ON THE CONTROLLABILITY OF SYSTEMS ON COMPACT LIE GROUPS AND QUANTUM MECHANICAL SYSTEMS

Domenico D'Alessandro
Department of Mathematics
Iowa State University
Ames, IA 50011, USA
Tel. (+1) 515 294 8130
email: daless@iastate.edu

Abstract

In this paper, we develop some general results on the properties of the reachable sets for right invariant bilinear systems with state varying on compact Lie groups. The main results consist of a characterization of the set of states reachable in arbitrary time from the identity of the group. This, under suitable assumptions, is proved to be a Lie subgroup of the underlying Lie group. We apply these results to the analysis of the controllability of particles with spin. For these systems we also obtain estimates of the time after which every state is reachable from the identity. The results are motivated by the problem of controlling a two-level quantum system in implementations of quantum computers.

Keywords: Control of Quantum Mechanical Systems, Reachable sets, Systems on Lie Groups, Particles with Spin.

1 Introduction

In recent years there has been a large amount of interest in the development and application of techniques from control theory for the manipulation of the state of quantum mechanical systems (see e.g. [3], [7], [10], [16]). In typical laboratory experiments, an electro-magnetic field is used to drive the state of a quantum system to a desired value. The electro-magnetic field is seen as a control and classical issues of control theory have a natural physical interpretation in this setting. Systems like the spin $\frac{1}{2}$ particles in electro-magnetic field have recently been used to perform logic operation in implementation of quantum computers [8]. For systems of this type, the describing model is given by Schrödinger equation with the control multiplying the state variable. This is a right-invariant bilinear system whose state varies on a compact Lie group.

Controllability of systems on Lie groups was dealt with in the classical paper [12] and in a number of following papers. The survey article [18] and the book [11] give an up-to-date account of the main results and we refer to them for further references on this topic. It is a problem of great fundamental and practical importance to characterize the set of states that can be obtained in arbitrary small time. This set is in general not the whole set of reachable states even if the control is allowed to be a general Lebesgue measurable function (see Example 8.1 in [12]). In the present paper, we present a study of this set and we apply the results to the controllability analysis of quantum mechanical systems with an arbitrary value of the spin. This generalizes the results of [4] which concerned two-level systems.

The paper is organized as follows. Sections 2 through 4 are of general interest since they deal with controllability properties of general right invariant bilinear systems on compact Lie groups. In particular in Section 2 we describe the mathematical model we want to study and give the basic definitions. We also give a sufficient condition for the set of states reachable at any arbitrary time from the identity to be empty. This motivates the study of this set in the following two sections. In Section 3, we state that this set is either empty or it is dense in a Lie subgroup of the underlying Lie group and in Section 4 we relate the property of this set to be a connected Lie subgroup to the small time local controllability of the identity of the group. If this is verified, given the correspondence between connected Lie subgroup and Lie subalgebras, the problem of characterizing the set of states reachable from the identity at any arbitrary time can be approached studying the structure of the Lie algebra. Sections 5 and 6 contain application of these results to the system of a particle with spin in an electro-magnetic field.

2 Systems on compact Lie groups

In this section we study general systems of the form

$$\dot{X} = AX + \sum_{i=1}^{m} B_i X u_i. \tag{1}$$

The state X varies on a compact matrix Lie group G while the matrices A, B_i, $i = 1, ..., m$, are constant matrices belonging to the corresponding Lie algebra $\mathcal{G}$. The restriction to *matrix* Lie groups is not necessary in the following and is considered here only for sake of concreteness. The control functions u_i, $i = 1, ..., m$, are piecewise continuous functions defined on some interval of $\mathbf{R}^+$. The matrices B_i, $i = 1, ..., m$, are assumed to be linearly independent although this is done without loss of generality. In fact, it is always possible to reduce the analysis of the behavior of the system (1) to this case by opportunely redefining the control functions. System (1) is right invariant in that if $X(t)$ is a solution corresponding to the initial condition equal to the identity matrix, the solution corresponding to the initial condition F is given by $X(t)F$.

