Nonlinear H_∞-Control via Sampled-Data Measurement Feedback: Time-Scale Conversion to Continuous Measurement Case

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Abstract

Nonlinear H_∞-controller synthesis is presented for time-varying systems via sampled - data measurement feedback. The H_∞-control problem for a nonlinear system with a sampled-data measurement output is shown to be equivalent to the H_∞-control problem for a certain nonlinear system with a time-continuous measurement output. Special mathematical tools are developed to straightforwardly derive a local solution of the sampled-data measurement feedback H_∞-control problem from that of the time-continuous measurement feedback H_∞-control problem.

1 Introduction

We consider a time-varying nonlinear system governed by

$$\begin{aligned} \dot{x} &= f(x(t),t) + g_1(x(t),t)w + g_2(x(t),t)u && \text{(1a)} \\ z &= h_1(x(t),t) + k_{12}(x(t),t)u && \text{(1b)} \\ y_j &= h_2(x(\tau_j),\tau_j) + k_{21}(x(\tau_j),\tau_j)w(\tau_j), && \text{(1c)} \end{aligned}$$

where $x \in R^n$ is the state vector, $t \in R^1$ is the time variable, $u \in R^m$ is the control input, $w \in R^r$ is the unknown disturbance, $z \in R^l$ is the unknown output to be controlled, $y = (y_0, y_1, ...)$ is the only available measurement on the system with the discrete measurements $y_j \in R^p$, $j = 0, 1, ...$, taken at possibly nonuniformly spaced time instants τ_j, which have no finite limiting points. The functions $f(x,t)$, $g_1(x,t)$, $g_2(x,t)$, $h_1(x,t)$, $h_2(x,t)$, $k_{12}(x,t)$, $k_{21}(x,t)$ are assumed to be continuous in t for all x and twice continuously differentiable in x for all t, whereas their first and second order state derivatives are assumed to be continuous and uniformly bounded in t. It is also assumed that $f(0,t) = 0$, $h_1(0,t) = 0$ and $h_2(0,t) = 0$ for all t.

A causal dynamic feedback compensator

$$u = \mathcal{K}(y,t), \qquad (2)$$

with internal state $\xi \in R^q$, is said to be a globally (locally) admissible controller if the closed-loop system (1), (2) is globally (uniformly) asymptotically stable when $w = 0$.

Given a real number $\gamma > 0$, it is said that system (1), (2) has L_2/l_2-gain less than γ if the response z, resulting from w for initial state $x(t_0) = 0, \xi(t_0) = 0$, satisfies

$$\int_{t_0}^{t_1} \|z(t)\|^2 \, dt < \gamma^2 [\int_{t_0}^{t_1} \|w(t)\|^2 \, dt + \sum_{\tau_j \in [t_0,t_1]} \|w(\tau_j)\|^2] \qquad (3)$$

for all $t_1 > t_0$ and all continuous functions $w(t)$. The right-hand side in (3) should be viewed as a mixed L_2/l_2-norm on the uncertain signals affecting the system and the sampled-data measurements.

The nonlinear H_∞-control problem over sampled-data measurements is to find a globally admissible controller (2) such that L_2/l_2-gain of the closed-loop system (1), (2) is less than γ. In turn, a local solution to the H_∞-control problem is defined as follows. A locally admissible controller (2) is said to be a local solution of the H_∞-control problem if there exists a neighborhood U of the equilibrium such that inequality (3) is satisfied for all $t_1 > t_0$ and all continuous functions $w(t)$ for which the state trajectory of the closed-loop system starting from the initial point $(x(t_0), \xi(t_0)) = (0,0)$ remains in U for all $t \subset [t_0, t_1]$.

In the present work our investigation is confined to seeking a local solution of the H_∞-control problem. The following contribution is made to the existing literature on nonlinear H_∞-control of dynamic systems.

The H_∞-control problem for a nonlinear (time-invariant or time-varying) system with a sampled-data measurement output is constructively shown to be equivalent to the well-studied H_∞-control problem for a certain nonlinear (definitely, time-varying) system with a time-continuous measurement output. In fact, the paper formalizes, with the help of the discontinuous time scale method developed, an intuitively clear idea that a sampled-data measurement may be viewed as a time-continuous measurement made for a short time period, whereas a time-continuous measurement may be represented as a continuum of sampled-data measurements. Based on this equivalence, a local solution of the sampled-data measurement feedback H_∞-control

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problem is derived at the same level of generality as that of the time-continuous measurement feedback H$_\infty$-control problem. Special mathematical tools are developed to carry out a formal derivation of a solution of the sampled-data measurement feedback H$_\infty$-control problem from that of the time-continuous measurement feedback H$_\infty$-control problem.

