of all states reachable in a finite number of steps $t \geq n$ always coincides with the reachability subspace at time n as a consequence of Cayley-Hamilton theorem [8]. All the efforts (as indicated in [2] and, to our knowledge, to date) of applying the Cayley-Hamilton theorem to study the reachability property of positive linear systems have not been successful. The question is whether the reachability property of PLDS can be recognised by examining the reachability matrix R_n only, i.e. for $t = n$. The answer to this question is affirmative. It is proved in [2] for the case when $A \geq 0$ contains a diagonal (a selection of n non-zero entries, one from each row and column). The work [2] presents also an interesting sufficient type result (as a conjecture) namely every i-monomial column (a scalar non-zero multiple of the unit vector e_i is called a monomial vector) that appears in R_t for $t > n$ appears in R_n, i.e. for $t = n$. The proof given in [3] exploits a graph-theoretic technique. As a consequence, the pair $(A ,b) \geq 0$ is reachable if the reachability matrix $R_n(A, b)$ is a monomial matrix.

It is proved in this paper that the aforementioned condition is not only sufficient but also necessary for reachability of the pair $(A, b) \geq 0$. We provide also a new algebraic proof of sufficiency and develop complete characterisations of reachable non-negative pair (A, b) in both algebraic and graph-theoretic forms.

2. Reachability

We give simple proofs of the known results given in Lemma 1 and Lemma 2 included for the sake of completeness.

Lemma 1 *The pair* $(A, b) \geq 0$ *is reachable if and only if for some* $t \geq n$ *the reachability matrix* R_t *contains a monomial submatrix.*

Proof: *If part.* Let R_t for some $t \geq n$ contains a monomial submatrix (i.e. n linearly independent monomial columns). Then, since any other column of $R_t \geq$

0 can be expressed as a non-negative linear combination of these monomials, the reachable set $\Re_t$ (the set of all states reachable from the origin by non-negative controls)

$$\Re_t = \{x \,/\, x = \sum_{i=1}^{n} \alpha_i e_i \;,\; \alpha_i \geq 0\}, \quad t \geq n \,,$$

that is $\Re_t = R^+_n$ for that t, and so for any $x \subseteq R^+_n$ there exists a non-negative control sequence of length t such that $x(t) = x$. The pair (A, b) is reachable. To prove the *only if part* assume that for some $t \geq n$ the pair (A, b) is a reachable. Then, by definition, $\Re_t = R^+_n$. Two polyhedral cones coincide if their edges coincide. Hence, with necessity there are n linearly independent monomials in the sequence

$$b, Ab, A^2b, \ldots, A^{t-1}, \qquad (5)$$

i.e. R_t contains a monomial submatrix.

Lemma 2. *The pair* $(A, b) \geq 0$ *is reachable if and only if for some* $t \geq n$ the reachable set $\Re_t = R^+_n$.

Proof: Obviously, $t \geq n$ since otherwise all states belonging to $R^+_n - \Re_k$, $k < n$, cannot be reached by non-negative controls. The *if part* is trivial. To prove *the only if* part assume that the non-negative pair (A, b) is reachable for some finite $t \geq n$ but $\Re_t \subset R^+_n$. Then any non-negative state that belongs to $R^+_n - \Re_t$ can not be reached by a non-negative control sequence of length t. A contradiction.

The smallest integer ν_o for which the reachability matrix R_t contains a monomial submatrix M is called *reachability index* $\nu_o = \min \{t \,/\, M \subseteq R_t\}$or, equivalently, $\nu_o = \min \{t \,/\, \Re_t = R^+_n\}$.Clearly, $\nu_o \geq n$. We show that $\nu_o = n$.

Lemma 3.[9] *For any* $A \geq 0$ *and* $b \geq 0$, *all monomial columns that appear in the sequence* (5) *appear also in the pair* (A, b), or, equivalently, *for any* $A \geq 0$ *and* $b \geq 0$, *all monomial columns that are in* R_t *are in the pair* (A, b).

The following result contributes to the understanding of the structure of reachable pairs (A, b).

