Stationary Riccati Equation for Linear Minimum Mean Square Error Estimator of Markovian Jump Systems

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Abstract

In this paper we obtain sufficient conditions for the convergence of the error covariance matrix to a stationary value for the linear minimum mean square error estimator (LMMSE) of discrete time linear systems subject to abrupt changes in the parameters modeled by a Markov chain $\theta(k)\epsilon\{1,\cdots,N\}$ (MJLS). Under the assumption of mean square stability of the MJLS and ergodicity of the associated Markov chain it is shown that there exists a unique solution for the stationary Riccati filter equation, and moreover this solution is the limit of the error covariance matrix of the LMMSE. This result is suitable for designing a time-invariant stable suboptimal filter of LMMSE for MJLS.

Keywords: Kalman filter, Riccati equation, jump systems, Markov parameters.

1 Introduction

Several dynamic systems are inherently vulnerable to abrupt changes in their structures due, for instance, to component and/ or interconnections failures, abrupt environment changes, etc. A common approach for analyzing systems in this class is through the multiple model approach, which assumes that the real system can be represented by a finite number of possible models. An usual hypothesis is to assume that the models are discrete-time linear systems and transitions among them follow a Markov chain $\theta(k)\epsilon\{1,\cdots,N\}$. For this class of systems, known as Markovian jump linear systems (MJLS), the problem of optimal and sub-optimal filtering has been addressed in [1], [3], [5], [2], [11], [12] among other authors, under the hypothesis of Gaussian distribution for the disturbances. Since the optimal estimator requires exponentially increasing memory and computation with the time, sub-optimal algorithms are required. In [6] it was obtained the LMMSE for MJLS, based on estimating $x(k)1_{\{\theta(k)=i\}}$ instead of estimating directly $x(k)$, where $1_{\{.\}}$ stands for the Dirac measure. The advantage of this formulation is that it can be applied to a broader class of systems than the linear models with output uncertainly studied in [9] e [10].

In [6] the filter equation is a function of the estimation error covariance matrix. This covariance was expressed as the difference between two recursive equations: one associated with the second moment matrix of the state and the other one associated with the second moment matrix of the estimator. Initially in this paper we write the error covariance matrix in terms of recursive Riccati difference equations. After that we present conditions to guarantee the convergence of the error covariance matrix to the stationary solution of an algebraic Riccati equation (ARE). Moreover it is proved stability of the stationary filter. These results allow us to design a time-invariant stable suboptimal filter of LMMSE for MJLS.

This paper is organized as follows. In section 2 is presented the problem formulation, some assumptions, notation, auxiliary results and the main theorem in [6], deriving the LMMSE. In section 3 a recursive Riccati equation for the error covariance is obtained. The proof for convergence of the error covariance matrix to the stationary solution of an algebraic Riccati equation is presented in section 4. The paper is concluded in section 5 with some final comments.

2 Preliminaries

Consider the following discrete-time markovian jump linear system.

$$x(k+1) = A_{\theta(k)}x(k) + C_{\theta(k)}\xi(k) \tag{1a}$$
$$y(k) = H_{\theta(k)}x(k) + G_{\theta(k)}\nu(k) \tag{1b}$$

Here $\{x(k)\}$ denotes the $\mathbb{R}^n$-valued state sequence, $\{\xi(k)\}$ and $\{\nu(k)\}$ are random disturbances in $\mathbb{R}^{q1}$ and $\mathbb{R}^{q2}$ respectively, $\{y(k)\}$ is the $\mathbb{R}^m$-valued output sequence, $\{\theta(k)\}$ is a discrete-time Markov chain with finite state space $\{1,\cdots,N\}$, and transition probability matrix $\mathbf{P}=[p_{ij}]$, and $A_i, C_i, H_i, G_i, i=1,\cdots,N$ are matrices of appropriated dimensions. The following assumptions will be made:

A1) $G_iG_i' > 0$ (where ' stands for transpose) for all $i = 1,\cdots,N$.

A2) $\{\xi(k)\}$ and $\{\nu(k)\}$ are null mean second-order, independent wide sense stationary sequence mutually independent with covariance matrices equal to the identity.

A3) $x(0)1_{\theta(0)=i}, i = 1,\cdots,N$ are second order random vectors with $\mathrm{E}(x(0)1_{\theta(0)=i}) = \mu_i$ and $\mathrm{E}(x(0)x(0)'1_{\theta(0)=i}) = V_i, i=1,\cdots,N$.

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A4) $x(0)$ and $\{\theta(k)\}$ are independent of $\{\xi(k)\}$ and $\{\nu(k)\}$.

