Generalized LQG Design by Filter and Controller Model Selection

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Abstract

This paper investigates a generalization of the conventional approach to LQG control design. First we investigate removing the assumption that the Kalman filter as the observer is necessarily based on the same model as the best plant model. The controller gain matrix design is performed as usual, based on the optimal solution to the deterministic design for the best model of the real-world plant. For the next case, we also remove this controller design restriction to investigate robustness to uncertainties in the plant model. The filter and controller gain matrices are both determined by models possibly other than the plant model. We relate the plant model to the filter and controller design models by a position correlation (mean square error on output) measure in order to determine optimal performance.

1. Introduction

Linear Quadratic Gaussian (LQG) is one established approach to optimal conventional control in state space. The design approach for the LQG controller is based on the principle of certainty equivalence [3] in which the filter is designed separately from the full-state feedback controller. The two components are put together to yield the LQG control algorithm. It is conventionally assumed that both the filter and full-state feedback controller are designed based on the truth model in order to minimize a mean squared error criterion.

Given that the filter design model for the system is described in terms of state $\mathbf{x}^{\mathrm{f}}$ (the superscript f denoting "filter model"), measurement $\mathbf{z}$, and controlled variable $\mathbf{y}$ as:

$$\mathbf{x}^{\mathrm{f}}(t_{i+1}) = \mathbf{\Phi}^{\mathrm{f}}(t_{i+1},t_i)\mathbf{x}^{\mathrm{f}}(t_i) + \mathbf{B}_{\mathrm{d}}^{\mathrm{f}}(t_i)\mathbf{u}(t_i) + \mathbf{G}_{\mathrm{d}}^{\mathrm{f}}(t_i)\mathbf{w}_{\mathrm{d}}^{\mathrm{f}}(t_i) \quad (1)$$

$$\mathbf{z}(t_i) = \mathbf{H}^{\mathrm{f}}(t_i)\mathbf{x}^{\mathrm{f}}(t_i) + \boldsymbol{v}^{\mathrm{f}}(t_i) \quad (2)$$

$$\mathbf{y}(t_i) = \mathbf{C}^{\mathrm{f}}(t_i)\mathbf{x}^{\mathrm{f}}(t_i) \quad (3)$$

the control law is given by

$$\mathbf{u}(t_i) = -\mathbf{G}_{\mathrm{c}}\hat{\mathbf{x}}^{\mathrm{f}}(t_i^+) \quad (4)$$

where $\hat{\mathbf{x}}$ is the output of the Kalman filter given by

$$\hat{\mathbf{x}}^{\mathrm{f}}(t_{i+1}^-) = \mathbf{\Phi}^{\mathrm{f}}\hat{\mathbf{x}}^{\mathrm{f}}(t_i^+) - \mathbf{B}_{\mathrm{d}}^{\mathrm{f}}\mathbf{G}_{\mathrm{c}}\hat{\mathbf{x}}^{\mathrm{f}}(t_i^+) \quad (5)$$

propagated between samples, with a sampled-data update given by

$$\hat{\mathbf{x}}^{\mathrm{f}}(t_i^+) = \hat{\mathbf{x}}^{\mathrm{f}}(t_i^-) + \mathbf{K}^{\mathrm{f}}[\mathbf{z}(t_i) - \mathbf{H}^{\mathrm{f}}\hat{\mathbf{x}}^{\mathrm{f}}(t_i^-)] \quad (6)$$

The dynamics matrices of the Kalman filter in Equations (5)-(6) are those from the system matrices in (1). For the conventional design, it is then a matter of determining $\mathbf{K}$ and $\mathbf{G}_{\mathrm{c}}$ via Riccati equations. $\mathbf{K}$ is the observer gain matrix as found for the standard Kalman filter. $\mathbf{G}_{\mathrm{c}}$ is the optimal control gain matrix from the solution to the deterministic LQ regulator portion of the control problem.

The form of the observer in Equation (5) is conventionally based on the dynamics of the system. This is a very logical approach for an estimator since the goal of the optimization is to minimize the error between the actual state and the estimate. Likewise for control, it would seem that this estimation error should be minimal and so the best estimate is desired. However, for the regulator control problem, the actual goal is to minimize the mean squared output. The observer, though it may have a *perfect* estimate, creates a lag in the response and thus contributes to the error in regulating the controlled output. To correct this, it may be possible to set up an optimization approach that sacrifices best estimates for faster response to yield better control. This implies that the dynamics model of the filter is not necessarily matched to the best plant model which is based on a parameter set that will be denoted as $\mathbf{a}_{\mathrm{plant}}$.

