Output Feedback Guaranteed Cost Control for Stochastic Uncertain Systems with Multiplicative Noise [1]

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Abstract

This paper is concerned with the existence of a guaranteed cost controller for an uncertain system which is subject to structured uncertainty. The uncertainty in the system is assumed to have a stochastic character and satisfy certain stochastic integral quadratic constraints. It is shown that a guaranteed cost output feedback controller for a stochastic system can be synthesized as an output feedback controller absolutely stabilizing this system. For each initial state of the system, this controller can be found by parametric optimization of solutions of a pair of parameter-dependent generalized matrix Riccati inequalities or matrix Riccati equations arising in stochastic H_∞ theory.

Keywords: Robust control, robust performance, structured uncertainty, stochastic H_∞ control, stochastic absolute stability, multiplicative noise.

1 Introduction

Stochastic approach to uncertain systems modeling has received much attention in the recent control literature. In parallel with the areas of robust filtering where stochastic processes traditionally form a major instrument of research, stochastic uncertainty models have proved being useful in robust control design. Stochastic uncertainty description may allow to capture some features of uncertainty which can not be captured otherwise or would lead to an excessively conservative controller if were modeled using deterministic models. In particular it was noted in [13, 8] that stochastic uncertainty modeling may provide a possible approach to the problem of non-worst case robust control design. Recall that the standard deterministic worst case robust control design presumes that all uncertainties have an equal chance of occuring, so that one does not expect that certain uncertainty inputs are more or less likely than others. Although the worst-case design methodology has proved its efficacy in various engineering problems, it suffers from the disadvantage that the designer lacks the opportunity to discriminate between "expected" uncertainties and those uncertainties which are known to seldom occur. Examples presented in [12, 13, 8] show that in some cases, this drawback of the standard worst-case design methodology can be overcome using an appropriate stochastic uncertainty description.

One particular robust control problem where stochastic uncertainty modeling has been demonstrated to lead to a practically useful and nonconservative result was considered in [13]. The reference [13] was concerned with a robust state feedback control problem for a class of uncertain systems in which the uncertainty was characterized using a stochastic integral quadratic constraint. In this paper, we address an output feedback version of this problem. As in [13, 8], we use the stochastic integral quadratic constraint uncertainty description which naturally extends the deterministic structured uncertainty model (cf., [5, 6, 9, 10, 14]) allowing for stochastic perturbations. This opens new avenues for robust control design with guaranteed performance and in many instances, allows to reduce the conservatism of existing techniques.

The main problem addressed is to find a linear output feedback controller yielding a guaranteed level of performance in the face of stochastic structured uncertainty in the system. In [13], such a controller was found as a minimax optimal controller which minimizes the maximum (over all admissible uncertainties) value of a cost functional. The class of controllers considered were state feedback controllers, which absolutely stabilized the system. The construction of the optimal controller found in [13] was based on the stabilizing solution of a *generalized* Riccati equation related to a stochastic H_∞ control problem and to a stochastic differential game considered recently in [12, 13]; also, see [8]. As we consider output feedback control in this paper, the key point of the guaranteed cost controller proposed here is that its construction exploits solutions to a pair of generalized Riccati inequalities or Riccati equations.

Notation We use the notation $\mathbf{R}^n$, $\mathbf{R}^{n\times q}$ to denote the n-dimensional real Euclidean vector space and the space of real $n \times q$-matrices equipped with the Euclidean matrix norm. We shall use symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to denote, respectively, the norm of vectors and matrices, and the inner product of vectors. Furthermore, given a positive definite

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symmetric matrix $Q \in \mathbf{R}^{q\times q}$, $\mathrm{tr}\Theta_1 Q\Theta_2'$ defines an inner product on $\mathbf{R}^{n\times q}$. Hence, this space can be considered as a subspace in the Hilbert space of Hilbert-Schmidt operators.