Denote by $\mathcal{F}_k$ the $\sigma-$field generated by $\{x(t), y(t), \theta(t); t = 0, \cdots, k\}$. For any sequence of second-order random vectors $\{r(k)\}$, it is possible to define the "centered" random vector $r^c(k) := r(k) - E(r(k))$, $\hat{r}^c(k|l)$ as the best linear estimator of $r^c(k)$ given $y^c(0), \cdots, y^c(l)$, $l \leq k$ and $\tilde{r}^c(k|l) := r^c(k) - \hat{r}^c(k)$. It is well known [8] that $\hat{r}(k|l) = \hat{r}^c(k|l) + E(r(k))$ and, in particular, $\tilde{r}^c(k|l) = \tilde{r}(k|l)$. Denote by $\mathcal{L}((y^c)^k)$ the linear subspace spanned by $(y^c)^k := (y^c(k)', \cdots, y^c(0)')'$ (see [8]). The best linear estimator $\hat{r}^c(k|l) = (\hat{r}^c_1(k|l), \cdots, \hat{r}^c_\iota(k|l))'$ of the random vector $r^c(k) = (r^c_1(k), \cdots, r^c_\iota(k))'$ in $\mathcal{R}^\iota$ is the projection of $r^c(k)$ onto the subspace $\mathcal{L}((y^c)^l)$ and satisfies the properties (5), (6) and (7) of [6], where (7) is:
If $cov((y^c)^l)$ is nonsingular then

$$\hat{r}^c(k|l) = E(r^c(k)((y^c)^l)')cov((y^c)^l)^{-1}(y^c)^l. \tag{2}$$

We also have that

$$\begin{aligned} \hat{r}^c(k|k) =& \hat{r}^c(k|k-1) + E(r^c(k)\tilde{y}(k|k-1)') \\ & E(\tilde{y}(k|k-1)\tilde{y}(k|k-1)')^{-1} \\ & \cdot (y^c(k) - \hat{y}^c(k|k-1)). \end{aligned} \tag{3}$$

For $k \geq 0$ and $j \in \{1, \cdots, N\}$ define $z(k,j) := x(k)1_{\{\theta(k)=j\}} \in \mathbb{R}^n$ where $1_{\{\theta(k)=j\}}$, $z(k) := (z(k,1)', \cdots, z(k,N)')' \in \mathbb{R}^{Nn}$. Define also $\hat{z}(k|k-1)$ as the projection of $z(k)$ onto $\mathcal{L}(y^{k-1})$ and $\tilde{z}(k|k-1) := z(k) - \hat{z}(k|k-1)$. The second-moment matrix associated to the above variables are

$$\begin{aligned} Z(k) :=& E(z(k)z(k)')\epsilon\mathbb{R}^{N(n)\times N(n)} \\ Z_i(k) :=& E(z(k,i)z(k,i)')\epsilon\mathbb{R}^{n\times n}, i = 1, \cdots, N \\ \hat{Z}(k|l) :=& E(\hat{z}(k|l)\hat{z}(k|l)')\epsilon\mathbb{R}^{N(n)\times N(n)}, l \leq k \\ \tilde{Z}(k|l) :=& E(\tilde{z}(k|l)\tilde{z}(k|l)')\epsilon\mathbb{R}^{N(n)\times N(n)}, l \leq k. \end{aligned} \tag{4}$$

We consider the following augmented matrices

$$A := \begin{bmatrix} p_{11}A_1 & \cdots & p_{N1}A_N \\ \vdots & \cdots & \vdots \\ p_{1N}A_1 & \cdots & p_{NN}A_N \end{bmatrix} \tag{5}$$

$$\begin{aligned} G(k) :=& [G_1\pi_1(k)^{1/2} \cdots G_N\pi_N(k)^{1/2}] \\ H :=& [H_1 \cdots H_N] \end{aligned}$$

where $\pi_j(k) := P(\theta(k) = j)$. The filter equation obtained in a recursive form in [6] was derived from geometric arguments following (2) and (3). The main result of [6] is Theorem 1, which states that for a system represented by (1a), (1b) with assumptions A1-A4, the LMMSE $\hat{x}(k|k)$ is given by

$$\hat{x}(k|k) = \sum_{i=1}^{N} \hat{z}(k,i|k) \tag{6}$$

where $\hat{z}(k|k)$ satisfies the recursive equation

$$\begin{aligned} \hat{z}(k|k) =& \hat{z}(k|k-1) + \tilde{Z}(k|k-1)H' \\ & \cdot(H\tilde{Z}(k|k-1)H' + G(k)G(k)')^{-1} \\ & \cdot(y(k) - H\hat{z}(k|k-1)) \end{aligned} \tag{7}$$

where

$$\begin{aligned} \hat{z}(k|k-1) &= A\hat{z}(k-1|k-1), k \geq 1 \\ \hat{z}(0|-1) &= \zeta(0), \zeta(k) = E(z(k)) \\ \zeta(0) &= (\zeta_1', \cdots, \zeta_N'), \\ \zeta_i &= \mu_i. \end{aligned} \tag{8}$$