For ease of reference, a time-continuous measurement feedback and a sampled-data measurement feedback will further be abbreviated as CMF and SMF, respectively.

2 Local H_∞-Control Synthesis via Sampled-Data Measurements

The following assumptions inherited from the standard H$_\infty$-control problem are made throughout

$$h_1^T(x,t)k_{12}(x,t) = 0, k_{12}^T(x,t)k_{12}(x,t) = I,$$

$$k_{21}(x,t)g_1^T(x,t) = 0,$$

$$k_{21}(x(\tau_j),\tau_j)k_{21}^T(x(\tau_j),\tau_j) = I, \quad j = 0, 1, ... \tag{4}$$

Under these assumptions we derive a local solution to the SMF H$_\infty$-control problem in question. Relaxing these assumptions is indeed possible, but it would substantially complicate the formulas to be worked out. Along with the standard Ricatti differential equation

$$-\dot{P}_\varepsilon = P_\varepsilon(t)A(t) + A^T(t)P_\varepsilon(t) + C_1^T(t)C_1(t)$$

$$+P_\varepsilon(t)[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T](t)P_\varepsilon(t) + \varepsilon I, \tag{5}$$

the solution invokes the differential equations with jumps:

$$\dot{Z}_\varepsilon = \tilde{A}_\varepsilon(t)Z_\varepsilon(t) + Z_\varepsilon(t)\tilde{A}_\varepsilon^T(t) + B_1(t)B_1^T(t)$$

$$+\gamma^{-2}Z_\varepsilon(t)P_\varepsilon(t)B_2(t)B_2^T(t)P_\varepsilon(t)Z_\varepsilon(t) + \varepsilon I \tag{6}$$

$$Z_\varepsilon(\tau_j+) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)]^{-1} \tag{7}$$

$$\dot{\xi} = f(\xi,t) + [\frac{1}{\gamma^2}g_1(\xi,t)g_1^T(\xi,t) - g_2(\xi,t)g_2^T(\xi,t)]P_\varepsilon(t)\xi \tag{8}$$

$$\xi(\tau_j+) = \zeta_j(1), \quad j = 0, 1, ... \tag{9}$$

where $\varepsilon \geq 0, \widetilde{A}_\varepsilon(t) = A(t) + \frac{1}{\gamma^2}B_1(t)B_1^T(t)P_\varepsilon(t)$, and $\zeta_j(t)$ satisfies

$$\dot{\zeta}_j(t) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)t]^{-1}C_2^T(\tau_j)$$

$$\times[y_j - h_2(\zeta_j(t),\tau_j)], \quad \zeta_j(0) = \xi(\tau_j-). \tag{10}$$

These jumps describe instantaneous changes in the dynamics of the H$_\infty$-controller at the sampling time moments τ_j, $j = 0, 1, ...$ In order to state the main result of this section, the following conditions are introduced:

A1) under $\varepsilon = 0$ there exists a bounded positive semidefinite symmetric solution $P(t)$ of equation (5) such that the system

$$\dot{x} = [A - (B_2B_2^T - \gamma^{-2}B_1B_1^T)P](t)x(t) \tag{11}$$

is exponentially stable;

A2) under $\varepsilon = 0$ there exists a bounded, positive semidefinite, symmetric solution $Z(t)$ of (6), (7) such that the system

$$\dot{x} = [\tilde{A}_{\varepsilon=0} + \gamma^{-2}ZPB_2B_2^TP](t)x(t) \tag{12}$$

with the jumps

$$x(\tau_j+) = x(\tau_j-) - Z(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z(\tau_j-)]^{-1}$$

$$\times C_2^T(\tau_j)C_2(\tau_j)x(\tau_j-) \tag{13}$$

is exponentially stable.

Theorem 1 *Let conditions A1) and A2) be satisfied. Then there exists $\varepsilon_0 > 0$ such that system (5)-(7) has a unique, continuous from the left, bounded, positive definite, symmetric solution $(P_\varepsilon(t), Z_\varepsilon(t))$ for each $\varepsilon \in (0, \varepsilon_o)$ and the output feedback*

$$u = -g_2^T(\xi,t)P_\varepsilon(t)\xi \tag{14}$$

with the external state $\xi(t)$, governed by (8)-(10), is a local solution of the SMF H_∞-control problem. Conversely, conditions A1) and A2) are satisfied if the nonlinear H_∞-control problem (1) has a local exponentially stabilizing solution.

The proof of Theorem 1 is given in subsection 2.2.Various sufficient conditions for the existence of a solution of the H$_\infty$−control problem for a nonlinear time-invariant system with sampled-data measurements may be found in [4] and [3].