Let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a complete probability space and let $w(t)$ be a Wiener process in $\mathbf{R}^q$ with covariance matrix Q. Also, $w(0) = 0$. Let $\mathcal{F}_t$ denote the increasing sequence of Borel sub-σ-fields of $\mathcal{F}$, generated by $\{w(s), 0 \le s < t\}$. Also, let $\mathbf{E}$ denote the expectation.

Let L_2^s denote the Hilbert space $L_2(\Omega, \mathcal{F}_s, \mathcal{P}; \mathbf{R}^n)$ of $\mathcal{F}_s$-measurable random variables $\Omega \to \mathbf{R}^n$, which is complete with respect to the norm $\left(\mathbf{E}\|\cdot\|^2\right)^{1/2}$. For $T \le \infty$, let $L_2(s, T; \mathbf{R}^n)$ denote the Hilbert space generated by the (t,ω)-measurable $\{\mathcal{F}_t, t \ge 0\}$-non-anticipating processes $x(t,\omega)\colon [s,T]\times\Omega \to \mathbf{R}^n$ and complete with respect to the norm $|||\cdot||| = \left(\int_s^T \mathbf{E}\|\cdot\|^2 dt\right)^{1/2}$. We shall write $L_2(s; \mathbf{R}^n)$ for $L_2(s, +\infty; \mathbf{R}^n)$.

Given a symmetric positive definite $q\times q$-matrix Q, let $\mathbf{R}_Q^{n\times q}$ denote the Hilbert space of $n \times q$-matrices, with the inner product $\mathrm{tr}\Theta_1 Q\Theta_2'$.

We consider an uncertain stochastic system described by the following stochastic differential Ito equation:

$$\begin{aligned} dx &= (Ax(t) + B_1 u(t) + B_2\xi(t))dt + Hx(t)dw(t), \qquad (1)\\ z(t) &= C_1 x(t) + D_1 u(t),\\ y(t) &= C_2 x(t) + D_2\xi(t), \end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^{m_1}$ is the control input, $y(t) \in \mathbf{R}^l$ is the measured output, $z(t) \in \mathbf{R}^p$ is a vector assembling all uncertainty outputs and $\xi(t) \in \mathbf{R}^{m_2}$ is a vector assembling all uncertainty inputs. Here A, B_1, B_2, C_1, C_2, D_1, D_2 are matrices of corresponding dimensions, and H is a linear bounded operator $\mathbf{R}^n \to \mathbf{R}_Q^{n\times q}$.

1.1 System uncertainty

In a typical situation, the plant may contain several uncertain feedback loops. In our notation, this is described by the decomposition of uncertainty output vector z and uncertainty input vector ξ into several blocks of reduced dimensions as follows:

$$z = [z_1', \ldots, z_k']', \quad \xi = [\xi_1', \ldots, \xi_k']'. \qquad (2)$$

This in turn induces a corresponding block decomposition on the matrices C_1, D_1, D_2, B_2 in equation (1):

$$\begin{aligned} &C_1 = \begin{bmatrix} C_{1,1}\\ \ldots \\ C_{1,k}\end{bmatrix}, \quad D_1 = \begin{bmatrix} D_{1,1}\\ \ldots \\ D_{1,k}\end{bmatrix},\\ &B_2 = \left[\, B_{2,1} \ \ldots \ B_{2,k}\,\right],\\ &D_2 = \left[\, D_{2,1} \ \ldots \ D_{2,k}\,\right], \end{aligned} \qquad (3)$$

The uncertainty in the above system (1) is then described by the equation

$$\xi_i(t) := \phi_i(t, x(\cdot)|_0^t, u(\cdot)|_0^t); \quad i = 1, \ldots, k. \qquad (4)$$

We suppose that this uncertainty satisfies the following Stochastic Integral Quadratic Constraint introduced in [13]; also, see [8].