Moreover

$$\tilde{Z}(k|k-1) = Z(k) - \hat{Z}(k|k-1) \tag{9}$$

where $Z(k) = diag(Z_j(k))$ with

$$\begin{aligned} Z_j(k+1) =& \sum_{i=1}^{N} p_{ij}A_iZ_i(k)A_i' + \sum_{i=1}^{N} p_{ij}\pi_i(k)C_iC_i' \\ Z_j(0) =& V_j \end{aligned} \tag{10}$$

and,

$$\begin{aligned} \hat{Z}(k|k) &= \hat{Z}(k|k-1) + \hat{Z}(k|k-1)H' \\ &\quad \cdot(H\tilde{Z}(k|k-1)H' + G(k)G(k)')^{-1} \\ &\quad \cdot H\hat{Z}(k|k-1) \\ \hat{Z}(k|k-1) &= A(k-1)\hat{Z}(k-1|k-1)A(k-1)' \\ \hat{Z}(0|-1) &= \zeta(0)\zeta(0)'. \end{aligned} \tag{11}$$

It is interesting to notice that in the filter equation (7) the term $\tilde{Z}(k|k-1)$ is obtained from (9), where the two terms that form this equation are derived separately. The goal of the next section is to obtain a Ricatti recursive equation for $\tilde{Z}(k|k-1)$.

3 The Riccati Difference Equation

The following Lemma consider the system represented by (1a), (1b) and the assumptions A1-A4.

Lemma 1 *Recursive equation for $\tilde{Z}(k|k-1)$ is given by:*

$$\begin{aligned} \tilde{Z}(k+1|k) &= A\tilde{Z}(k|k-1)A' + \mathcal{V}(Q(k)) \\ &+ diag[\sum_{i=1}^{N} \pi_i(k)p_{ij}C_iC_i'] - A\tilde{Z}(k|k-1)H' \\ &\cdot (H\tilde{Z}(k|k-1)H' + G(k)G(k)')^{-1}H\tilde{Z}(k|k-1)A' \end{aligned} \tag{12}$$

where $\mathcal{V}(\cdot)$ define a semi-positive monotonic operator, that is:

$$\begin{aligned} \mathcal{V}(Q(k)) = diag[\sum_{i=1}^{N} p_{ij}A_iQ_i(k)A_i'] - \\ Adiag[Q_i(k)]A'. \end{aligned}$$

setting $Q_i(k) = E(z(k,i)z(k,i)')$, and $Q(k) = (Q_1(k), \cdots, Q_N(k))$.

The equation (12) is equivalent to:

$$\begin{aligned} \tilde{Z}(k+1|k) &= (A - T(k)H)\tilde{Z}(k|k-1)(A - T(k)H)' \\ &+ \mathcal{V}(Q(k)) + diag[\sum_{i=1}^{N} \pi_i(k)p_{ij}C_iC_i'] \\ &+ T(k)G(k)G(k)'T(k)' \end{aligned} \tag{13}$$

where we define

$$\begin{aligned} T(k) =& A\tilde{Z}(k|k-1)H'(H\tilde{Z}(k|k-1)H' \\ &+G(k)G(k)')^{-1}. \end{aligned} \tag{14}$$

□

The equations (12) and (13) describe a Riccati recursive equation for $\tilde{Z}(k|k-1)$.

Proof. Writing (1a) in terms of $z(k)$ we obtain

$$\begin{aligned} z(k+1,j) =& 1_{\{\theta(k+1)=j\}}[A_1 \cdots A_N]z(k) \\ &+1_{\{\theta(k+1)=j\}}C_{\theta(k)}\xi(k) \end{aligned} \tag{15}$$

and adding and subtracting $[p_{ij}A_1 \cdots p_{Nj}A_N]z(k)$ we get

$$\begin{aligned} &z(k+1,j) = [p_{ij}A_1 \cdots p_{Nj}A_N]z(k)+ \\ &[(1_{\{\theta(k+1)=j\}} - p_{1j})A_1 \cdots (1_{\{\theta(k+1)=j\}} - p_{Nj})A_N] \\ &\cdot z(k) + 1_{\{\theta(k+1)=j\}}C_{\theta(k)}\xi(k). \end{aligned} \tag{16}$$