Lemma 4. *The pair* $(A, b) \geq 0$ *is reachable only if* b *is monomial.*

Proof: Let (A, b) be reachable. Then, it readily follows from Lemma 1 that for some $t \geq n$ the reachability matrix R_t contains a monomial submatrix, i.e. n linearly independent monomial columns. Lemma 3 tells us that all of these monomials are in (A, b). Assume they are in A and the vector b has at least two non-zero entries. Then, since all columns of A are linearly independent monomials, the vector Ab has at least two non-zero entries and so does

$$A^s b = A\,(A^{s-1}b)\;,\; s = 2, \ldots, t\text{-}1.$$

Hence, the reachability matrix R_t does not contain a monomial submatrix. A contradiction. The case $b = 0$ is trivial. The lemma is proved.

Lemma 4 simply states that the *only class of reachable positive systems with scalar controls* is the class of single-input systems. We are now going to prove that if the scalar-input positive linear discrete-time system is reachable in $t > n$ steps it is reachable in n steps, that is *t-steps reachability implies reachability in n steps.*

Theorem 1. *If the reachability matrix* R_t *of the pair* $(A, b) \geq 0$ *for some* $t \geq n$ *contains a monomial submatrix then* R_n *is a monomial matrix.*

Proof: Let for some $t > n$ the reachability matrix R_t contains a monomial submatrix, i.e. n linearly independent monomials. Then the non-negative pair (A, b) is reachable (Lemma 1) and b is monomial (Lemma 4). So the matrix A contains at least $(n-1)$ monomial columns. Let b be an i_1-monomial (the only non-zero entry is in the i_1 position) and assume that for some $1 \leq s < n\text{-}1$ the vectors $A^j b$, $0 \leq j \leq s\text{-}1$, are s linearly independent i_j- monomials, where $i_j \in S = \{ i_1, i_2, \ldots, i_s\} \subset N = \{1, 2, \ldots, n\}$ and $i_k \neq i_l$ for $k \neq l$, $k, l = 1, 2, \ldots, s$. Then , since $A^{j+1}b = A\,(A^j b)$, the i_jth column of A is i_{j+1}-monomial for $0 \leq j \leq s\text{-}1$ and $1 \leq s < n\text{-}1$. Consider now the i_s-column of A. It can be either (a) a monomial column, or (b) not. If it is not a monomial column it can be either (b1) a 0-vector or (b2) it can have at least two non-zero entries. If (b1), then the i_{s+1} column of A is also a 0-column and A has less than $(n\text{-}1)$ linearly independent monomial columns. A *contradiction*. If (b2), then the i_{s+1} -column of A has at least two non-zero entries and the number of monomial columns in A is once again less than $(n\text{-}1)$ which *contradicts* the assumption of reachability of $(A, b) \geq 0$. So the i_s-column of A must be an i_{s+1}-monomial, that is the case (a). If $s < n\text{-}1$ and $i_s = i_k$ for some $i_k \in S$, then the i_s-column of A is an i_k-monomial and it cycles through the subset of monomial columns $\{i_k, i_{k+1}, \ldots, i_s\}$, i.e. the number of linearly independent monomial columns in A is less than $(n\text{-}1)$. A *contradiction*, once again. Therefore $i_{s+1} \notin S$ and s can be increased up to $s = n\text{-}1$, thus proving the theorem.

Corollary 1. *The pair* $(A, b) \geq 0$ *is reachable if and only if its reachability matrix* R_n *is monomial.*

Theorem 1 (and Corollary 1) tells us that the reachability index of the pair (A, b) is $\nu_o\,(A, b) = n$. The proof of Corollary 1 follows directly from Theorem 1 and Lemma 1. Corollary 1 is a *reachability criterion* for identifying the reachability property of PLDS with scalar controls. From a computational point of view this criterion is easier to implement than the reachability criterion for general linear systems when no restrictions are imposed on the system. Note that the reachability matrix R_n of reachable PLDS with scalar controls is of full rank, i.e. rank $(R_n) = n$. This condition, necessary and sufficient for the reachability of general linear systems, is not sufficient in the case of PLDS because of the positivity properties of the system.

3. Generic structure of reachable pairs

Theorem 2. *The pair* $(A, b) \geq 0$ *is reachable if and only if there exists a permutation* $\sigma = \{ i_1, i_2, \ldots, i_n \}$ *of the numbers* $N = \{1, 2, \ldots, n\}$ *such that* b *is an* i_1*-monomial, all the* i_s*-columns of* A , $s = 1, 2, \ldots, n\text{-}1$, *are, respectively,*

i_{s+1}-monomials and the i_n-column of A is an arbitrary non-negative column.