To this end, we note that value (9) from the right of the controller internal state ξ at the sampling time moments τ_j, $j = 0, 1, ...$ are determined through the evolution of the auxiliary dynamic system (10). Particularly, when dealing with a linear observation $h_2(x,t) = C_2(t)x$, (10) admits the analytical representation

$$\zeta_j(t) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)t]^{-1}$$

$$\times[Z_\varepsilon^{-1}(\tau_j-)\xi(\tau_j-) + C_2^T(\tau_j)y_jt]$$

and (9) can be rewritten in the explicit form

$$\xi(\tau_j+) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)]^{-1}$$

$$\times[Z_\varepsilon^{-1}(\tau_j-)\xi(\tau_j-) + C_2^T(\tau_j)y(\tau_j)]$$

$$= \xi(\tau_j-) + Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)]^{-1}$$

$$\times C_2^T(\tau_j)[y_j - C_2(\tau_j)\xi(\tau_j-)], \tag{15}$$

like that for the variable Z_ε (cf. (7)).

2.1 Conversion into the H_∞-Control Synthesis via Continuous Measurements.

We demonstrate now how to form an equivalent CMF H_∞-control problem and then, based on such an equivalence, we shall prove Theorem 1.

To begin with, let us introduce the functions

$$\chi(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}, \quad v(t) = \sum_{j=0}^{\infty} \chi(t - \tau_j),$$

$$\alpha(t) = t + v(t), \quad \beta(s) = \inf\{t : \alpha(t) > s\},$$

$$\psi(s) = \begin{cases} s - t, \text{ if } \beta(s) = t \in D \\ v(\beta(s)), \text{ otherwise} \end{cases}$$

where $D = \{\tau_j\}_{j=0}^{\infty}$. It is clear, that the functions $\beta(s)$ and $\psi(s)$ have the piecewise continuous derivatives

$$\dot{\beta}(s) = \begin{cases} 0, \text{ if } \beta(s) \in D \\ 1, \text{ otherwise} \end{cases}, \quad \dot{\psi}(s) = \begin{cases} 1, \text{ if } \beta(s) \in D \\ 0, \text{ otherwise} \end{cases},$$

and

$$\beta(\alpha(t)) = t \text{ for all } t \in R^1, \tag{16}$$

$$\alpha(\beta(s)) = s \text{ for all } s \in R^1 \text{ such that } \beta(s) \notin D$$

$$\begin{aligned} \alpha(\beta(s)-) &= \inf\{s' : \beta(s') = \beta(s)\} \\ \text{for all } s &\in R^1 \text{ such that } \beta(s) \in D, \end{aligned}$$

$$\begin{aligned} \alpha(\beta(s)+) &= \alpha(\beta(s)-) + 1 \\ \text{for all } s &\in R^1 \text{ such that } \beta(s) \in D. \end{aligned} \tag{17}$$

Along with the SMF H_∞-control problem for the nonlinear system (1), we shall consider the standard CMF H_∞-control problem for the auxiliary system

$$\begin{aligned} \dot{\widehat{x}} &= \widehat{f}(\widehat{x}, s) + \widehat{g}_1(\widehat{x}, s)\widehat{w}(s) + \widehat{g}_2(\widehat{x}, s)\widehat{u}(s) \\ \widehat{z}(s) &= \widehat{h}_1(\widehat{x}, s) + \widehat{k}_{12}(\widehat{x}, s)\widehat{u}(s) \\ \widehat{y}(s) &= \widehat{h}_2(\widehat{x}, s) + \widehat{k}_{21}(\widehat{x}, s)\widehat{w}(s) \end{aligned} \tag{18}$$

where $\widehat{x} \in R^n$ is the state vector, $s \in R^1$ is the time variable, $\widehat{u} \in R^m$ is the control input, $\widehat{w} \in R^r$ is the unknown disturbance, $\widehat{z} \in R^l$ is the unknown output to be controlled, $\widehat{y} \in R^p$ is the only available continuous-time measurement on the system;

$$\widehat{f}(\widehat{x}, s) = f(\widehat{x}, \beta(s))\dot{\beta}(s), \widehat{g}_1(\widehat{x}, s) = g_1(\widehat{x}, \beta(s))\dot{\beta}(s),$$

$$\widehat{g}_2(\widehat{x}, s) = g_2(\widehat{x}, \beta(s))\dot{\beta}(s), \widehat{h}_1(\widehat{x}, s) = h_1(\widehat{x}, \beta(s))\dot{\beta}(s),$$

$$\widehat{h}_2(\widehat{x}, s) = h_2(\widehat{x}, \beta(s))\dot{\psi}(s), \ \widehat{k}_{12}(\widehat{x}, s) = k_{12}(\widehat{x}, \beta(s))\dot{\beta}(s),$$

$$\widehat{k}_{21}(\widehat{x}, s) = k_{21}(\widehat{x}, \beta(s))\dot{\psi}(s). \tag{19}$$

As stated below, the latter problem is equivalent to the former one. This equivalence will further allow us to deduce a local solution of the SMF H_∞-control problem from that of the CMF H_∞-control problem.