We define

$$\begin{aligned} M(k+1,j) =& [(1_{\{\theta(k+1)=j\}} - p_{1j})A_1 1_{\{\theta(k)=1\}} \\ &\cdots (1_{\{\theta(k+1)=j\}} - p_{Nj})A_N 1_{\{\theta(k)=N\}}] \end{aligned} \tag{17}$$

and

$$M(k+1) = \begin{bmatrix} M(k+1,1) \\ \vdots \\ M(k+1,N) \end{bmatrix}. \tag{18}$$

Recall that the main objetive is to obtain a recursive expression for $\tilde{Z}(k|k-1)$, we have from (16) that:

$$z(k+1) = Az(k) + M(k+1)z(k) + \vartheta(k) \tag{19}$$

where

$$\vartheta(k) = \begin{bmatrix} 1_{\{\theta(k+1)=1\}}C_{\theta(k)}\xi(k) \\ \vdots \\ 1_{\{\theta(k+1)=N\}}C_{\theta(k)}\xi(k) \end{bmatrix}. \tag{20}$$

Replacing (7) in (8) we get that

$$\begin{aligned} \hat{z}(k+1|k) =& A\hat{z}(k|k-1) + A\tilde{Z}(k|k-1)H' \\ &\cdot(H\tilde{Z}(k|k-1)H' + G(k)G(k)')^{-1} \\ &\cdot(y(k) - H\hat{z}(k|k-1). \end{aligned} \tag{21}$$

Since

$$y(k) - H\hat{z}(k|k-1) = H\tilde{z}(k|k-1) + G_{\theta(k)}\nu(k) \tag{22}$$

and considering (14), we obtain that

$$\begin{aligned} \hat{z}(k+1|k) =& A\hat{z}(k|k-1) + T(k)H\tilde{z}(k|k-1) \\ &+T(k)G_{\theta(k)}\nu(k) \end{aligned} \tag{23}$$

and thus, from (19) and (23)we get that:

$$\begin{aligned} \tilde{z}(k+1|k) =& (A - T(k)H)\tilde{z}(k|k-1) + \vartheta(k) \\ &+ M(k+1)z(k) - T(k)G_{\theta(k)}\nu(k). \end{aligned} \tag{24}$$

Using (24) most be calculated $\tilde{Z}(k|k-1)$. We have:

$$E(M(k+1,j)|\mathcal{F}_k) = 0,$$

Therefore, the covariance expression takes the following form:

$$\begin{aligned} \tilde{Z}(k+1|k) =& (A - T(k)H)\tilde{Z}(k|k-1)(A - T(k)H)' \\ &+ E(M(k+1)z(k)z(k)'M(k+1)') \\ &+ E(\vartheta(k+1)\vartheta(k+1)') \\ &+T(k)E(G_{\theta(k)}\nu(k)\nu(k)'G'_{\theta(k)})T(k)'. \end{aligned} \tag{25}$$

Notice that

$$\begin{aligned} &E(M(k+1)z(k)z(k)'M(k+1)') = \\ &E(\left[E(M(k+1)z(k)z(k)'M(k+1)'|\mathcal{F}_k)\right]). \end{aligned} \tag{26}$$

We also have that:

$$\begin{aligned} &M(k+1,j)z(k)z(k)'M(k+1,l)' = \\ &\left(\sum_{t=1}^{N}(1_{\{\theta(k+1)=j\}} - p_{tj})A_t z(k,t)\right) \\ &\left(\cdot\sum_{s=1}^{N}(1_{\{\theta(k+1)=l\}} - p_{sl})A_s z(k,s)'\right) \\ &= \sum_{t=1}^{N}\sum_{s=1}^{N}\left((1_{\{\theta(k+1)=j\}} - p_{tj})\right)A_t z(k,t)z(k,s)' \\ &\quad \cdot A'_s\left((1_{\{\theta(k+1)=l\}} - p_{sl})\right). \end{aligned} \tag{27}$$

The product $z(k,t)z(k,s)'$ will be different from zero only if $t = s$, since otherwise one of the terms will be zero. Thus the expression (27) can be reduced to

$$\begin{aligned} &M(k+1,j)z(k)z(k)'M(k+1,l)' = \\ &\sum_{i=1}^{N}(1_{\{\theta(k+1)=j\}}1_{\{\theta(k+1)=l\}} - p_{ij}1_{\{\theta(k+1)=l\}} \\ &-p_{il}1_{\{\theta(k+1)=j\}} + p_{ij}p_{il})A_i z(k,i)z(k,i)'A'_i. \end{aligned} \tag{28}$$