The proof of Theorem 2 is similar to the proof of Theorem 1. Theorem 2 reveals the *structure* of reachable non-negative pairs (A, b). The pair $(A, b) \geq 0$ is reachable if and only if

$$b = \alpha_1 e_{i_{11}} \tag{6a}$$

$$a_{i_{s+1},i_s} > 0 \quad \text{for } s = 1, 2, ..., n\text{-}1 \tag{6b}$$

$$a_{i,i_n} \geq 0 \quad \text{for } i = 1, 2, ..., n \text{ , and} \tag{6c}$$

$$a_{ij} = 0 \text{ , } \quad \text{otherwise,} \tag{6d}$$

where $\sigma = \{i_1, i_2, ..., i_n\}$ is any permutation of the numbers $N = \{1, 2, ..., n\}$. The relations (6) give a *complete algebraic characterisation* of reachable scalar-input positive linear discrete-time systems.

Corollary 2. *For any given monomial reachability matrix R_n the permutation σ that generates a reachable pair of the form* (6) *is unique and*

$$col_{i_s} A = \frac{1}{\alpha_s} col_{s+1} R_n \text{ , } s = 1, 2, ..., n\text{-}1,$$

$$col_{i_n} A \geq 0 \text{ (arbitrary)}$$

with $\alpha_{s+1} = \alpha_s \, a_{i_{s+1},i_s}$ for $s = 1, 2, ..., n\text{-}1$.

Corollary 2 gives the relation between the matrices A and R_n of reachable pairs $(A, b) \geq 0$. Its proof follows from the proof of Theorem 2. Now, let $A \geq 0$ be a full-cycle monomial matrix, that is $a_{i_{s+1},i_s} > 0$ for $s = 1, 2,..., n\text{-}1$, $a_{i_1,i_n} > 0$ for $i = 1, 2,..., n$, and $a_{ij} = 0$, otherwise. Let $b \geq 0$ be, say, an i_k –monomial. Then a simple renumeration $i_1 := i_k$, ..., $i_{n-k+1} := i_n$,..., $i_n := i_{k-1}$ does the job and the conditions of Theorem 2 are satisfied. So the following results takes place.

Corollary 3. *The pair $(A, b) \geq 0$ is reachable for any monomial b if and only if the matrix A is a full-cycle monomial matrix.*

Corollary 4. *The single-input single-output PLDS (SISO PLDS), represented by the pair (A_c, e_n),*

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ : & : & : & & : & : \\ : & : & : & & : & : \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_o & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{bmatrix} \geq 0, \; e_n = \begin{bmatrix} 0 \\ 0 \\ : \\ : \\ 0 \\ 1 \end{bmatrix}$$

and $a_o \neq 0$, is reachable if and only if $a_j = 0$ for $j = 1, 2, ..., n\text{-}1$.

Corollary 4 is proved also in [7] by using a different construction and a different technique. In [7] a recursive expression for $col_n A_c^k$, $k = 2, 3, ..., n\text{-}1$, is found, first, and then used to derive the reachable (and controllable) structure of A_c. If $a_o = 0$, then the matrix A_c of a reachable SISO PLDS is a nil-potent matrix with nil-potency index n. Thus, the SISO PLDS is reachable if and only if A_c is either a full-cycle monomial matrix or a nil- potent matrix with a nil-potency index n.

The reachability property of non-negative pairs (A, b) can be recognised by examining the digraph of A. Let $D(A)$ be the digraph of an $n \times n$ nonnegative matrix A constructed as follows: the set of vertices of $D(A)$ is denoted as $N = \{1, 2, ..., n\}$ and there is an arc (i, j) in $D(A)$ if and only if $a_{ji} > 0$; the set of all arcs is denoted by U . A *walk* in $D(A)$ is an alternating sequence of vertices and arcs of D (see, for example [4]): $\{ i_1, (i_1, i_2), i_2, ..., (i_k, i_{k+1}), i_{k+1}\}$. The walk is called *closed* if $i_1 = i_{k+1}$ and *spanning* if it passes through all the vertices of $D(A)$, i.e. $\{ i_1, i_2, ..., i_n\} = N$. A walk is said to be a *path* if all of its vertices are distinct, and a *cycle* if it is a closed path. The path length is defined to be equal to the number of arcs in the path. If there exists a path in $D(A)$ from one of its vertices i to another j , then the vertex j is said to be *reachable* from the vertex i. The number of arcs directed away from a vertex i is called the *outdegree* of i and is written *od(i)*. The number of arcs directed towards a *vertex i* is called the *indegree* of i and is written *id(i)*. A path $\{i_1, (i_1, i_2), i_2, ..., (i_k, i_{k+1}), i_{k+1}\}$ in $D(A)$ is called *i_1-monomial path* of length k if and only if $a_{j,i_s} = 0$ for $j \neq i_{s-1}$ and $s = 2, ..., k+1$. It is not difficult to see that if a vector b is an i_1-monomial and in $D(A)$ contains a *i_1-monomial path* of length k then the columns $Ab, A^2b, ..., A^kb$ are monomial and