Definition 1 *Let $W_i > 0$, $i = 1, \ldots, k$ be given matrices. Then an uncertainty of the form (4) is said to be admissible if the following conditions hold.*

1. *If $u(\cdot) \in L_2(s, T; R^{m_1})$, $x(s) = h \in L_2^s$, then there exists a unique solution to (1), (4) that belongs to $L_2(s, T; R^n)$;*
2. *There exists a sequence $\{t_j\}_{j=1}^\infty$ such that $t_j > s$, $t_j \to \infty$ as $j \to \infty$ and the following condition holds. If $u(\cdot) \in L_2(s, t_j; R^{m_1})$ and $x(\cdot) \in L_2(s, t_j; R^n)$, then $\xi_\phi(\cdot) \in L_2(s, t_j; R^{m_2})$, and*

$$\begin{aligned} \mathbf{E}^{s,h}\int_s^{t_j} &\left(\|z_i(t)\|^2 - \|\xi_{\phi,i}(t)\|^2\right) dt\\ &\ge -\langle h, W_i h\rangle, \quad i = 1, \ldots, k. \end{aligned} \qquad (5)$$

Here, $\mathbf{E}^{s,h}$ denotes the expectation conditioned on the initial condition $x(s) = h$.

We use the notation Φ to denote the set of admissible uncertainties. Observe that the trivial uncertainty $\xi(\cdot) \equiv 0$ satisfies the above constraint. In the sequel, we shall refer to the system corresponding to this uncertainty as the *nominal* system.

The uncertainty constraint given by (5) extends the integral quadratic constraints such as given in [9, 10, 11] to stochastic systems with multiplicative noise. As in these references, this uncertainty description allows for the uncertainty input ξ to depend dynamically on the uncertainty outputs. Also note, that the uncertainty description in the form of the constraint (5) encompasses the standard norm-bounded uncertainty description. We refer to [13, 8] for a further discussion of Definition 1.

1.2 Guaranteed cost control problem

As has been mentioned above, the main problem addressed in this paper is to find a linear output feedback controller ensuring a guaranteed performance in the face of uncertainty in the system (1). In this section, we set up this problem.

Let $R \in \mathbf{R}^{n\times n}$, $G \in \mathbf{R}^{m_1\times m_1}$, $R' = R > 0$, $G' = G > 0$, be given matrices. Associated with the uncertain system (1), (4), consider the cost functional,

$$J^{s,h}(u(\cdot)) = \int_s^\infty \mathbf{E}^{s,h}\left(\langle x(t), Rx(t)\rangle + \langle u(t), Gu(t)\rangle\right) dt, \qquad (6)$$

defined on the set of output feedback controllers of the form

$$\begin{aligned} \dot{\hat{x}} &= A_c\hat{x} + B_c y, \quad \hat{x}(s) = h\\ u &= C_c\hat{x} + D_c y \end{aligned} \qquad (7)$$

In (6), $x(t)$ is the solution to (1), (4) satisfying the initial condition $x(s) = h$. Also, we introduce the notation:

$$\bar{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A + B_1 D_c C_2 & B_1 C_c \\ B_c C_2 & A_c \end{bmatrix},$$
$$\bar{H} = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} B_2 + B_1 D_c D_2 \\ B_c D_2 \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} C_1 + D_1 D_c C_2 & D_1 C_c \end{bmatrix}, \quad \bar{D} = D_1 D_c D_2. \tag{8}$$

Definition 2 *Given the cost functional (6), a controller of the form (7) is said to be a* guaranteed cost controller *for the uncertain system (1), (4) with cost functional (6) and initial condition h, if it satisfies the following condition:*