From (28) we have two possibilities:

- Case 1: Include the terms outside the main diagonal, where $j \neq l$. In this case $1_{\{\theta(k+1)=j\}}1_{\{\theta(k+1)=l\}} = 0$. Moreover ,$-E(p_{ij}1_{\{\theta(k+1)=l\}}|\mathcal{F}_k) = -p_{ij}p_{il}$ and replacing in (28) we get that:

$$E(M(k+1,j)z(k)z(k)'M(k+1,l)'|\mathcal{F}_k) \tag{29}$$

$$= -\sum_{i=1}^{N}p_{ij}p_{il}A_i z(k,i)z(k,i)'A'_i$$

$$= -\left[p_{1j}A_1 \cdots p_{Nj}A_N\right]z(k)z(k)'\begin{bmatrix} p_{1l}A'_1 \\ \vdots \\ p_{Nl}A'_N \end{bmatrix}.$$

- Case 2:Include the terms in the main diagonal, where $j = l$. In this case:

$$E(M(k+1,j)z(k)z(k)'M(k+1,l)'|\mathcal{F}_k) \tag{30}$$

$$= \sum_{i=1}^{N}(p_{ij}\left(1 - p_{ij}\right)A_i z(k,i)z(k,i)'A'_i$$

$$= \left[\sqrt{p_{1j}}A_1 \cdots \sqrt{p_{Nj}}A_N\right] z(k)z(k)' \begin{bmatrix} \sqrt{p_{1j}}A_1' \\ \vdots \\ \sqrt{p_{Nj}}A_N' \end{bmatrix}$$
$$- \left[p_{1j}A_1 \cdots p_{Nj}A_N\right] z(k)z(k)' \begin{bmatrix} p_{1j}A_1' \\ \vdots \\ p_{Nj}A_N' \end{bmatrix}.$$

Combining both cases in one expression, we get that:

$$E\big(M(k+1)z(k)z(k)'M(k+1)'|\mathcal{F}_k\big) = diag\left[\sum_{i=1}^{N} p_{ij}A_i z(k,i)z(k,i)'A_i'\right] - Az(k)z(k)'A'.$$

Therefore from the previous result and (26):

$$E(M(k+1)z(k)z(k)'M(k+1)') = diag\left[\sum_{i=1}^{N} p_{ij}A_iQ_i(k)A_i'\right] - Adiag[Q_i(k)]A'. \tag{31}$$

The other terms of the covariance take the following form:

$$E(\vartheta(k+1)\vartheta(k+1)') = diag\left[\sum_{i=1}^{N} p_{ij}C_iC_i'\pi_i(k)\right] \tag{32}$$

and:

$$E(G_{\theta(k)}\nu(k)\nu(k)'G_{\theta(k)}) = G(k)G(k)'. \tag{33}$$

Replacing (31), (32) and (33) into equation (25), we get (12). ■

4 Asymptotic Analysis

In this section we analyze the behaviour of $\tilde{Z}(k|k-1)$ given by equation (12) and shall establish its convergence when $k \to \infty$. We shall assume that:

H1: System (1a) is mean square stable (MSS) according to the definition in [7].

H2: The Markov chain is ergodic.

Notice first that the term

$$diag\left[\sum_{i=1}^{N} \pi_i(k)p_{ij}C_iC_i'\right]$$

is convergent since p_{ij} and C_i do not depend on k and for an ergodic Markov chain it follows that $\lim_{k\to\infty} P(\theta(k) = i)$ exists and is independent from $\theta(0)$. Define$\pi_i = \lim_{k\to\infty} P(\theta(k) = i) = \lim_{k\to\infty} \pi_i(k)$.

Let us look at the following term;

$$\mathcal{V}(Q(k)) = diag\left[\sum_{i=1}^{N} p_{ij}A_iQ_i(k)A_i'\right] - Adiag[Q_i(k)]A'.$$

From H1 and H2, and according to Proposition 8 in [7], $Q(k) \to Q$ whenever $k \to \infty$, where $Q = (Q_1, \cdots, Q_N)$ satisfies:

$$Q_j = \sum_{i=1}^{N} p_{ij}(A_iQ_iA_i' + \pi_iC_iC_i'), j = 1, \cdots, N.$$

Therefore

$$\mathcal{V}(Q(k)) \overset{k\to\infty}{\to} diag\left[\sum_{i=1}^{N} p_{ij}A_iQ_iA_i'\right] - Adiag[Q_i]A' \geq 0 = \mathcal{V}(Q). \tag{34}$$