The following Lie algebras and Lie groups are associated to the system in (1)

- $\mathcal{L}$ is the Lie algebra generated by $\{A, B_1, ..., B_m\}$ and $e^{\mathcal{L}}$ is the corresponding connected Lie subgroup of G.
- $\mathcal{L}_0$ is the ideal in $\mathcal{L}$ generated by $\{B_1, ..., B_m\}$ and $e^{\mathcal{L}_0}$ is the corresponding connected Lie subgroup of G.
- $\mathcal{B}$ is the Lie algebra generated by $\{B_1, ..., B_m\}$ and $e^{\mathcal{B}}$ is the corresponding connected Lie subgroup of G.

Notice that $\mathcal{L}_0$ has co-dimension 0 or 1 in $\mathcal{L}$ according to whether or not A is in $\mathcal{L}_0$.

The following sets of states reachable from the identity I are associated to the system (1).

- $R(T)$; The set of states reachable from the identity at time T.
- $\mathcal{R}(\leq T) := \cup_{0 \leq t \leq T} R(t)$.
- $\mathcal{R} := \cup_{0 \leq t < \infty} R(t)$.

We have $R(0) = \mathcal{R}(\leq 0) = \{I\}$, and, by right invariance, the sets of states reachable *from* a point $X \in G$ are given by $R(T)X$, $\mathcal{R}(\leq T)X$ and $\mathcal{R}X$, respectively. Therefore a study of the states reachable from the identity gives information on the states reachable from any other point.

From the results of [12], we have that $\mathcal{R} = G$ if and only if $\mathcal{L} = \mathcal{G}$ and, more in general, $\mathcal{R} = e^{\mathcal{L}}$. Moreover there exists a time T such that $\mathcal{R}(\leq T) = e^{\mathcal{L}}$. At every time t, $R(t) \subseteq e^{At}e^{\mathcal{L}_0}$, and the interior of $R(t)$ with respect to the topology of $e^{At}e^{\mathcal{L}_0}$ is not empty. It also follows from a result in [13] that, if $\mathcal{B} = \mathcal{L}$, then $R(t) = e^{\mathcal{L}}$, for every $t > 0$. This is the case for homogeneous controllable systems ($A = 0$) [2].

The main topic of the following two sections is the study of the set of states reachable at any arbitrary time in the cases where the above recalled condition does not guarantee it to be equal to $e^{\mathcal{L}}$. More specifically, we are interested in the study of the set

$$\mathcal{A} := \cap_{t>0} R(t). \tag{2}$$

The examples in [12] [4] show that the set $\mathcal{A}$ might not be the whole $e^{\mathcal{L}}$. In fact, using the property $R(t) \subseteq e^{\mathcal{L}}$, it is possible to show the following proposition.

Proposition 2.1. *If $\mathcal{L}_0$ has co-dimension 1 in $\mathcal{L}$ then $\mathcal{A}$ is empty.*

In the following, we consider the system (1) as varying on $e^{\mathcal{L}}$ and the topology on $e^{\mathcal{L}}$ is the one induced by the one of G. Since we will be studying the set $\mathcal{A}$, we can assume, from the previous proposition, that $\mathcal{L}_0 = \mathcal{L}$. Since the interior of $R(t)$ is not empty in $e^{At}e^{\mathcal{L}_0}$ [12], for every t, we have, in our case, that $R(t)$ has nonempty interior in $e^{\mathcal{L}}$ for every t.

3 Set of states reachable at any arbitrary time

In the following three theorems, we assume that $\mathcal{A}$ is not empty. This can be checked, for example, by constructing a class of controllers (for example constant controls) steering to a fixed point in G (for example the identity) in arbitrary time.