(i) This controller stabilizes the nominal system exponentially in mean-square sense, i.e. the resulting closed-loop nominal system

$$d\bar{x}(t) = \bar{A}\bar{x}(t)dt + \bar{H}x(t)dw(t),$$
$$\bar{x}(s) = [h', h']', \tag{9}$$

satisfies the condition: There exist constants $c > 0$, $\alpha > 0$ such that

$$\mathbf{E}^{s,h}\|x(t)\|^2 \leq c\|h\|^2 e^{-\alpha(t-s)}.$$

(ii) For all $h \in L_2^s$, the corresponding solution to the closed-loop uncertain system (1), (5), (7)

$$d\bar{x}(t) = \left(\bar{A}\bar{x}(t) + \bar{B}_2\xi(t)\right)dt + \bar{H}x(t)dw(t), \tag{10}$$
$$z = \bar{C}\bar{x} + \bar{D}\xi(t),$$

with any admissible uncertainty input (4) and initial condition $\bar{x}(s) = [h' \ h']'$ lies in $L_2(s, \mathbf{R}^n)$. Furthermore, the corresponding control input $u(\cdot)$ and admissible uncertainty input $\xi(\cdot)$ lie in $L_2(s, \mathbf{R}^{m_1})$, $L_2(s, \mathbf{R}^{m_2})$, respectively.

(iii) The corresponding value of the cost functional (6) is uniformly bounded for all admissible uncertainties:

$$\sup_{\phi(\cdot)\in\Phi} J^{s,h}(u(\cdot)) \leq \gamma \tag{11}$$

for a certain $\gamma > 0$.

In the sequel, we will assume that the following assumptions hold:

$$D_{1,i}' D_{1,i} > 0, \quad D_{2,i} D_{2,i}' > 0 \quad i = 1, \ldots, k.$$

Also, for the sake of simplicity we will assume that

$$D_{1,i}' C_{1,i} = 0, \quad B_{2,i} D_{2,i}' = 0. \tag{12}$$

These assumptions are standard assumptions arising in H_∞ control; see, e.g., [1].

We now are in a position to formulate a result which establishes a connection between the above guaranteed cost control problem and a stochastic H^∞ control problem. The stochastic H^∞ state feedback control problem was considered in [12, 13, 3]. Also, the stochastic H^∞ output feedback control problem was considered in [4] using a matrix inequalities approach.

Let $\tau_1 > 0, \ldots, \tau_k > 0$ be some positive constants and

$$\tilde{B}_2 := \begin{bmatrix} \frac{1}{\sqrt{\tau_1}} B_{2,1} & \cdots & \frac{1}{\sqrt{\tau_k}} B_{2,k} \end{bmatrix},$$
$$\tilde{D}_2 := \begin{bmatrix} \frac{1}{\sqrt{\tau_1}} D_{2,1} & \cdots & \frac{1}{\sqrt{\tau_k}} D_{2,k} \end{bmatrix},$$
$$\tilde{C}_1 = \begin{bmatrix} R^{1/2} \\ 0 \\ \sqrt{\tau_1} C_{1,1} \\ \vdots, \\ \sqrt{\tau_k} C_{1,k} \end{bmatrix}, \quad \tilde{D}_1 = \begin{bmatrix} 0 \\ G^{1/2} \\ \sqrt{\tau_1} D_{1,1} \\ \vdots, \\ \sqrt{\tau_k} D_{1,k} \end{bmatrix},$$

Note that $\tilde{D}_1' \tilde{D}_1 > 0$ and $\tilde{D}_2 \tilde{D}_2' > 0$ for any collection of positive constants τ_i. Also,

$$\tilde{C}_1' \tilde{D}_1 = 0, \quad \tilde{B}_2 \tilde{D}_2' = 0.$$

Consider the system

$$dx = (Ax(t) + B_1 u(t) + \tilde{B}_2 \tilde{\xi}(t))dt + Hx(t)dw(t) \tag{13}$$
$$\tilde{z}(t) = \tilde{C}_1 x(t) + \tilde{D}_1 u(t),$$
$$y(t) = C_2 x(t) + \tilde{D}_2 \tilde{\xi}(t),$$

driven by a disturbance input $\tilde{\xi}(\cdot) \in L_2(s; \mathbf{R}^{m_2})$. Associated with the system (13), consider the following stochastic H^∞ control problem: Find an output feedback controller of the form (7) satisfying the following conditions.

