

EXISTENCE OF A DENSITY FOR THE FILTER ASSOCIATED TO HILBERT–SPACE VALUED SYSTEMS

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Abstract

The purpose of this paper is to prove that the unnormalized filter associated to a Hilbert space valued nonlinear filtering problem with correlated noises, which solves a parabolic stochastic partial differential equation, the Zakai equation, admits a density with respect to a given measure on the Hilbert space. We also compute the stochastic differential equation solved by this density.

1 Introduction

The purpose of this paper is to study a correlated filtering problem, when both the signal and the observation processes evolve in infinite dimensional Hilbert spaces H and V .

Our aim is to prove that the filter associated to this filtering problem is absolutely continuous with respect to a given reference measure μ on H . In fact, we prove that the Zakai equation, solved by the unnormalized filter, has a solution which admits a density with respect to μ and we compute the equation solved by this density.

Filtering problems with a signal process evolving in a infinite dimensional Hilbert space and a finite dimensional observation process have been studied by N.Ahmed, M.Fuhrman and J.Zabczyk [1]. In particular, the existence of an absolutely continuous solution of the Zakai equation solved by the unnormalized filter is established in [1]. We generalize this result when both the signal and the observation processes also evolve in a infinite dimensional Hilbert space.

This paper is organized as follow. In the second section, we define the filtering problem we are dealing with and in the third section we state our main result, that is the existence of a solution of the Zakai equation which admits a density with respect to the choosen measure μ on H .

2 Setting of the problem

Let H and U be two real separable Hilbert spaces, $(W_1(t))_{t \geq 0}$ and $(W_2(t))_{t \geq 0}$ be two independent cylindrical Wiener processes defined on H and U respec-

tively and $(X_t, Y_t)_{t \geq 0}$ be a stochastic process on $H \times U$, solution of the following filtering problem

$$\begin{cases} dX_t = (AX_t + F(X_t)) dt + R dW_1(t) + B dW_2(t) \\ dY_s = G(X_t) dt + dW_2(t) \end{cases} \quad (1)$$

where :

(H₁) X_0 is given in H and $Y_0 = 0$.

(H₂) The semi-group $(S(t))_{t \geq 0}$ generated by the operator A is such that $\|S(t)\| \leq M e^{-\alpha t}$ for M and $\alpha > 0$.

(H₃) $F : H \rightarrow H$ is Lipschitz continuous, $G : H \rightarrow U$ is continuous and bounded and R is a non-negative self-adjoint operator.

(H₄) The operators $Q_t = \int_0^t S(s)Q S(s)^* ds$, $t \geq 0$, with $Q = RR^* + BB^*$, are trace class.

(H₅) The operator $Q_\infty = \int_0^\infty e^{tA} Q e^{tA^*} dt$ is nuclear.

Let K be a real separable Hilbert space and denote by $UC_b^k(H, K)$, $k \in \mathbb{N}$, the linear space of all mappings ϕ from H into K which are bounded and uniformly continuous as well as all their Fréchet derivatives up to order k .

Furthermore, following [1], denote by \mathcal{D}_0 the space $\mathcal{D}_0 = \{ \phi \in UC_b^0(H, \mathbb{R}) / \phi_A \in UC_b^0(H, \mathbb{R}), \phi_{xx} \text{ and } (\phi_A)_{xx} \in UC_b^0(H, L_1(H, H)) \}$.

Let $(Z_t)_{t \geq 0}$ be the stochastic process defined by :

$$Z_t = \exp \left(\int_0^t \langle G(X_s), dY_s \rangle_V - \frac{1}{2} \int_0^t \|G(X_s)\|_V^2 ds \right).$$

Then, since G is bounded, one can prove easily that $E(Z_t^{-1}) = 1$ for every $t \geq 0$, and hence, denoting by \bar{P} the probability on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ by the Radon–Nikodym derivative $\left. \frac{d\bar{P}}{dP} \right|_{\mathcal{F}_t} = Z_t^{-1}$, the version of Girsanov’s theorem proved by J.Y.Ouvrard in [3] implies

that $(Y_t)_{t \geq 0}$ is, under \bar{P} , a $(\mathcal{F}_t)_{t \geq 0}$ -Wiener process independent of $(W_1(t))_{t \geq 0}$.

Then, one can define for any $\phi \in \mathcal{D}_0$ the filter and the unnormalized filter associated to the nonlinear filtering problem (1) by $\pi_t(\phi) = E[\phi(X_t)/\mathcal{Y}_t]$ and $\rho_t(\phi) = \bar{E}[\phi(X_t)Z_t/\mathcal{Y}_t]$ respectively.

Furthermore these processes are related by the Kallianpur–Striebel formula (see [2]) $\pi_t(\phi) = \frac{\rho_t(\phi)}{\rho_t(1)}$.