Theorem 3.1. *Assume $\mathcal{A}$ is not empty, then it is a semigroup and $\bar{\mathcal{A}}$ is a Lie subgroup of G, in particular it contains the identity I.*

Theorem 3.2 *Assume $\mathcal{A}$ is not empty. If $t_1 < t_2$, then $\bar{R}(t_1) \subseteq \bar{R}(t_2)$ and $intR(t_1) \subseteq intR(t_2)$.*

Theorem 3.3 *Assume $\mathcal{A}$ is not empty. Then, $e^{\mathcal{B}} \subseteq \cap_{t>0}\bar{R}(t)$.*

4 Consequences of Small Time Local Controllability

The three Theorems in the previous section can all be sharpened if we assume *Small Time Local Controllability* for the identity element I of the group ($STLCI$). $STLCI$ means that there exists a time $T > 0$ such that the identity is in the interior of the reachable set $R(t)$ for every t, $0 < t \leq T$. This is readily seen to imply that the identity is in the interior of the reachable set $R(t)$ for every $t > 0$.

The following result contains an improvement of Theorem 3.1 in the case where $STLCI$ is verified.

Theorem 4.1. *$STLCI$ implies that $\mathcal{A}$ is not empty and it is a closed Lie subgroup of G*

The following results sharpen Theorems 3.2 and 3.3, in the case where $STLCI$ is verified.

Theorem 4.2. *Assume STLCI is verified. Then $t_1 < t_2$ implies that $R(t_1) \subseteq R(t_2)$*

Theorem 4.3. *Assume STLCI is verified. Then $\cap_{t>0}\bar{R}(t) = \cap_{t>0}R(t) := \mathcal{A}$. As a consequence, $e^{\mathcal{B}} \subseteq \mathcal{A}$.*

Theorem 4.4. *If STLCI is verified, then $\mathcal{A}$ is a closed, connected Lie subgroup of G whose Lie algebra contains $\mathcal{B}$.*

From Theorem 4.4, it follows that, once $STLCI$ is proved, one can approach the problem of characterizing $\mathcal{A}$ at the Lie algebra level. In fact $\mathcal{A}$ is a Lie group whose Lie algebra contains $\mathcal{B}$. One can consider all the Lie algebras containing $\mathcal{B}$. In some cases, as in the case of systems with spin angular momentum considered in the next section, the only Lie algebra containing $\mathcal{B}$ is $\mathcal{L}$. Therefore, in these cases, if one proves that not every state can be reached in arbitrary time then it immediately follows $\mathcal{A} = e^{\mathcal{B}}$.

There have been many studies concerning the property of Small Time Local Controllability for a given point in the state space of a nonlinear system. Many results (see e.g. [18]) deal with the case where the point is an equilibrium point. The following simple criterion, based on the Maximum Principle [1] [18] will be applied for the systems of interest here. The proof is a generalization of the one used in [4] for the case of two-level quantum systems.

Theorem 4.5. *Assume there exists a time T such that, for every $\tau \leq T$, there exists a piecewise constant control u_τ steering to the identity in time τ. Denote the values assumed by the function u_τ by $\mathcal{U}_\tau := \{u_1, u_2, ..., u_{k(\tau)}\}$. For a value u_j define the matrix*

$$F_j := A + \sum_{i=1}^{m} B_i u_{ij}, \tag{3}$$

where u_{ij}, $i = 1, ..., m$, are the components of u_j. Assume that for every τ, there exists a $u_j \in \mathcal{U}_\tau$ such that [1]

$$ad^n_{F_j} B_i, \qquad n = 0, 1, 2, ..., k, \qquad i = 1, ..., m, \tag{4}$$

span the whole Lie Algebra $\mathcal{L}$. Here k is the dimension of the Lie group. Then, the system has the STLCI property.

5 Particles with spin in an electro-magnetic field

In this and the following section we apply the results obtained in the previous sections and perform the controllability analysis of a class of quantum systems. We consider a particle with spin and all the other degrees of freedom ignored under the action of an externally applied electro-magnetic field. We review the basic facts about the mathematical model in this section (see e.g. [17]) and perform the controllability analysis in the next section.