Theorem 2 *There exists a (local and/or exponentially stabilizing) solution of the SMF H_∞-control problem for the nonlinear system (1) if and only if there exists a (local and/or exponentially stabilizing) solution of the CMF H_∞-control problem for the nonlinear system (18). Moreover, if $u(t)$ is a (local and/or exponentially stabilizing) solution of the former problem then $\widehat{u}(s) = u(\beta(s))$ is a (local and/or exponentially stabilizing) solution of the latter problem. Conversely, if $\widehat{u}(s)$ is a (local and/or exponentially stabilizing) solution of the latter problem then $u(t) = \widehat{u}(\alpha(t))$ is a (local and/or exponentially stabilizing) solution of the former problem.*

Proof. Let $\widehat{x}(s)$ be a trajectory of (18) subjected to an external disturbance $\widehat{w}(s)$ and driven by an admissible dynamic CMF feedback controller $\widehat{u}(s)$ with the internal state $\widehat{\xi}(s)$. Then it is straightforward to check that in accordance with (16) and (19), $x(t) = \widehat{x}(\alpha(t))$ is a trajectory of (1a) subjected to the external disturbance $w(t) = \widehat{w}(\alpha(t))$ and driven by the admissible dynamic SMF feedback controller $u(t) = \widehat{u}(\alpha(t))$ with the internal state $\xi(t) = \widehat{\xi}(\alpha(t))$. Hence, in spite of discontinuities in $\alpha(t)$, the function $\widehat{x}(\alpha(t))$ is continuous for all t, and furthermore, the following relations

$$\int_{t_0}^{t_1} \|z(t)\|^2 \, dt = \int_{s_0}^{s_1} \|\widehat{z}(s)\|^2 \, ds,$$

$$\int_{t_0}^{t_1} \|w(t)\|^2 \, dt + \sum_{\tau_j \in [t_0, t_1]} \|w(\tau_j)\|^2 = \int_{s_0}^{s_1} \|\widehat{w}(s)\|^2 \, ds \tag{20}$$

are satisfied for all $s_0, s_1 \in R^1$ and $t_0 = \beta(s_0), t_1 = \beta(s_1)$.

Analogously, if $x(t)$ is a trajectory of (1a) enforced by an external disturbance $w(t)$ and an admissible dynamic SMF feedback controller $u(t)$ with the internal state $\xi(t)$ then $\widehat{x}(s) = x(\beta(s))$ is a solution of (18) enforced by the external disturbance $\widehat{w}(s) = w(\beta(s))$ and the admissible dynamic CMF feedback controller $\widehat{u}(s) = u(\beta(s))$ with the internal state $\widehat{\xi}(s) = \xi(\beta(s))$. Along with this, relations (20) remain true.

Due to the above one-to-one correspondence between the trajectories of the original system (1) and those of the auxiliary system (18) a (local and/or exponentially stabilizing) solution $u(t)$ of the SMF H_∞-control problem for (1) generates the (local and/or exponentially stabilizing) solution $\widehat{u}(s) = u(\beta(s))$ of the CMF H_∞-control problem for (18), whereas a (local and/or exponentially stabilizing) solution $\widehat{u}(s)$ of the latter problem generates the (local and/or exponentially stabilizing) solution $u(t) = \widehat{u}(\alpha(t))$ of the former problem. Theorem 2 is thus proven. ♠

We conclude this subsection with providing necessary and sufficient conditions for a local solution of the H_∞-control problem for the auxiliary system (18) to exist. These conditions are as follows:

Ã1) There exists a bounded, positive semidefinite, symmetric solution of the equation

$$-\dot{\widehat{P}}(s) = \widehat{P}(s)A(\beta(s))\dot{\beta}(s) + A^T(\beta(s))\widehat{P}(s)\dot{\beta}(s)$$
$$+C_1^T(\beta(s))C_1(\beta(s))\dot{\beta}(s) + \widehat{P}(s)[\frac{1}{\gamma^2}B_1(\beta(s))B_1^T(\beta(s))$$
$$-B_2(\beta(s))B_2^T(\beta(s))]\widehat{P}(s)\dot{\beta}(s) \qquad (21)$$

such that the system

$$\dot{\widehat{x}}(s) = \{A(\beta(s)) - [B_2(\beta(s))B_2^T(\beta(s))$$
$$-\gamma^{-2}B_1(\beta(s))B_1^T(\beta(s))]\widehat{P}(s)\}\dot{\beta}(s)\widehat{x}(s) \qquad (22)$$

is exponentially stable;