(i) The closed loop nominal system with $\tilde{\zeta} = 0$ corresponding to this controller is exponentially stable.

(ii) The closed loop system (13), (7) with $x(0) = 0$, $\hat{x}(0) = 0$ satisfies the stochastic H^∞ norm condition

$$\mathbf{E}\int_0^{+\infty} \left(\|\tilde{z}(t)\|^2 - \|\tilde{\xi}(t)\|^2\right) dt \leq -\varepsilon \mathbf{E}\int_0^{+\infty} \|\tilde{\xi}(t)\|^2 dt. \tag{14}$$

The solution to the guaranteed cost control problem considered in this paper relies on the following result.

Lemma 1 *Suppose for a given $\gamma > 0$ and the initial condition h, there exists a guaranteed cost controller of the form (7). Then there exist $\tau_1 > 0, \ldots, \tau_k > 0$ such that this controller solves the above stochastic H^∞ control problem (13), (14).*

The proof of the above lemma follows the lines proving of the corresponding result in [13, 8] and is based on the so-called S-procedure [15, 8].

In the sequel, we will show that the solution to the stochastic H^∞ output feedback control problem (13), (14) leads to a guaranteed cost controller.

2 Stochastic H^∞ control

In this section we review solutions to the stochastic H^∞ output feedback control problem available from the literature; see [12, 13, 8, 4, 2, 3]. The problem considered in this section is the following. Consider the system (1) driven by a disturbance input $\xi(\cdot) \in L_2(0;\mathbf{R}^{m_2})$. The closed loop system corresponding to the system (1) and an output feedback controller (7) is described by the stochastic differential equation (10). We wish to find an output feedback controller of the form (7) such that:

(i) The closed loop nominal system (9) corresponding to this controller is exponentially stable.

(ii) The closed loop system (10) with $x(0)=0$, $\hat{x}(0)=0$ satisfies the stochastic H^∞ norm condition

$$\mathbf{E}\int_0^{+\infty}\left(\|z(t)\|^2 - \|\xi(t)\|^2\right)dt \le -\varepsilon\mathbf{E}\int_0^{+\infty}\|\xi(t)\|^2 dt \quad (15)$$

for each $\xi(\cdot) \in L_2(0;\mathbf{R}^{m_2})$. Here, the output $z(\cdot)$ is defined by the system (10).

Lemma 2 ([4]) *The controller of the form (7) that solves the above stochastic H^∞ control problem exists if and only if the pair of Riccati inequalities*

$$\begin{aligned} &YA + A'Y + H^*YH + YB_2B_2'Y \\ &-C_2'(D_2D_2')^{-1}C_2 + C_1'C_1 < 0, \end{aligned} \quad (16)$$

$$\begin{aligned} &A'X + XA + H^*YH + C_1'C_1 \\ &-X\left(B_1D_1'D_1)^{-1}B_1' - B_2B_2'\right)X < 0. \end{aligned} \quad (17)$$

admits symmetric solutions $Y > 0$ and $X > 0$ such that $Y > X$. If inequalities (16), (17) admit such solutions, then the solution of the above H^∞ control problem is given by a controller of the form (7) in which

$$\begin{aligned} A_c &= A - B_cC_2 + B_1C_c + B_2B_2'X, \\ B_c &= (Y-X)^{-1}C_2'(D_2D_2')^{-1}, \\ C_c &= -(D_1'D_1)^{-1}B_1'X, \\ D_c &= 0. \end{aligned} \quad (18)$$