The next Theorem will provide the proof for the asymptotic convergence of $\tilde{Z}(k|k-1)$ satisfying (12):

Theorem 1 *Consider algebraic Riccati equation (ARE) given by:*

$$\begin{aligned} Z =& AZA' + diag\left[\sum_{i=1}^{N} \pi_i p_{ij}C_iC_i'\right] - \\ & AZH'(HZH' + GG')^{-1}HZA' + \mathcal{V}(Q) \\ =& (A - T(Z)H)Z(A - T(Z)H)' + \mathcal{V}(Q) + \\ & diag\left[\sum_{i=1}^{N} \pi_i p_{ij}C_iC_i'\right] + T(Z)GG'T(Z)' \end{aligned} \tag{35}$$

where:

$$\begin{aligned} T(Z) =& AZH'(HZH' + GG')^{-1} \\ G =& [G_1\pi_1^{1/2} \cdots G_N\pi_N^{1/2}]. \end{aligned}$$

Under the hypothesis H1 e H2 there exists a unique positive semi-definite solution P to the ARE given by (35). Moreover,the spectral radius $r_\sigma(A - T(P)H) < 1$ and for any $Q(0) = (Q_1(0), \cdots, Q_N(0))$ with $Q_i(0) \geq 0$, $i = 1, \cdots, N$, and $\tilde{Z}(0|-1) = diag[Q_i(0)] - E(z(0))E(z(0))' \geq 0$ we have that

$$\tilde{Z}(k+1|k) \overset{k\to\infty}{\to} P.$$

□

Proof. From MSS of (1a), we have from Proposition 5 of [7] that $r_\sigma(A) < 1$ and thus according to standard results for ARE there exists a unique positive semi-definite solution P to (35) and moreover $r_\sigma(A - T(P)H) < 1$ (see [4]). Define

$$\begin{aligned} P(k+1) =& (A - T(P)H)P(k)(A - T(P)H)' \\ & + \mathcal{V}(Q(k)) + diag\left[\sum_{i=1}^{N} \pi_i(k)p_{ij}C_iC_i'\right] \\ & + T(P)G(k)G(k)'T(P)' \end{aligned} \tag{36}$$
$$P(0) = \tilde{Z}(0|-1)$$

where:

$$T(P) = APH'(HPH' + GG')^{-1}.$$

Since:

$$\begin{aligned} \tilde{Z}(k+1|k) =& \mathcal{V}(Q(k)) + diag\left[\sum_{i=1}^{N} \pi_i(k)p_{ij}C_iC_i'\right] \\ & + (A - T(P)H)\tilde{Z}(k|k-1)(A - T(P)H)' \\ & + T(P)G(k)G(k)'T(P)' - (T(k) - T(P)) \\ & \cdot (H\tilde{Z}(k|k-1)H' + G(k)G(k)')(T(k) - T(P))' \end{aligned} \tag{37}$$

we have from (36) and (37) that:

$$\begin{aligned}&(P(k+1)-\tilde{Z}(k+1|k)) =\\ &(A-T(P)H)(P(k)-\tilde{Z}(k|k-1))(A-T(P)H)'+\\ &(T(k)-T(P))(H\tilde{Z}(k|k-1)H'+G(k)G(k)')\\ &(T(k)-T(P))'.\end{aligned} \tag{38}$$

By definition $P(0)=\tilde{Z}(0|-1)$. Suppose that $P(k)\geq\tilde{Z}(k|k-1)$. From (38) we have that $P(k+1)\geq\tilde{Z}(k+1|k)$. Therefore we have shown by induction that $P(k)\geq\tilde{Z}(k|k-1)$ for all $k=0,1,2,\cdots$. From H1 and H2 we have that $Q(k)\overset{k\to\infty}{\to}Q$, $G(k)\overset{k\to\infty}{\to}G$ and $diag\left[\sum_{i=1}^{N}\pi_i(k)p_{ij}C_iC_i'\right]\overset{k\to\infty}{\to}diag\left[\sum_{i=1}^{N}\pi_ip_{ij}C_iC_i'\right]$ exponentially fast. From $r_\sigma(A-T(P)H)<1$ and Proposition 2 in [7] we get that $P(k)\overset{k\to\infty}{\to}\overline{P}$, where $\overline{P}$ satisfies:

$$\begin{aligned}\overline{P}=&(A-T(P)H)\overline{P}(A-T(P)H)'+\mathcal{V}(Q)+\\ &diag\left[\sum_{i=1}^{N}\pi_ip_{ij}C_iC_i'\right]+T(P)GG'T(P)'.\end{aligned} \tag{39}$$