Ã2) there exists a bounded, positive semidefinite, symmetric solution to the equation

$$\dot{\widehat{Z}}(s) = \widehat{A}(s)\widehat{Z}(s)\dot{\beta}(s) + \widehat{Z}(s)\widehat{A}^T(t)\dot{\beta}(s)$$
$$+B_1(\beta(s))B_1^T(\beta(s))\dot{\beta}(s)$$
$$+\widehat{Z}(s)[\frac{1}{\gamma^2}\widehat{P}(s)B_2(\beta(s))B_2^T(\beta(s))\widehat{P}(s)\dot{\beta}(s)$$
$$-C_2^T(\beta(s))C_2(\beta(s))\dot{\psi}(s)]\widehat{Z}(s) \qquad (23)$$

such that the system

$$\dot{\widehat{x}}(s) = \{\widehat{A}(s)\dot{\beta}(s) - \widehat{Z}(s)[C_2^T(\beta(s))C_2(\beta(s))\dot{\psi}(s)$$
$$-\gamma^{-2}\widehat{P}(s)B_2(\beta(s))B_2^T(\beta(s))\widehat{P}(s)\dot{\beta}(s)]\}\widehat{x}(s) \qquad (24)$$

is exponentially stable and $\widehat{A}(s) = A(\beta(s)) + \frac{1}{\gamma^2}B_1(\beta(s))B_1^T(\beta(s))\widehat{P}(s)$.

The corresponding result is stated below in the form of lemma.

Lemma 3 *Let conditions Ã1) and Ã2) be satisfied. Then there exists $\varepsilon_o > 0$ such that system*

$$-\dot{\widehat{P}}_\varepsilon(s) = \widehat{P}_\varepsilon(s)A(\beta(s))\dot{\beta}(s) + A^T(\beta(s))\widehat{P}_\varepsilon(s)\dot{\beta}(s)$$
$$+C_1^T(\beta(s))C_1(\beta(s))\dot{\beta}(s)$$
$$+\widehat{P}_\varepsilon(s)[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T](\beta(s))\widehat{P}_\varepsilon(s)\dot{\beta}(s) + \varepsilon I, \quad (25)$$
$$\dot{\widehat{Z}}_\varepsilon(s) = \widehat{A}(s)\widehat{Z}_\varepsilon(s)\dot{\beta}(s) + \widehat{Z}_\varepsilon(s)\widehat{A}^T(s)\dot{\beta}(s)$$
$$+B_1(\beta(s))B_1^T(\beta(s))\dot{\beta}(s)$$
$$+\widehat{Z}_\varepsilon(s)[\frac{1}{\gamma^2}\widehat{P}_\varepsilon(s)B_2(\beta(s))B_2^T(\beta(s))\widehat{P}_\varepsilon(s)\dot{\beta}(s)$$
$$-C_2^T(\beta(s))C_2(\beta(s))\dot{\psi}(s)]\widehat{Z}_\varepsilon(s) + \varepsilon I \qquad (26)$$

has a unique bounded, positive definite, symmetric solution $(\widehat{P}_\varepsilon(s), \widehat{Z}_\varepsilon(s))$ for each $\varepsilon \in (0, \varepsilon_o)$ and a solution of the H_∞-control problem is given by

$$\dot{\widehat{\xi}}(s) = \widehat{f}(\widehat{\xi}, s) + (\frac{1}{\gamma^2}\widehat{g}_1(\widehat{\xi}, s)\widehat{g}_1^T(\widehat{\xi}, s)$$
$$-\widehat{g}_2(\widehat{\xi}, s)\widehat{g}_2^T(\widehat{\xi}, s))\widehat{P}_\varepsilon(s)\widehat{\xi}$$
$$+\widehat{Z}_\varepsilon(s)C_2^T(\beta(s))\dot{\psi}(s)[\widehat{y}(s) - \widehat{h}_2(\widehat{\xi}, s)], \qquad (27)$$
$$\widehat{u} = -\widehat{g}_2^T(\widehat{\xi}, s)\widehat{P}_\varepsilon(s)\widehat{\xi}(s).$$

Conversely, conditions Ã1) and Ã2) are satisfied if the H_∞-control problem for (18) has a local exponentially stabilizing solution.

The proof of Lemma 3 follows the standard line of reasoning, used in nonlinear H_∞-control (see, e.g., that of [2] for time-varying setting), and it is therefore omitted. The above lemma is subsequently used in proving Theorem 1.

2.2 Proof of Theorem 1

We first demonstrate that conditions A1) and A2) are equivalent to Ã1) and Ã2), respectively.