Guaranteed cost controllers based on Riccati inequalities and associated LMIs are often known to be excessively conservative. Also in the deterministic case and in the stochastic state-feedback case, the design of minimax optimal controllers which guarantee optimal performance requires that a solution to a corresponding H^∞ control problem be expressed in terms of algebraic Riccati equations rather than Riccati inequalities [8]. This motivates us to proceed from considering the Riccati inequalities (16), (17) to considering corresponding Riccati equations. The next result extends the Strict Bounded Real Lemma [7] to the realm of stochastic systems with multiplicative noise. It shows that the internal stability of the system (19) along with the satisfaction of a bound condition on the norm of the system operator is equivalent to the solvability of a certain generalized Riccati equation or generalized Riccati inequality. As a by-product of this result, the equivalence between the considered generalized Riccati equation and generalized Riccati inequality is established. This is an important fact since, in a general case, algebraic matrix equations and matrix inequalities arising in stochastic H^∞ control are not equivalent; see, e.g., [4].

Lemma 3 (Stochastic Strict Bounded Real Lemma) *Consider the system (1) where $u \equiv 0$:*

$$\begin{aligned} &dx = (Ax(t) + B_2\xi(t))dt + Hx(t)dw(t), \\ &z(t) = C_1x(t). \end{aligned} \quad (19)$$

The following conditions are equivalent:

(i) The linear system

$$dx(t) = Ax(t)dt + Hxdw(t) \quad (20)$$

which corresponds to the system (19) driven by the uncertainty input $\xi(\cdot) \equiv 0$, is exponentially mean-square stable. Also, the system (19) satisfies the following stochastic H_∞-norm condition: There exists a constant $\varepsilon > 0$ such that condition (15) is satisfied for each $\xi(\cdot) \in L_2(0;\mathbf{R}^{m_2})$.

(ii) There exists a nonnegative definite solution to the generalized Riccati equation

$$A'X + XA + H^*XH + C_1'C_1 + XB_2B_2'X = 0 \quad (21)$$

such that the system

$$dx(t) = (A + B_2B_2'X)xdt + Hxdw(t) \quad (22)$$

is exponentially mean-square stable.

(iii) There exists a positive definite matrix X satisfying the matrix inequality

$$A'X + XA + H^*XH + C_1'C_1 + XB_2B_2'X < 0. \quad (23)$$

The equivalence of conditions (i) and (ii) was established in [12, 13, 3]; also, see [8]. The proof of the fact that conditions (i) and (iii) are equivalent can be found in [4].

We now present a result which addresses the stochastic H^∞ output feedback control problem using the generalized Riccati equations approach. This result makes use of the following generalized Riccati equations:

$$\begin{array}{r}YA+A'Y+H^*YH+YB_2B_2'Y\\ -C_2'(D_2D_2')^{-1}C_2+C_1'C_1=0,\end{array}\quad(24)$$

$$\begin{array}{r}A'X+XA+H^*YH+C_1'C_1\\ -X\left(B_1D_1'D_1)^{-1}B_1'-B_2B_2'\right)X=0.\end{array}\quad(25)$$

Suppose there exists a unique positive definite solution Y to the Riccati equation (24) such that the system

$$dx=(A+B_2B_2'Y)xdt+Hxdw(t)\quad(26)$$

is exponentially stable. Also, suppose that for that Y, there exists a minimal nonnegative definite solution X to the Riccati equation (25) such that the matrix $A-(B_1(D_1'D_1)^{-1}B_1'-B_2B_2')X$ is stable and $Y>X$.