We also have from Proposition 2 in [7] that $\overline{P}$ is the unique solution of (39). Recalling that P satisfies (35), we have that P is also a solution of (39) and from uniqueness, $\overline{P}=P$. Therefore

$$\tilde{Z}(k|k-1)\leq P(k) \tag{40}$$

and $P(k)\overset{k\to\infty}{\to}P$. Define now $\alpha_i(k)=\underset{\ell\geq k}{inf}\pi_i(\ell)$, e $\overline{Q}(k)=(\overline{Q}_1(k),\cdots,\overline{Q}_N(k))$ with

$$\begin{aligned}\overline{Q_j}(k+1)=\sum_{i=1}^{N}p_{ij}(A_i\overline{Q_i}(k)A_i'+\alpha_i(k)C_iC_i')\\ \overline{Q_j}(0)=0, j=1,\cdots,N.\end{aligned}$$

Obviously

$$\begin{aligned}\pi_i(k)\geq\alpha_i(k)\geq\alpha_i(k-1), &k=1,2,\cdots,\\ &i=1,\cdots,N\end{aligned} \tag{41}$$

and $\alpha_i(k)\overset{k\to\infty}{\to}\pi_i$ exponentially fast. Therefore from Proposition 8 in [7], we get that $\overline{Q}(k)\overset{k\to\infty}{\to}Q$. The next sep is to show by induction in k that:

$$Q(k)\geq\overline{Q}(k)\geq\overline{Q}(k-1). \tag{42}$$

For $k=1$ the result is immediate, since $Q(0)\geq 0=\overline{Q}(0)$ and $\pi_i(0)\geq\alpha_i(0)$, $i=1,\cdots,N$. Suppose that (42) holds for k. Then from (41) and (42) we have that

$$\begin{aligned}&Q_j(k+1)=\sum_{i=1}^{N}p_{ij}(A_iQ_i(k)A_i'+\pi_i(k)C_iC_i')\geq\\ &\sum_{i=1}^{N}p_{ij}(A_i\overline{Q_i}(k)A_i'+\alpha_i(k)C_iC_i')=\overline{Q}_j(k+1)\geq\\ &\sum_{i=1}^{N}p_{ij}(A_i\overline{Q_i}(k-1)A_i'+\alpha_i(k-1)C_iC_i')=\overline{Q}_j(k)\end{aligned}$$

completing the induction argument in (42). Define now:

$$\begin{aligned}R(k+1)=&AR(k)A'+\mathcal{V}(\overline{Q}(k))+\\ &diag\left[\sum_{i=1}^{N}\alpha_i(k)p_{ij}C_iC_i'\right]+\\ &AR(k)H'(HR(k)H'+\overline{G}(k)\overline{G}(k)')^{-1}HR(k)A'\end{aligned}$$

where:

$$\begin{aligned}&R(0)=0\\ &\overline{G}(k)=[G_1\alpha_1(k)^{1/2},\cdots,G_N\alpha_N(k)^{1/2}].\end{aligned}$$

We want to show by induction in k that:

$$0\leq R(k)\leq R(k+1)\leq\tilde{Z}(k+1|k). \tag{43}$$

Setting

$$S(k)=AR(k)H'(HR(k)H'+\overline{G}(k)\overline{G}(k)')^{-1}$$

it follows that, if $R(k)\leq\tilde{Z}(k|k-1)$, then from (41) and (42):

$$\begin{aligned}&R(k+1)=(A-S(k)H)R(k)(A-S(k)H)'+\\ &\mathcal{V}(\overline{Q}(k))+diag\left[\sum_{i=1}^{N}\alpha_i(k)p_{ij}C_iC_i'\right]+\\ &S(k)\overline{G}(k)\overline{G}(k)'S(k)'=\\ &(A-T(k)H)R(k)(A-T(k)H)'+\mathcal{V}(\overline{Q}(k))+\\ &diag\left[\sum_{i=1}^{N}\alpha_i(k)p_{ij}C_iC_i'\right]+T(k)\overline{G}\overline{G}'T(k)'-\\ &(T(k)-S(k))(HR(k)H'+\overline{G}(k)\overline{G}(k)')(T(k)-S(k))'\\ &\leq(A-T(k)H)\tilde{Z}(k|k-1)(A-T(k)H)'+\mathcal{V}(Q(k))\\ &+diag\left[\sum_{i=1}^{N}\pi_i(k)p_{ij}C_iC_i'\right]+T(k)G(k)G(k)'T(k)'\\ &=\tilde{Z}(k+1|k).\end{aligned}$$