Indeed, if $\widehat{P}(s)$, $\widehat{Z}(s)$ are solutions of (21), (23), then it is straightforward to check that in continuity intervals of $\alpha(t)$ the functions $P(t) = \widehat{P}(\alpha(t))$, $Z(t) = \widehat{Z}(\alpha(t))$ satisfy (5), (6) under $\varepsilon = 0$. Moreover, for $s \in [\alpha(\tau_j-), \alpha(\tau_j+)]$ with $\tau_j \in D$ and $\alpha(\tau_j+) = \alpha(\tau_j-)+1$, we have that $\dot{\beta}(s) = 0$, $\dot{\psi}(s) = 1$, the following equations holds

$$\dot{\widehat{P}}(s) = 0,$$
$$\dot{\widehat{Z}}(s) = -\widehat{Z}(s)C_2^T(\tau_j)C_2(\tau_j)\widehat{Z}(s) \qquad (28)$$

and the functions

$$\widehat{P}(s) = \widehat{P}(\alpha(\tau_j-)),$$
$$\widehat{Z}(s) = \widehat{Z}(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z(\tau_j-)s]^{-1} \qquad (29)$$

are solutions of (28). Thus, in spite of discontinuities in $\alpha(t)$, the function $\widehat{P}(\alpha(t))$ is continuous for all t, thereby satisfying (5) for all t, whereas the jumps

$$\widehat{Z}(\alpha(\tau_j+)) - \widehat{Z}(\alpha(\tau_j-)) =$$
$$\widehat{Z}(\tau_j-)[I+C_2^T(\tau_j)C_2(\tau_j)Z(\tau_j-)]^{-1} - \widehat{Z}(\tau_j-), j = 0, 1, ...$$

of the function $\widehat{Z}(\alpha(t))$ are the same as those of (6), (7) and therefore $\widehat{Z}(\alpha(t))$ is a solution of the differential equation (6) with jumps (7).

Analogously, one can prove that if $(P(t), Z(t))$ is a solution of (5)-(7) under $\varepsilon \equiv 0$ then $\widehat{P}(s) = P(\beta(s))$, $\widehat{Z}(s) = Z(\beta(s))$ satisfy (21), (23). In order to conclude the equivalence of conditions Ã1), Ã2) and A1), A2), it remains to note that the same relations are also in force between solutions of (11)-(13) and (22), (24).

By applying Theorem 2 and Lemma 3, it follows that conditions A1), A2) are necessary (certainly, under the additional assumption that there exists a local exponentially stabilizing solution of the SMF H_∞-control problem) and sufficient for a local solution of the SMF H_∞-control problem for (1) to exist. Moreover, if these conditions are satisfied, the conditions $\tilde{\mathrm{A}}1$), $\tilde{\mathrm{A}}2$) are satisfied as well, and due to Lemma 3, there exists $\varepsilon_0 > 0$ such that system (25), (26) has a unique bounded, positive definite, symmetric solution $(\widehat{P}_\varepsilon(s), \widehat{Z}_\varepsilon(s))$ for each $\varepsilon \in (0, \varepsilon_o)$. These functions were shown to generate the unique bounded, positive definite, symmetric solution $(P_\varepsilon(t) = \widehat{P}_\varepsilon(\alpha(t)), Z_\varepsilon(t) = \widehat{Z}_\varepsilon(\alpha(t)))$ of (5)-(7) (the uniqueness is guaranteed by the invertibility of the time substitution in the above relations: $\widehat{P}_\varepsilon(s) = P_\varepsilon(\beta(s))$, $\widehat{Z}_\varepsilon(s) = Z_\varepsilon(\beta(s))$). Furthermore, according to Theorem 2 the CMF solution (27), given by Lemma 3, generates the solution

$$\begin{aligned} u(t) &= \widehat{u}(\alpha(t)) = -\widehat{g}_2^T(\widehat{\xi}(\alpha(t)), \alpha(t)) P_\varepsilon(\alpha(t)) \widehat{\xi}(\alpha(t)) \\ &= -g_2^T(\widehat{\xi}(\alpha(t)), t) P_\varepsilon(t) \widehat{\xi}(\alpha(t)) \end{aligned}$$

of the SMF H_∞-control problem in question.

