A Disk "Optimal" Robust Pole Assignment using Genetic Algorithms

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Abstract

This paper presents a method for determining the maximum allowable perturbation, for a m.i.m.o. linear time-invariant uncertain system, such that the closed loop poles are assigned in a specified disk by static output feedback; the uncertainty being norm-2 bounded. A sufficient condition for *d*-stabilizability is derived. Hence, in order to solve the related optimization problem a genetic-like algorithm is performed. An illustrative example shows the effectiveness of the proposed procedure.

Keywords: *D*-pole assignment, robustness, genetic alg. .

I Introduction

The dynamic response of a linear time-invariant system can be modified, by means of a static output feedback, placing the poles in desired locations of the complex plane. In practice, due to the uncertain on the parameters on the model plant, the exact pole location is not required and it is often sufficient to locate the closed-loop poles in a prescribed region of the complex plane. In [2], a sufficient condition for disk location is obtained by modification of Lyapunov matrix equation. For norm-bounded uncertain systems, necessary and sufficient conditions in terms of parameter-dependent Riccati equations are given in [3]. This paper deals with the problem of pole assignment in a specified disk by static output feedback for linear time-invariant uncertain systems, the uncertainty being of norm-2 bounded type. This note extend the concept of quadratic stabilizability to the case of disk pole location in a similar manner as proposed in [3]. In order to characterize the uncertainty, we refer to the theory of "well-conditioning" related to the solution of the pole assignment problem [4]. The above problem is formalized as an optimization one. In order to solve it, we propose a numerical procedure based on the genetic algorithms, [6]-[7].

II Problem Formulation and Preliminaries

Let us consider a continuous system described by

$$\dot{x} = (A + \delta A)x + Bu; \;\; y = Cx \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, e $y \in \mathbf{R}^p$. The triplet (A, B, C) is controllable and observable with B and C of full rank and $mp > n$ and $p \geq m$. The uncertainty δA belongs to the set: $\mathcal{T} = \left\{ \delta A \in \mathcal{R}^{n \times n} : \|\delta A\|_2 < \gamma^* \right\}$. If the control law $u = Ky$ is applied to (1), the closed-loop system becomes

$$\dot{x} = (A + \delta A + BKC)x$$

Let us introduce the following notations: $A_c = A + BKC$, $A_{c\,\alpha} = (A + BKC - \alpha I)/r$, $\delta A_{c\,\alpha} = \delta A / r$.
In the paper [1], in order to assign a desired symmetric set of complex numbers, a method is reported that allows to characterize a parametric set of matrices, that is

$$K = K(\overline{\Theta}), \qquad \overline{\Theta} \in \mathcal{R}^{mp-n}$$

Problem: Given α and r scalars, $\alpha < 0$ and $r < |\alpha|$, and a set of symmetric complex number $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ within the disk $D(\alpha, r)$, find a gain matrix K such that

$$\gamma^* = \max_{\Theta_{n-p} \in \mathcal{S}_{n-p}} \gamma(\Theta_{n-p})$$

under the constraints: $eig(A_c) = \Lambda$ and $eig(A_c + \delta A) \in D(\alpha, r)$, $\forall \delta A \in \mathcal{T}$, where $\mathcal{S}_{n-p}$ is a compact set in $\mathcal{R}_{n-p}$, therefore the optimization on $\gamma(\Theta_{n-p})$ also gives the corresponding "optimal" gain matrix with respect to the constraints. In the sequel we recall some well known results that allow to prove a novel sufficient condition on the quadratically stabilizability.

Theorem 1: *[2] Consider the following matrix equation:*

$$\alpha A_c^* P + \alpha P A_c - A_c^* P A_c - (\alpha^2 - r^2) P = Q, \tag{2}$$

where Q is positive definite. Then the eigenvalues of A_c are within $D(\alpha, r)$ if and only if there exists a positive definite solution P satisfying (2).

Definition 1: *The system (1) is a "quadratically d stabilizable" by a gain output feedback $u = Ky$ if and only if there exists a symmetric matrix $P \in \mathcal{R}^{n \times n}$ such that:*

$$X' \begin{pmatrix} -P^{-1} & A_{c\,\alpha} + \delta A_{c\,\alpha} \\ A'_{c\,\alpha} + \delta A'_{c\,\alpha} & -P \end{pmatrix} X < 0, \tag{3}$$

where $X \in \mathcal{R}^{2n}$ and $\delta A \in \mathcal{T}$.

Theorem 2: *[3] If the system (1) is quadratically d stabilizable then it is quadratically stabilizable and P is a Lyapunov matrix for all the systems in the uncertainty domain.*

Theorem 3: *[4] If the output feedback matrix K, that assigns the set Λ, then the perturbed closed-loop system matrix $A + BKC + \delta A$ remains stable for all disturbances δA which satisfy*

$$\|\delta A\|_2 < \min_{s = j\omega} \sigma_n \left\{ sI - (A + BKC) \right\} \equiv \gamma, \tag{4}$$

where a lower bound on γ is $\gamma \geq \min_i \mathrm{Re}(-\lambda_i)/\kappa_2(\Omega)$, σ_n is the lowest singular value, Ω is the modal closed-loop matrix and $\kappa_2(\Omega)$ is the condition number [5].