Consider the closed loop system (10) consisting of the system (1) and the controller given by equations (7), (18). As in [4], we let

$$\bar{X}:=\begin{bmatrix}Y & X-Y\\ X-Y & Y-X\end{bmatrix}.\quad(27)$$

It is straightforward to verify that the matrix $\bar{X}$ is nonnegative definite and satisfies the algebraic Riccati equation

$$\bar{A}'\bar{X}+\bar{X}\bar{A}+\bar{H}^*\bar{X}\bar{H}+\bar{C}'\bar{C}+X\bar{B}_2\bar{B}_2'\bar{X}=0\quad(28)$$

where the matrices $\bar{A}$, $\bar{B}_2$, $\bar{C}$ and the operator $\bar{H}$ are defined by equations (8), (18); see [4]. Also, the system

$$d\bar{x}=(\bar{A}+\bar{B}_2\bar{B}_2'\bar{X})\bar{x}dt+\bar{H}\bar{x}dw(t)\quad(29)$$

is exponentially mean-square stable. Indeed, equation (29) can be written as follows:

$$\begin{aligned}dx &= ((A+B_2B_2'Y)x\\ &\quad -((B_1(D_1'D_1)^{-1}B_1'-B_2B_2')X+B_2B_2'Y)\hat{x})dt\\ &\quad +Hxdw(t),\\ \dot{\hat{x}} &= (A-(B_1(D_1'D_1)^{-1}B_1'-B_2B_2')X)\hat{x}.\end{aligned}$$

Hence, this system is exponentially mean-square stable due to the stabilizing properties of the matrices Y and X mentioned above. We have verified all of the conditions needed to apply the Stochastic Strict Bounded Real Lemma; see Lemma 3. From Lemma 3, it follows that the output feedback controller (7), (18) solves the stochastic H^∞ control problem considered in this section. We summarize the above discussion in the following result.

Theorem 1 *Suppose there exist solutions to the Riccati equations (24) and (25) as described in the above. Then, the output feedback controller (7), (18) solves the stochastic H^∞ control problem associated with the system (1).*

Note that the deterministic H^∞ control is recovered by setting $H=0$. In this particular case, equations (25) and (24) turn into standard algebraic Riccati equations arising in H^∞ control [1]; equation (24) must be pre- and post-multiplied by Y^{-1}. Also, the controller (7), (18) becomes the controller solving the deterministic H^∞ control problem for the system (1) with $H=0$.

It is worth noting that in addition to coupling through the condition $Y>X$, equations (24) and (25) are also coupled through the term H^*YH in the second equation.

3 Design of a guaranteed cost controller

The solution to the guaranteed cost control problem considered in this paper relies on the following scaled versions of the generalized Riccati inequalities (16), (17)

$$\begin{array}{r}\tilde{Y}A+A'\tilde{Y}+H^*\tilde{Y}H+\tilde{Y}\tilde{B}_2\tilde{B}_2'\tilde{Y}\\ -C_2'(\tilde{D}_2\tilde{D}_2')^{-1}C_2+\tilde{C}_1'\tilde{C}_1<0,\end{array}\quad(30)$$

$$\begin{array}{r}A'\tilde{X}+\tilde{X}A+H^*\tilde{Y}H+\tilde{C}_1'\tilde{C}_1\\ -\tilde{X}\left(B_1(\tilde{D}_1'\tilde{D}_1)^{-1}B_1'-\tilde{B}_2\tilde{B}_2'\right)\tilde{X}<0\end{array}\quad(31)$$

and the scaled versions of the generalized Riccati equations (24), (25)

$$\begin{array}{r}YA+A'Y+H^*YH+Y\tilde{B}_2\tilde{B}_2'Y\\ -C_2'(\tilde{D}_2\tilde{D}_2')^{-1}C_2+\tilde{C}_1'\tilde{C}_1=0,\end{array}\quad(32)$$

$$\begin{array}{r}A'X+XA+H^*YH+\tilde{C}_1'\tilde{C}_1\\ -X\left(B_1(\tilde{D}_1'\tilde{D}_1)^{-1}B_1'-\tilde{B}_2\tilde{B}_2'\right)X=0.\end{array}\quad(33)$$

We present two versions of the controller design. The first result given in Theorem 2 makes use of the solution of stochastic H^∞ control problem based on Lemma 2. Theorem 2 presents a necessary and sufficient condition for the guaranteed cost control problem under consideration is solvable. The second result given in Theorem 3 presents a tractable procedure for guaranteed control design. We conjecture that this procedure leads to a controller less conservative than the controller of Theorem 2.