The (time varying) Hamiltonian describing the system is given by

$$H(t) := \gamma \mathbf{J} \cdot \mathbf{B} := \gamma(J_x B_x(t) + J_y B_y(t) + J_z B_z(t)). \tag{5}$$

In (5) γ is the *gyromagnetic ratio* of the particle, $J_{x,y,z}$ are the x, y, z components of the spin angular momentum operators and $B_{x,y,z}$ are the (time varying) components of the electro-magnetic field which play the role of control. $J_{x,y,z}$ are Hermitian operators on the underlying Hilbert space which satisfy the *fundamental commutation relations*

$$[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x, \quad [J_z, J_x] = i\hbar J_y. \tag{6}$$

The theory of angular momentum in quantum mechanics originates from these relations (see e.g. [17] Chpt. 3). The evolution (rotation) operator X is obtained by solving Schrödinger equation

$$i\hbar \dot{X}(t) = H(t)X(t), \tag{7}$$

with initial condition $X(0)$ given by the identity operator. The Hamiltonian is given in (5), and we are interested here in a controllability analysis of this system, namely we want to investigate what are the rotations that can be achieved in a particular configuration for system (7).

The spin of a particle may assume a value j which is either a positive integer or a positive half integer. For a particle with spin j the operators J_x,, J_y, J_z can be represented by $2j+1 \times 2j+1$ Hermitian matrices which we still denote by J_x, J_y, J_z. Defining $S_{x,y,z} := \frac{-iJ_{x,y,z}}{\hbar}$, we can write Schrödinger equation (7), (5) for the evolution matrix as

$$\dot{X}(t) = \gamma(S_x B_x(t) + S_y B_y(t) + S_z B_z(t))X(t), \tag{8}$$

which has to be solved with $X(0) = I_{2j+1 \times 2j+1}$. The matrices S_x, S_y, S_z satisfy the commutation relations corresponding to (6)

$$[S_x, S_y] = S_z, \quad [S_y, S_z] = S_x, \quad [S_z, S_x] = S_y. \tag{9}$$

They are skew-Hermitian and it follows immediately from (9) that they have zero trace. Therefore, they span a three-dimensional subalgebra of the Lie algebra $su(2j+1)$ of skew-Hermitian $2j+1 \times 2j+1$ matrices with zero trace. We denote this 3-dimensional Lie algebra by $\mathcal{G}_j$ and the corresponding connected Lie subgroup of $SU(2j+1)$ by G_j. An inner product $< \cdot, \cdot >$ can be defined in $\mathcal{G}_j$ by

$$< A, B > := Trace(AB^*), \tag{10}$$

where B^* denotes the conjugate transpose of the matrix B. The Lie algebra $\mathcal{G}_j$ is semisimple (it is not Abelian and it has no Abelian ideal) and the Lie subgroup is compact.

A result about the isomorphism between the Lie group G_j and $SU(2)$ or $SO(3)$ is crucial to the controllability analysis for spin angular momentum systems that will follow. This result reduces the study to essentially two cases: the Lie group $SU(2)$ and the Lie group $SO(3)$. This result appeared in a study by E. P. Wigner ([19] pp. 163-168). We state it in the following theorem.

Theorem 5.1 *G_j is isomorphic to $SO(3)$ for j integer and isomorphic to $SU(2)$ for j half-integer.*

---

[1] $ad^0_X Y := Y, ad^k_X Y = [X, ad^{k-1}_X Y]$

6 Controllability of Spin Angular Momentum

We refer to the system in the general form (1) that we repeat here

$$\dot{X} = AX + \sum_{i=1}^{m} B_i X u_i, \tag{11}$$

where it is now understood that the matrices $A, B_1, ..., B_m$ are in the Lie algebras $\mathcal{G}_j$, as defined in the previous Section. This general form include all the possible geometric configurations that can be realized in a laboratory. For example, one may apply a constant electro-magnetic field and a time varying one at an angle of 30^o in the $x-y$ plane so that both the components of the field in the x and y directions have a constant component (modeled by the matrix A) and a time varying component.