Finally, we recall that for every ϕ in \mathcal{D}_0 , the unnormalized filter ρ_t associated to the filtering problem (1) is solution of the following s.p.d.e, the Zakai equation

$$d\rho_t(\phi) = \rho_t(\mathcal{L}_0\phi) dt + \langle \rho_t(\mathcal{L}_1\phi), dY_t \rangle_V$$

where

$$\mathcal{L}_0\phi(X) = \frac{1}{2} \text{Tr}(Q \phi_{xx}(X)) + \langle X, A^* \phi_x(X) \rangle_H + \langle F(X), \phi_x(X) \rangle_H \text{ and } \mathcal{L}_1\phi(X) = G(X)\phi(X) + B^* \phi_x(X).$$

3 Existence of a density for the filter

We consider as a reference measure μ on H the Gaussian measure with mean 0 and covariance operator Q_∞ , denoted by $\mu = \mathcal{N}(0, Q_\infty)$.

Our aim is to prove the existence, under appropriate assumptions, of a process \mathcal{Y}_t -adapted, $(q_t)_{t \geq 0}$, with value in $\mathcal{H} = L^2(H, \mu)$, such that the process $(\sigma_t)_{t \geq 0}$, defined by $\sigma_t(dx) = q_t(x)\mu(dx)$, $t \geq 0$, solves the Zakai equation. We also give the equation satisfied by the process $(q_t)_{t \geq 0}$.

With this aim, suppose that the following assumption is satisfied

(H₆) The law ν of X_0 is absolutely continuous with respect to μ and $q_0 = \frac{d\nu}{d\mu} \in L^2(H, \mu)$.

Let (e_k) be an orthonormal basis of H consisting of eigenvectors of Q_∞ and denote by $C_b^\infty(\mathbb{R}^m)$ the set of all functions from \mathbb{R}^m into \mathbb{R} with continuous and bounded derivatives of all orders. Then, we say that a function $\phi : H \rightarrow \mathbb{R}$ belongs to $\mathcal{F}C_b^\infty(H)$ if there exist $m \in \mathbb{N}$ and $f \in C_b^\infty(\mathbb{R}^m)$ such that $\phi(x) = f(\langle x, e_1 \rangle, \dots, \langle x, e_m \rangle)$ for $x \in H$.

Then we define the Hilbert space $\mathcal{V} = W_Q^{1,2}(\mu)$ as the completion of $\mathcal{F}C_b^\infty(H)$ with respect to the norm $\|\phi\|_{\mathcal{V}}^2 = \int_H \|Q^{\frac{1}{2}}\phi_x(x)\|^2 \mu(dx) + \int_H \phi(x)^2 \mu(dx)$ and \mathcal{V} can be identified as a subspace of $L^2(H, \mu)$, the imbedding being dense and continuous.

Furthermore, define a bilinear form \mathcal{G} on $\mathcal{V} \times \mathcal{V}$ by

$$\mathcal{G}(\phi, \psi) = \int_H \langle \phi_x(x), A Q_\infty \psi_x(x) \rangle$$

$$+ \langle F(x), \phi_x(x) \rangle \mu(dx)$$

and an operator \mathcal{B} from \mathcal{V} to $L_2(V, L^2(H, \mu))$ by $(\mathcal{B}(q)u)(x) = \langle G(x)q(x), u \rangle_V - \langle B_1^* q_x(x), u \rangle_V + \ll Q_\infty^{-\frac{1}{2}} B_1 u, Q_\infty^{-\frac{1}{2}} x \gg q(x)$.

Then, we obtain the main theorem of this paper.

Theorem Assume that hypotheses (H₁) to (H₆) hold and that

(i) Image $Q_\infty \subset \text{Dom}(A)$ and there exists $K > 0$ such that for every $x, y \in H$

$$| \langle x, A Q_\infty y \rangle | \leq K |Q^{\frac{1}{2}} x| |Q^{\frac{1}{2}} y|.$$

(ii) There exists $C > 0$ such that for every $x \in H$

$$|Q^{-\frac{1}{2}} F(x)| \leq C.$$

(iii) Image $B_1 \subset \text{Image } Q_\infty^{\frac{1}{2}} \cap \text{Image } R_1^{\frac{1}{2}}$.

Then Zakai's equation has a solution of the form

$$\sigma_t(dx) = q_t(x)\mu(dx), \quad t \geq 0$$

and the $L^2(H, \mu)$ -continuous process $(q_t)_{t \geq 0}$ is the unique weak solution of the s.p.d.e

$$dq_t = \mathcal{L}^* q_t dt + \mathcal{B} q_t dY_t.$$

Example Suppose that $H = L^2(0, 1)$ and consider the stochastic heat equation

$$dX(t, \xi) = \left(\frac{\partial^2 X}{\partial \xi^2}(t, \xi) + f(X(t, \xi)) \right) dt + dW_1(t, \xi) + b(\xi) dW_2(t) \text{ with } X(t, 0) = X(t, 1) = 0 \text{ and } X(0, \cdot) = X_0(\cdot).$$

We suppose that the observation of the filtering problem solves the following equation

$$dY_t = G(X_t) dt + dW_2(t) \text{ with } Y_0 = 0.$$

Then, if f is a bounded lipschitz function from \mathbb{R} to \mathbb{R} , if G is a continuous bounded function from H to V and if $b \in H_0^1(0, 1)$, the Zakai equation has a solution of the form $\sigma_t(dx) = q_t(x)\mu(dx)$, $t \geq 0$ where q_t is a \mathcal{Y}_t -adapted process, with value into $L^2(H, \mu)$.

References

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