To complete the proof let us demonstrate that the function $\xi(t) = \widehat{\xi}(\alpha(t))$ satisfies the differential equation (8) with jumps (9), (10). In continuity intervals of $\alpha(t)$ we have $\dot{\psi}(s) = 0$ and by inspection $\widehat{\xi}(\alpha(t))$ is a solution of (8). If $s \in [\alpha(\tau_j-), \alpha(\tau_j+)]$ where $\tau_j \in D$ and $\alpha(\tau_j+) = \alpha(\tau_j-) + 1$, then the relations $\dot{\beta}(s) = 0$, $\dot{\psi}(s) = 1$ and (29) are in force and, consequently,

$$\dot{\widehat{\xi}}(s) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j) C_2(\tau_j) Z_\varepsilon(\tau_j-)s]^{-1} C_2^T(\tau_j)$$

$$\times [y(\tau_j) - h_2(\widehat{\xi}(s), \tau_j)], \quad \alpha(\tau_j-) \le s \le \alpha(\tau_j-) + 1.$$

Thus, the jumps $\widehat{\xi}(\alpha(\tau_j+)) - \widehat{\xi}(\alpha(\tau_j-))$, $j = 0, 1, ...$ of the function $\xi(t) = \widehat{\xi}(\alpha(t))$ are the same as those defined by (9), (10) and therefore $\widehat{\xi}(\alpha(t))$ satisfies (8)-(10) for all t. Theorem 1 is proven.

3 Distribution Formalism

As shown, the SMF H_∞-control problem, rewritten in a new time variable obtained through a certain time substitution (absolutely continuous, non-invertible substitution $t = \beta(s)$), becomes a CMF H_∞-control problem, and vice versa (the discontinuous time substitution $s = \alpha(t)$ is employed for the corresponding CMF H_∞-control problem conversion). A distribution formalism in nonlinear setting is subsequently developed to make such a connection between the SMF and CMF H_∞-control problems transparent.

To begin with, let us consider a nonlinear differential equation of the form

$$\dot{\varkappa} = \psi(\varkappa, t) + \omega(\varkappa, t)\nu(t), \varkappa(t_0) = \varkappa^0 \qquad (30)$$

where n-vector functions ψ and ω are sufficiently smooth and a scalar input

$$\nu(t) = \sum_{j=0}^{\infty} \delta(t - \tau_j) \qquad (31)$$

is a combination of Dirac functions. Since a solution $\varkappa(t)$ of (30) is expected to undergo discontinuities at the impact time instants $\tau_j, j = 0, 1, ...$ the linear theory of Schwartz distributions is inapplicable to the differential equation (30) because of the irregularity of a product of the impulsive input $\nu(t)$ and the discontinuous (in t) function $\omega(\varkappa(t), t)$. By virtue of this, the meaning of (30) is modified as follows.

Definition 1 *A function $\varkappa(t)$ is said to be a vibroimpact solution of (30) if the weak* convergence of an arbitrary sequence of integrable inputs $\nu_i(t), i = 1, 2, ...$ to the input function $\nu(t)$ implies the weak* convergence of the corresponding standard solutions $\varkappa_i(t)$ of (30) with $\nu(t) = \nu_i(t)$, regardless of a choice of an approximating sequence $\{\nu_i(t)\}$.*

Recall that a sequence of integrable (peaking) functions $\nu_i(t)$, $i = 1, 2, \dots$ converges to $\nu(t) = \sum_{j=0}^{\infty} \delta(t - \tau_j)$ in weak* topology iff

$$\lim_{i \to \infty} \int_{-\infty}^{\infty} \nu_i(t) \varphi(t) dt = \Sigma_{j=0}^{\infty} \varphi(\tau_j)$$

for any continuous function $\varphi(t)$ with compact support.

The concept of a vibroimpact solution was introduced in [1] for nonlinear systems whose impulse response depends upon the approximation of the impulse. The following result is extracted from [1].

Proposition 4 *Vibroimpact solutions of (30) are well-posed for arbitrary initial conditions $\varkappa^0 \in R^n$, satisfy the differential equation $\dot{\varkappa} = \psi(\varkappa, t)$ for all t but the impact time moments $\tau_j, j = 0, 1, ...,$ and their values at $t = \tau_j$ are determined through the relation*

$$\varkappa(\tau_j+) = \rho(1) \qquad (32)$$

where $\rho(t)$ satisfies

$$\dot{\rho} = \omega(\rho, t), \rho(0) = \varkappa(\tau_j-). \qquad (33)$$

In the remainder, vibroimpact solutions of nonlinear differential equations in distributions are utilized to unify the SMF and CMF H_∞-controller equations and to straightforwardly deduce a local solution of the SMF H_∞-control problem from that of the CMF H_∞-control problem.

