RISK SENSITIVE FILTERING EQUATIONS IN INFINITE DIMENSIONAL SPACES WITH COUNTING OBSERVATION

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Abstract

The purpose of this paper is to compute the risk–sensitive filtering equations when the state process, given as the solution of a stochastic differential equation on an infinite dimensional Hilbert space, is observed through a counting observation.

1 Introduction

The filtering equations for finite dimensional systems with a counting observation have been established by Kliemann, Koch and Marchetti [4] and Elliott [2].

More recently, nonlinear filtering problems described by the solution of an infinite dimensional system observed in Gaussian noise have been studied by Ahmed, Furhman and Zabczyk [1] and Elliott and Moore [3]. The method used in the last cited paper relies on a change of measure technique which makes the observation a "white noise".

Risk–sensitive filtering with Poisson process observation has been studied by Malcom, James and Elliott in [5] for finite dimensional systems.

2 The model

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(\mathcal{F}_t)_{t\geq 0}$ a complete right–continuous filtration defined on this space.

Denote by $(X_t)_{t\geq 0}$ the stochastic process with values in a real separable Hilbert space $\mathcal{H}$ defined as the solution of the evolution equation written in the sense of Itô,

$$X_t = X_0 + \int_0^t (AX_s + F(X_s))ds + \int_0^t G(X_s)dW(s) \quad (1)$$

where

1. $(W(t))_{t\geq 0}$ is a cylindrical Wiener process with values in $\mathcal{H}$, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, with increments $W(t) - W(s)$ independent of $\mathcal{F}_s$ for $t \geq s$.

2. A is an unbounded operator which generates a continuous semi–group $S(t)$ such that $||S(t)|| \leq M\mathrm{e}^{-\alpha t}$ for some M and $\alpha > 0$.

3. F is a bounded Lipschitz mapping from $\mathcal{H}$ to $\mathcal{H}$.

4. G is a linear, closed operator on its domain $D(G)$ with values in a space of Hilbert–Schmidt operators.

The observation process is the $\mathbb{R}$–valued stochastic process $(y_t)_{t\geq 0}$ given by

$$y_t = N_t \quad (2)$$

where $(N_t)_{t\geq 0}$ is a counting process with intensity $\lambda(X_t)$. That is, λ is a mapping from $\mathcal{H}$ to $\mathbb{R}_+$ such that

$$E\left(\int_0^{+\infty} \lambda(X_s)^2 ds\right) < +\infty$$

and, the stochastic process defined for any $t \geq 0$ by $N_t - \int_0^t \lambda(X_s)ds$ is a $(\mathcal{F}_t)_{t\geq 0}$ martingale.

Let l be a measurable function mapping $\mathbb{R}^n$ into $\mathbb{R}$ with than quadratic growth, and for any real number θ denote by D_t the exponential process defined by

$$D_t = \exp\left(\theta \int_0^t l(x_s)ds\right).$$

For any measurable function ϕ mapping $\mathbb{R}^n$ into $\mathbb{R}$ for which the expectation is finite, define the filter associated to the problem (1)–(2) by

$$\pi_t(\phi) = E\left[\phi(x_t)D_t/\mathcal{Y}_t\right]$$

where $\mathcal{Y}_t = \sigma(y_s \ / \ 0 \leq s \leq t)$ is the filtration generated by the paths of the observation process up to time t.

3 The reference probability measure

Denote by $(Z_t)_{t\geq 0}$ the stochastic process defined for any $t \geq 0$ by

$$Z_t = \exp\left(\int_0^t \ln(\lambda(X_s))dN_s - \int_0^t (\lambda(X_s)-1)ds\right).$$

Then, denoting by $\bar{P}$ the probability measure defined on $(\Omega, \mathcal{F})$ by the Radon–Nikodym derivative

$$\left.\frac{d\bar{P}}{dP}\right|_{\mathcal{F}_t} = Z_t^{-1}$$

one can deduce from Girsanov's theorem (see e.g. [6]) that, under the probability measure $\bar{P}$, the stochastic process $(N_t)_{t\geq 0}$ is a counting process with intensity 1.