Theorem 2 *Given an initial condition $x(0)=h$, a controller (7) solving a guaranteed cost control problem under consideration exists if and only if there exist constants $\tau_1>0,\ldots,\tau_k>0$ such that the corresponding Riccati inequalities (30), (31) admit symmetric positive definite solutions $\tilde{Y}$, $\tilde{X}$ satisfying $\tilde{Y}>\tilde{X}$. Furthermore, if such constants $\tau_1>0,\ldots,\tau_k>0$ exist, then the controller $\tilde{u}(\cdot)$ of the form (7), (18) where $X=\tilde{X}$, $Y=\tilde{Y}$ guarantees that*

$$\sup_{\xi\in\Phi}J^{s,h}(\tilde{u},\xi)\le h'(\tilde{X}+\sum_{i=1}^{k}\tau_iW_i)h.\quad(34)$$

Let $\mathcal{T}$ be the set of vectors $\tau = [\tau_1, \dots, \tau_k]'$, $\tau_1 > 0, \dots, \tau_k > 0$, such that the corresponding equations (32) and (33) satisfy the conditions of Theorem 1. Let Y_τ and X_τ denote the solutions to equations (32) and (33) corresponding to $\tau \in \mathcal{T}$.

Theorem 3 *Suppose the set $\mathcal{T}$ is non-empty. Let τ^* attain the infimum*

$$\inf_{\tau \in \mathcal{T}} h'(X_\tau + \sum_{i=1}^{k} \tau_i W_i)h, \quad h \in \mathbf{R}^n.$$

Then, the output feedback controller $u^*(\cdot)$ given by equations (7), (18) where $X = X_{\tau^*}$, $Y = Y_{\tau^*}$, guarantees that

$$\sup_{\xi_\phi \in \Phi} J^{s,h}(u^*, \xi_\phi) \leq \inf_{\tau \in \mathcal{T}} h'(X_\tau + \sum_{i=1}^{k} \tau_i W_i)h. \qquad (35)$$

Note that the guaranteed cost controllers constructed in Theorems 2 and 3 are absolutely stabilizing controllers for the uncertain system (1), (5). The definition of absolute stability given in [13, 8] is the following.

Definition 3 *A controller of the form (7) is said to absolutely stabilize the uncertain system (1), (5) if the following conditions hold:*

1. *The nominal closed-loop system is exponentially mean-square stable.*
2. *There exists a constant $c > 0$, independent of the initial condition and such that for any admissible uncertainty ϕ, the corresponding solution to the closed loop system (1), (5), (7) belongs to $L_2(s, \mathbf{R}^n)$, the corresponding uncertain input $\xi_\phi(\cdot)$ belongs to $L_2(s, \mathbf{R}^{m_2})$, and*

$$|||x(\cdot)|||^2 + |||\xi_\phi(\cdot)|||^2 \leq c\mathbf{E}\|h\|^2. \qquad (36)$$

Theorem 4 *Suppose the conditions of Theorem 2 or Theorem 3 are satisfied. Then, the controller constructed using Theorem 2 or Theorem 3 absolutely stabilizes the uncertain system (1), (5).*

4 Conclusions

This paper has been concerned with the existence of a guaranteed cost controller for an uncertain stochastic system subject to structured uncertainty. We have shown that for each initial state of the system, the linear output feedback controller yielding a guaranteed worst case performance in the face of stochastic structured uncertainty in the system, can be found by parametric optimization of solutions of the pair of parameter-dependent generalized matrix Riccati inequalities or matrix Riccati equations. The generalized Riccati inequalities and Riccati equations are of the type arising in stochastic H^∞ theory.

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