Let us now demonstrate that the SMF H_∞-controller equations (6)- (9), (12), (13) with jumps can be represented in the form (30), (31) as nonlinear differential equations in distributions. For this purpose let us specify (30) as follows:

$$\dot{Z}_\varepsilon = \widetilde{A}_\varepsilon(t)Z_\varepsilon(t) + Z_\varepsilon(t)\widetilde{A}_\varepsilon^T(t) + B_1(t)B_1^T(t) + \varepsilon I$$

$$+Z_\varepsilon(t)[\frac{1}{\gamma^2}P_\varepsilon B_2 B_2^T P_\varepsilon](t)Z_\varepsilon(t)$$

$$-Z_\varepsilon(t)C_2^T(t)C_2(t)Z_\varepsilon(t)\nu(t), Z_\varepsilon(t_0) = Z_\varepsilon^0 \tag{34}$$

$$\dot{\xi} = f(\xi,t) + [\frac{1}{\gamma^2}g_1(\xi,t)g_1^T(\xi,t) - g_2(\xi,t)g_2^T(\xi,t)]P_\varepsilon(t)\xi$$

$$+Z_\varepsilon(t)C_2^T(t)[y(t) - h_2(\xi,t)]\nu(t), \xi(t_0) = \xi^0 \tag{35}$$

$$\dot{x} = [\tilde{A}_{\varepsilon=0} + Z\gamma^{-2}PB_2B_2^TP)](t)x(t)$$

$$-Z(t)C_2^T(t)C_2(t)x(t)\nu(t), x(t_0) = x^0. \tag{36}$$

Due to Proposition 5 vibroimpact solutions of (34)-(36) are well-posed for arbitrary initial conditions, satisfy the differential equations (6), (8), (12) for all t but the sampling time moments, while their values at $t = \tau_j$, $j = 0, 1...$ are found through the relations

$$Z_\varepsilon(\tau_j+) = \mu_j(1), \quad \xi(\tau_j+) = \zeta_j(1), \quad x(\tau_j+) = \lambda_j(1) \tag{37}$$

by solving the auxiliary differential equations

$$\dot{\mu}_j(t) = -\mu_j(t)C_2^T(\tau_j)C_2(\tau_j)\mu_j(t), \mu_j(0) = Z_\varepsilon(\tau_j-), \tag{38}$$

$$\dot{\zeta}_j(t) = \mu_j(t)C_2^T(\tau_j)[y(\tau_j) - h_2(\zeta_j(t), \tau_j)], \zeta_j(0) = \xi(\tau_j-). \tag{39}$$

$$\dot{\lambda}_j(t) = -\mu_j(t)C_2^T(\tau_j)C_2(\tau_j)\lambda_j(t), \lambda_j(0) = x(\tau_j-). \tag{40}$$

Since equations (38) and (40) integrate to

$$\mu_j(t) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)t]^{-1},$$

$$\lambda_j(t) = \{I - Z(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z(\tau_j-)t]^{-1}$$

$$\times C_2^T(\tau_j)C_2(\tau_j)t\}x(\tau_j-),$$

relations (37)-(39) result in (7), (9), (10).

So, the vibroimpact solutions of (34), (35) with the impulsive input (31) coincide with the corresponding solutions of the differential equations (6), (8) with jumps (7), (9), (10) and the same initial conditions. In other words, the output feedback (14) with the external state $\xi(t)$, being a bounded, positive definite, symmetric vibroimpact solution of (35), coupled to (31), (34), represents another form of the local solution of the SMF H_∞-control problem whereas condition A2) is reformulated as follows:

A2') under $\varepsilon = 0$ there exists a bounded, positive semidefinite, symmetric vibroimpact solution $Z(t)$ of (34) such that the nonlinear differential equation (36) in distributions is exponentially stable.

Furthermore, substituting an arbitrary continuous strictly positive function $\sigma(t)$ for $\nu(t)$ in the SMF H_∞-controller equations (36), (14), (34), (35), we arrive at the standard H_∞-controller equations (cf. those of [2]) for the nonlinear system (1a), (1b) with the continuous-time measurements

$$y(t) = h_2(x(t),t) + k_{21}(x(t),t)w(t) \tag{41}$$

subject to

$$k_{21}(x,t)k_{21}^T(x,t) = \frac{1}{\sigma(t)}I, \tag{42}$$

while the last restriction in (4) is no longer in force. Conversely, the SMF H_∞-controller equations in distributions are straightforwardly obtained from the well-known CMF H_∞-controller equations by substituting distribution (31) for the function $\sigma(t)$ in the latter equations.

As a matter of fact, the SMF H_∞-controller thus constructed can be interpeted as a limiting result of the CMF H_∞-controllers under the weak* approximation of distribution (31) by continuous strictly positive peaking functions $\sigma(t)$. Indeed, Definition 4 admits such an interpretation of vibroimpact solutions of the SMF H_∞-controller equations in distributions. This interpetation, in turn, proves robustness of the H_∞-control synthesis to signal processing, regardless either sampled-data measurements or continuous measurements, made for a short time period, are accepted as an observation model.

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