Obviously $R(0)=0\leq\tilde{Z}(0|-1)$, showing that $R(k)\leq\tilde{Z}(k|k-1)$ for all $k=0,1,2,\cdots$. Similarly if $R(k)\geq R(k-1)$, then from (41) and (42):

$$\begin{aligned}&R(k)=(A-S(k)H)R(k-1)(A-S(k)H)'+\\ &\mathcal{V}(\overline{Q}(k-1))+diag\left[\sum_{i=1}^{N}\alpha_i(k-1)p_{ij}C_iC_i'\right]+\\ &S(k)\overline{G}(k-1)\overline{G}(k-1)'S(k)'-(S(k)-S(k-1))\\ &\cdot(HR(k-1)H'+\overline{G}(k-1)\overline{G}(k-1)')\\ &\cdot(S(k)-S(k-1))'\\ &\leq(A-S(k)H)R(k)(A-S(k)H)'+\mathcal{V}(Q(k))+\\ &diag\left[\sum_{i=1}^{N}\alpha_i(k)p_{ij}C_iC_i'\right]+S(k)\overline{G}(k)\overline{G}(k)'S(k)'\end{aligned}$$

and since $R(0)=0\leq R(1)$ the induction argument is completed for (43). From (40) and (43) it follows that:

$$0\leq R(k)\leq R(k+1)\leq P(k+1)$$

and thus we can conclude that $R(k)\uparrow R$ whenever $k\to\infty$ for some $R\geq 0$. Moreover, from the fact that $\alpha_i(k)\overset{k\to\infty}{\to}\pi_i$ and $\overline{Q}(k)\overset{k\to\infty}{\to}Q$ we have that R satisfies (35). From uniqueness of the positive semi-definite solution of (35) we can conclude that $R=P$. From (40) and (43),

$$R(k)\leq\tilde{Z}(k|k-1)\leq P(k)$$

and since $R(k)\uparrow P$, and $P(k)\to P$ as $k\to\infty$, we get that:

$$\tilde{Z}(k|k-1)\overset{k\to\infty}{\to}P.$$

■

5 Final Remarks

In this paper we have obtained sufficient conditions for the convergence of the error covariance matrix to a stationary value for the linear minimum mean square error estimator (LMMSE) of Markovian jump linear systems (MJLS). The LMMSE for MJLS was obtained in [6]. In this paper it was shown that if the MJLS is mean square stable (MSS) and the Markov chain is ergodic then the covariance matrix will converge to the unique positive semi-definite solution of an algebraic Riccati equation (ARE) associated to the problem. Moreover the filter error equation with the stationary gain will be stable.

References

[1] G.A. Ackerson and K.S. Fu. On the state estimation in switiching environments. *IEEE Trans. Automat. Contr.*, 15:10–17, 1970.

[2] Y. Bar-Shalom and X. Rong Li. *Estimation and Tracking*. Artech House, 1993.

[3] H.A.P. Blom and Y. Bar Shalom. The interacting multiple model algorithm for systems with markovian switching parameters. *IEEE Trans. Automat. Contr.*, 33:780–783, 1988.

[4] F. M. Callier and Ch. A Desoer. *Linear System Theory*. Springer-Verlag, 1991.

[5] C.G. Chang and M. Athans. State estimation for discrete systems with switching parameters. *IEEE Trans. Aerospace Eletctron. Syst.*, 14:418–424, 1978.

[6] O. L. V. Costa. Linear minimum mean square error estimation for discrete-time markovian jump linear systems. *IEEE Trans. on Automatic Control*, 39(8):1685–1689, 1994.

[7] O. L. V. Costa and M. D. Fragoso. Stability results for discrete-time linear systems with markovian jumping parameters. *J. Math. Anal. Appl.*, 179:154–178, 1993.

[8] M. H. A Davis and R. B. Vinter. *Stochastic Modelling and Control*. Chapman and Hall, 1985.

[9] M. T. Hadidi and C.S. Schwartz. Linear recursive state estimators under uncertain observations. *IEEE Trans. on Automatic Control*, 24:944–948, 1979.

[10] N.E. Nahi. Optimal recursive estimation with uncertain observation. *IEEE. Trans. Inform. Theory*, 15:457–462, 1969.

[11] J.K. Tugnait. Adaptive estimation and identification for discrete systems with markov jump parameters. *IEEE Trans. Automat. Contr.*, 27:1054–1064, 1982.

[12] J.K. Tugnait. Detection and estimation for abruptly changing systems. *Automatica*, 18:607–615, 1982.