First notice that if two (or more) inputs are available, $B_1, ...B_m$ generate the whole Lie algebra $\mathcal{G}_j$. This follows from the linear independence of $B_1, ..., B_m$. In this case, one can apply a result in [13] (Theorem 5.3) to conclude that $R(t) = G_j$ for every t. Therefore the only nontrivial case is the single-input one. We also assume that A and B_1 are linearly independent in this case which, in physical terms, means that there are at least two non parallel directions for the driving electro-magnetic field. If this is not the case then the solution of (11) is just $X(t) = e^{\int_0^t A + B_1 u(\tau) d\tau}$.

Consider now system (11) with a single input assuming the matrices A and B are in $su(2)$ (or $so(3)$) and the corresponding solution X in $SU(2)$ (or $SO(3)$). This is always possible because of the Lie group isomorphism of Theorem 5.1. Explicit expressions for this isomorphism are given for example in [9] (pp. 135-141). We write the system as

$$\dot{X} = AX + BXu. \tag{12}$$

Consider the constant input $u = -\frac{<A,B>}{<B,B>} + v$. The eigenvalues of the matrix $A + Bu$ are $0, \pm i\sqrt{v^2 + p^2}$, in the $so(3)$ case, and $\pm i\sqrt{\frac{<B,B>}{2} v^2 + p^2}$, in the $su(2)$ case. Here p is the magnitude of the purely imaginary conjugate eigenvalues of $A - \frac{<A,B>}{<B,B>} B$. These expressions show that the nonzero eigenvalues can be made arbitrarily large in magnitude by choosing v large, and therefore the corresponding solution of (12) returns to the identity in arbitrary small time. This shows that the identity is in $R(t)$ for each t. Define $F := A - \frac{<A,B>}{<B,B>} B + Bv$. Since A and B are assumed to be linearly independent so are B and F. Recalling that $su(2)$ (and $so(3)$) have no two-dimensional subalgebras, it is easily seen that B, $ad_F B$ and $ad_F^2 B$ span the whole $su(2)$ (or $so(3)$) so that we can apply Theorem 4.5 to conclude that the identity is in the interior of the reachable set $R(t)$ for every t. Using again the fact that $su(2)$ ($so(3)$) does not have two dimensional subalgebras we conclude from Theorem 4.4 that the set of states reachable in arbitrary time $\mathcal{A}$ for this system is either the whole group or the subgroup $e^{\mathcal{B}}$, where, in this case, $\mathcal{B}$ is the one dimensional subalgebra generated by B in (12). However, the set of states reachable in arbitrary time is not the whole group. An example of this phenomenon was given in [12] for $SO(3)$ (Example 8.1 in [12]) and, in fact, this example is somehow canonical since *every* system on $SO(3)$ with one input can be shown to have this property and the same thing is true for systems varying on $SU(2)$ [4]. Applying the isomorphism of Theorem 5.1, we conclude with the following Theorem.

Theorem 6.1 *Consider a system with spin under the action of an electro-magnetic field as described by equation (12). Then the set of rotations (states) that can be obtained in arbitrary time is given by the one dimensional Lie subgroup corresponding to the one dimensional Lie algebra generated by the matrix B.*

From an application point of view, it is of great importance to give an estimate of the time T_c after which every rotation can be performed. That such a time does in fact exist follows from an application of Theorem 4.5 to the case under consideration, since we have proved $STLCI$ in our case. From $R(t_1) \subseteq R(t_2)$, when $t_1 < t_2$, it follows that $R(t) = \mathcal{R}(\leq t)$, for every $t \geq 0$, and from the results of [12], there exists a time T_c such that $\mathcal{R}(\leq t)$ and therefore $R(t)$ is the whole group, for every $t > T_c$. An estimate of the time T_c can be obtained as follows.