Therefore, defining for any function ϕ mapping $\mathcal{H}$ to $\mathbb{R}$ the "modified" unnormalized filter associated to the system (1)–(2) by

$$\rho_t(\phi) = \bar{E}\left[\phi(x_t)D_t Z_t \;/\; \mathcal{Y}_t\right]$$

one can prove, by means of standard arguments in non-linear filtering theory, that for any $t \geq 0$, the following conditionnal likelihood ratio holds $\pi_t(\phi) = \dfrac{\rho_t(\phi)}{\rho_t(1)}$.

4 The risk–sensitive Zakai equation

For any real separable Hilbert space $\mathcal{K}$, denote by $UC_b^k(\mathcal{H},\mathcal{K})$ the linear space of mappings ϕ from $\mathcal{H}$ into $\mathcal{K}$ which are bounded and uniformly continuous with all its Fréchet derivatives up to order k. If $\phi \in UC_b^0(\mathcal{H},\mathbb{R})$ is such that the mapping $x \to \phi(Ax)$ defined on $D(A)$ has a continuous extension to all $\mathcal{H}$, then its extension, which is unique, is denoted by Φ_A.

Following [1], denote by $\mathcal{D}_0$ the set

$$\mathcal{D}_0 = \left\{\phi \in UC_b^2(\mathcal{H},\mathbb{R}) \;/\; \phi_A \in UC_b^2(\mathcal{H},\mathbb{R}), \phi_{xx} \right.$$
$$\left. \text{and } (\phi_A)_{xx} \in UC_b^0(\mathcal{H}, L_1(\mathcal{H},\mathcal{H}))\right\}.$$

where $L_1(\mathcal{H},\mathcal{H})$ denotes the space of all nuclear operators from $\mathcal{H}$ into $\mathcal{H}$ with the nuclear norm.

Denoting by $\mathcal{L}$ the infinitesimal generator of the stochastic process $(X_t)_{t\geq 0}$ one has :

Theorem 4.1 *For any function ϕ in $\mathcal{D}_0$, the unnormalized filter $\rho_t(\phi)$ solves the following stochastic partial differential equation*

$$\begin{aligned}\rho_t(\phi) \;=\; & E(\phi(X_0)) + \int_0^t \rho_s(\mathcal{L}\phi)ds + \int_0^t \rho_s(\theta l\phi)ds \\ & + \int_0^t \rho_{s-}((\lambda-1)\phi)(dN_s - ds).\end{aligned}$$

Proof By Itô's rule, one gets

$$\begin{aligned}\phi(X_t)D_tZ_t \;=\; & \phi(X_0) + \int_0^t \mathcal{L}\phi(X_s)D_sZ_s ds \\ & + \int_0^t Z_sD_s\phi_x(X_s)G(X_s)dW(s) \\ & + \int_0^t \theta Z_sD_s l(X_s)\phi(X_s)ds \\ & + \int_0^t Z_{s-}D_s(\lambda(X_s)-1)\phi(X_s)(dN_s - ds).\end{aligned}$$

Then, one gets the Zakai equation (3), by conditionning each side of the above equality on $\mathcal{Y}_t$, using the fact that, under $\bar{P}$, the stochastic integral with respect to $W(t)$ is a local martingale and the stochastic Fubini theorem proved in [6].

5 The Kushner–Stratonovitch equation

By application of Itô's formula, one can deduce from the Zakai equation that the filter $\pi_t(.)$ solves a "modified" Kushner–Stratonovitch equation.

Theorem 5.1 *For any function ϕ in $\mathcal{D}_0$, the filter $\pi_t(\phi)$ solves the following stochastic partial differential equation*

$$\begin{aligned}\pi_t(\phi) \;=\; & \pi_0(\phi) + \int_0^t \pi_s(\mathcal{L}\phi)ds + \int_0^t (\pi_s(\theta l\phi) - \pi_s(\phi)\pi_s(\theta l))\,ds \\ & + \int_0^t \frac{1}{\pi_{s-}(\lambda)}\left(\pi_{s-}(\lambda\phi) - \pi_{s-}(\lambda)\pi_{s-}(\phi)\right). \\ & .(dN_s - \pi_{s-}(\lambda)ds).\end{aligned}$$

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