III Main Results

Proposition 1:*Let ΔA a norm bounded matrix such that $\|\Delta A\|_2 = \overline{\gamma} < \gamma$, P a symmetric positive definite solution of the matrix equation (2) with Q symmetric and positive definite and $A_{cp} = A_c + \Delta A$. If*

$$\alpha A_{cp}^* P + \alpha P A_{cp} - A_{cp}^* P A_{cp} + (\alpha^2 - r^2)P > 0 \qquad (5)$$

the system (1) is quadratically stabilizable for all the systems in the uncertainty domain $\mathcal{T}$.

Proof: Since $\Lambda \in D(\alpha, r)$, and P is a positive definite and symmetric matrix, solution of the Riccati equation (2), it is evident that

$$\begin{pmatrix} -P^{-1} & A_{c\alpha} \\ A_{c\alpha}^T & -P \end{pmatrix} < 0$$

Moreover, Theorem 3 gives the bound γ such that $eig(A + \Delta A + BKC) \in \mathcal{C}^-$ for all the disturbance matrices with $\|\Delta A\|_2 < \gamma$, therefore, by hypothesis that $\overline{\gamma}$ is less than γ, one is guaranteed that, if (5) is hold true, the system (1) is quadratically stabilizable and P is a Lyapunov matrix. □

IV Numerical Procedure

The critical point that arises in this problem is the extremely large search space, therefore, in order to obtain a solution, exhaustive search methods appears unacceptable. Taking into account the structural nonlinearity of the pole placement problem and of its related ones, genetic algorithms [6] seem us a satisfactory way to face them. In this case, the objective (fitness) of the genetic algorithm is to find

$$f(\Theta_{n-p}) = \max \overline{\gamma}(\Theta_{n-p}) \qquad (6)$$

For sake of clearness the stabilization algorithm inside the genetic one is the following:

1. Solve the algebraic Riccati equation (2)
2. Check the condition (5). If it holds go to *Step 3*, otherwise zero value is assigned to the fitness
3. Compute the fitness $\overline{\gamma}(\Theta_{n-p})$

The key point of numerical optimization procedure is to define the subset in $\mathcal{R}^{n-p}$ to initialize the algorithm. The difficulty arises from the fact that Θ_{n-p} does not relate in any way to the structure of the problem to allow a restriction of search space. In order to overcome this obstacle, we have used the features of genetic algorithms: indeed, they go to some local minimum also if there is a huge parameter search interval. Hence, starting from any "initial condition" and applying recursively our genetic procedure with a low number of generations (usually ten), by a comparison operation we easily reduce the search space until finding the "optimal" one.

V Numerical Example

To illustrate D-pole assignment, we take a time continuous system as following

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The specification is to place the poles of the nominal closed loop system in $\Lambda = \{-5, -4, -3\}$ and to determine the bound γ^* such the closed-loop uncertain system has its poles within the disk with center $\alpha = -4$ and radius $r = 2.5$. The matrix Q is the identity matrix. The objective function depends on $\Theta_{n-p} = \{\theta_{21}, \theta_{31}\}$, with $\theta_{21} \in [-1.5, -1]$, $\theta_{31} \in [3.5, 4.5]$. By applying our genetic procedure, the maximum bound for the disturbance matrix is $\gamma^* = 0.04253$ and the feedback matrix is:

$$\overline{K} = \begin{bmatrix} -37.65687 & -13.00000 \\ -99.65687 & -22.34312 \end{bmatrix}.$$

Fig. 1 shows the closed-loop poles for 10000 disturbances δA with entries uniformly distributed in $[-1, 1]$ and such that $\|\delta A\|_2 < 0.0425$.

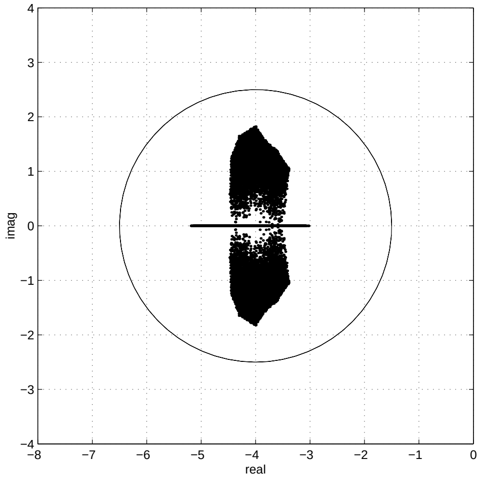

Figure 1: Closed-loop poles

VI Conclusions

In this note, a "optimal" problem of pole assignment in a disk for linear time-invariant uncertain systems has been addressed under the quadratic stabilizability framework. The uncertainty considered is of norm-2 bounded type and only the dynamical matrix has been considered uncertain. Moreover, a d-stabilization procedure is given and a genetic algorithm is used to find the desired maximum perturbation. A new sufficient condition for quadratic pole assignability in a disk has been given. It is worth noticing that if the genetic search is unsuccessful, it cannot makes any conclusions about stability or instability. Hence, in order to overcome such drawback, a further research is necessary.

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