For system (12), consider the control $u(t) :=$

$-\frac{<A,B>}{<B,B>} + v(t)$. The system (12) is written as

$$\dot{X} = (A - \frac{< A,B >}{< B,B >}B)X + BXv. \qquad (13)$$

The matrices $\tilde{A} := (A - \frac{<A,B>}{<B,B>}B)$ and B are orthogonal and applying the results of [14] [15] one can write every element X_f in $SU(2)$ or $SO(3)$ as

$$X_f := e^{Bt_1}e^{\tilde{A}t_2}e^{Bt_3}, \qquad (14)$$

for some coefficients t_1, t_2, t_3. Now, from the previous analysis the states e^{Bt_1} and e^{Bt_3} can be obtained in arbitrary time. The state $e^{\tilde{A}t_2}$ can be obtained by setting the control $v \equiv 0$ for a time t_2 in (14). It is clear that by allowing t_2 to vary in $[0, \frac{2\pi}{p})$, where p is the absolute value of the (nonzero) eigenvalue of $\tilde{A}$, we can obtain all the values in the closed one dimensional Lie group generated by $\tilde{A}$. In conclusion we have $T_c \leq \frac{2\pi}{p}$.

Constructive controllability for quantum mechanical systems based on decompositions of Lie groups such as the one in (14) is explored in [6]. An extended version of this paper including the proofs omitted here is given in [5].

References

[1] J. Bailleul, Geometric methods for nonlinear optimal control problems, *Journal of Optimization Theory and Applications,* Vol. 25, No. 4, pp. 519-548, August 1978.

[2] R. W. Brockett, System theory on group manifolds and coset spaces, *SIAM Journal on Control*, Vol. 10, No. 2, May 1972.

[3] A. G. Butkovskiy and Y. I. Samoilenko, *Control of Quantum-Mechanical Processes and Systems*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[4] D. D'Alessandro, Topological properties of reachable sets and the control of quantum bits, to appear in *Systems & Control Letters.*

[5] D. D'Alessandro, On the controllability of systems on compact Lie groups and spin angular momentum, Technical Report, Department of Mathematics, Iowa State University, 2000.

[6] D. D'Alessandro, Algorithms for quantum control based on decompositions of Lie groups, *Proceedings Conference on Decision and Control*, Sydney Australia, Dec. 2000.

[7] D. D'Alessandro and M. Dahleh, Optimal control of two-level quantum systems, in *Proceedings American Control Conference*, Chicago IL, June 2000.

[8] D. Di Vincenzo, Quantum Computation, *Science*, Vol. 270, 13 October 1995.

[9] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Krieger Publishing Company, Malabar, Florida, 1994.

[10] G. Harel and V. M. Akulin, Complete control of Hamiltonian quantum systems: engineering of Floquet evolution, *Physical Review Letters,* Vol. 82, No.1, 1999.

[11] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.

[12] V. Jurdjevic and H. Sussmann, Control systems on Lie Groups, *Journal of Differential Equations*, **12**, pg. 313-329, 1972.

[13] H. Kunita, Supports of diffusion processes and controllability problems, in K. Ito editor, *Proceedings Int. Symp. Stochastic Diff. Eqn.,* pp. 163-185. Wiley, 1978.

[14] F. Lowenthal, Uniform finite generation of the rotation group, *Rocky Mountain J. Math.* 1, pp. 575-586, 1971.

[15] F. Lowenthal, Uniform finite generation of $SU(2)$ and $SL(2,R)$, *Canad. J. Math.*, 24, pp. 713-727, 1972.

[16] V. Ramakrishna, M. V. Salapaka, M. Dahleh, H. Rabitz and A. Peirce, Controllability of molecular systems, *Physical Review A*, 51, no. 2, 1995.

[17] J. J. Sakurai, *Modern Quantum Mechanics*, Revised Edition, Addison-Wesley Pub. Co., Reading, Mass., 1994.

[18] H. J. Sussmann. Lie brackets, real analyticity and geometric control, in *Differential Geometric Control Theory*, R.W. Brockett, R. S. Millman and H. J. Sussmann eds., pp. 1-116, Birkhauser, Boston, 1983.

[19] E. P. Wigner, *Group Theory, and its Applications to the Quantum Mechanics of Atomic Spectra*, New York: Academic Press, 1959.