$$A^s b = \alpha_{s+1} e_{i_{s+1}} \quad \text{for } s = 0, 1, 2, ..., k.$$

Theorem 2 can be restated in the following a nice graph-theoretic form using the notion of monomial paths.

Theorem 3. *The pair $(A, b) \geq 0$ is reachable if and only if b is an i_1-monomial and the digraph $D(A)$ of A is a union of an i_1-monomial path of length n-1 that spans all the vertices, i.e. an i_1-monomial spanning path, and ,possibly, arcs $\{(i_n, i), i = 1,...,n\}$*

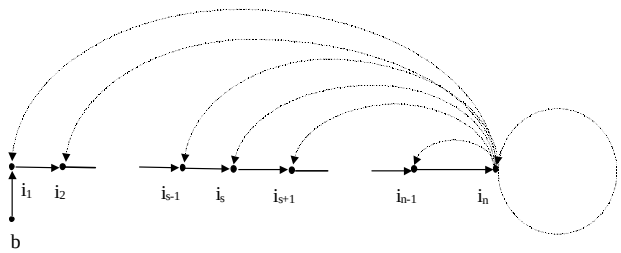

Figure 1. Generic digraph structure of reachable pair $(A, b) \geq 0$.

Theorem 3 gives a *complete graph-theoretic characterisation* of reachable PLDS with scalar controls. The generic graph-theoretic structure presented on Fig. 1 suggests the answer to the question of how many consecutive monomial columns A^sb can be generated by an i_1 – monomial. A number of useful ***observations*** can be

made analysing the digraph of reachable non-negative pairs (A, b) given in Fig.1 in relation with the sequence (5).

(i) $A^s b = \alpha_{s+1} e_{i_{s+1}}$ for $s = 0,1,2,....,n-1$, where $\alpha_s > 0$ is a real number, and hence all columns in the sequence (5) for $t= n$ are linearly independent monomials;

(ii) if $od(i_n) \geq 2$ then $A^n b$ is not a monomial column and so are $A^{n+s}b$ for $s = 1,2,...$;

(iii) if $od(i_n) = 1$ and an arc $(i_n, i_s) \in D(A)$ for some $s = 1,2,...,n$ then $D(A)$ contains a cycle (an orbit) of length $l = n-s$ and $A^s b = A^{s+kl}b$ for $k=0,1,2,...$;

(iv) the number of linearly independent monomial columns generated by a reachable pair (A, b) ≥ 0 in the sequence (5) is equal to the length of the i_1-monomial spanning path in $D(A)$ plus one (for b being an i_1-monomial), that is exactly equal to n (the number of vertices of the digraph $D(A)$);

(v) let b_{i_s} be an i_s-monomial for $s = 1,2,....,n$, the digraph $D(A)$ of A be as in Fig.1, and μ_s be the number of linearly independent monomial columns generated by the pair (A, b_{i_s}), $s = 1,2,\ldots,n$, in the sequence (5); then always $\mu_s < \mu_1$ for any $s =2,....,n$.

4. Final remarks

The reachability property of scalar-input discrete-time positive linear systems is a generic property. It depends on the zero-nonzero pattern of the pair (A, b) ≥ 0 only and does not depend on the specific values of their entries. Such a situation has no equivalent in the class of unconstrained linear systems. Thus, the reachable PLDS are not robust. This property is quite appealing and entails a number of important consequences in the real-life design problems where control system parameters are subject to measurement and modelling errors. The observations (i)-(v) as well as the new treatment of the reachability properties of the non-negative pairs (A, b) presented in the paper give some useful hints in the analysis of reachability of discrete-time positive linear system with vector controls.

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