2. Proposed Performance Measure

This research focuses on output correlation (mean squared regulation error) as the performance measure for the design of the best controller. Such a measure would be consistent with a "cheap control" version of LQG control in which the quadratic cost on states strongly dominates the quadratic on control; this is, in fact, the case of interest here. More generally, the design of the best controller could be based on the sum of the output correlation with a quadratic on control, consistent with the cost used to define the LQG controller in the first place, but that is not pursued here. The output correlation as a performance measure has been used in the multiple model adaptive control (MMAC) and is developed clearly by Sheldon [4,5]. We have generalized the output correlation computation to account for the three potentially different models: filter design model, full-state feedback controller design model and assumed truth. Thus, the model assumed by the state estimator or controller blocks may *not* directly correspond to $\mathbf{a}_{\mathrm{plant}}$ per se, but to values $\mathbf{a}_{\mathrm{filter}}$ and $\mathbf{a}_{\mathrm{controller}}$ that are *functions* of $\mathbf{a}_{\mathrm{plant}}$ but not necessarily to *equal* to $\mathbf{a}_{\mathrm{plant}}$. Also consider that the value of the true system parameter $\mathbf{a}_{\mathrm{true}}$ may not be equal to $\mathbf{a}_{\mathrm{plant}}$, but actually somewhere in some specifiable set around $\mathbf{a}_{\mathrm{plant}}$. For robustness, all possible $\mathbf{a}_{\mathrm{true}}$'s in that set would have to be considered when specifying the controller and filter blocks. For use in the filter and controller block specification, the following section develops the position error correlation based on the three separate models. This discussion lays the groundwork for the algorithm to select the $\mathbf{a}_{\text{filter}}$ and $\mathbf{a}_{\text{controller}}$ to yield best performance against either a true plant with parameter value of $\mathbf{a}_{\text{plant}}$ or a set of true plants based on a set of parameter values in the neighborhood of $\mathbf{a}_{\text{plant}}$.

Consider the conventional LQG controller specified in Equations (1)-(6) based on the model $\mathbf{a}_{\text{filter}}$ of the system dynamics in Equations (1)-(3). Now denote the controller gain $\mathbf{G}_c$ as $\mathbf{G}_c^c$ to indicate it is based on a controller model separate from the filter and truth models. So now the equation for true control becomes

$$\mathbf{u}^t(t_i) = -\mathbf{G}_c^c \hat{\mathbf{x}}^f(t_i^+) \tag{7}$$

The true system is modeled by

$$\mathbf{x}^t(t_{i+1}) = \mathbf{\Phi}^t \mathbf{x}^t(t_i) + \mathbf{B}_d^t \mathbf{u}^t(t_i) + \mathbf{G}_d^t \mathbf{w}_d^t(t_i) \tag{8}$$

Now after making all substitutions and simplifications, the augmented system description is given by

$$\begin{bmatrix} \mathbf{x}_{i+1}^t \\ \hat{\mathbf{x}}_{i+1}^{f\,-} \end{bmatrix} = \begin{bmatrix} (\mathbf{\Phi}^t - \mathbf{B}_d^t \mathbf{G}_c^c \mathbf{K}^f \mathbf{H}^t) & -\mathbf{B}_d^t \mathbf{G}_c^c (\mathbf{I} - \mathbf{K}^f \mathbf{H}^f) \\ (\mathbf{\Phi}^f - \mathbf{B}_d \mathbf{G}_c^c) \mathbf{K}^f \mathbf{H}^t & (\mathbf{\Phi}^f - \mathbf{B}_d^f \mathbf{G}_c^c)(\mathbf{I} - \mathbf{K}^f \mathbf{H}^f) \end{bmatrix} \begin{bmatrix} \mathbf{x}_i^t \\ \hat{\mathbf{x}}_i^{f\,-} \end{bmatrix} + \begin{bmatrix} -\mathbf{B}_d^t \mathbf{G}_c^c \mathbf{K}^f \\ (\mathbf{\Phi}^f - \mathbf{B}_d^f \mathbf{G}_c^c) \mathbf{K}^f \end{bmatrix} \mathbf{v}_i^t + \begin{bmatrix} \mathbf{G}_d^t \\ \mathbf{0} \end{bmatrix} \mathbf{w}_{d\,i}^t \tag{9}$$

Now if we define

$$E\left\{ \begin{bmatrix} \mathbf{w}_{d\,i} \\ \mathbf{v}_i \end{bmatrix} \begin{bmatrix} \mathbf{w}_{d\,j}^T & \mathbf{v}_j^T \end{bmatrix} \right\} = \begin{cases} \mathbf{Q_0}(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \tag{10}$$

where

$$\mathbf{Q_0}(t_i) = \begin{bmatrix} \mathbf{Q}_d^t(t_i) & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^t(t_i) \end{bmatrix} \tag{11}$$

and further define

$$\hat{\mathbf{O}} = \begin{bmatrix} (\mathbf{\Phi}^t - \mathbf{B}_d^t \mathbf{G}_c^c \mathbf{K}^f \mathbf{H}^t) & -\mathbf{B}_d^t \mathbf{G}_c^c (\mathbf{I} - \mathbf{K}^f \mathbf{H}^f) \\ (\mathbf{\Phi}^f - \mathbf{B}_d^f \mathbf{G}_c^c) \mathbf{K}^f \mathbf{H}^t & (\mathbf{\Phi}^f - \mathbf{B}_d^f \mathbf{G}_c^c)(\mathbf{I} - \mathbf{K}^f \mathbf{H}^f) \end{bmatrix} \tag{12}$$

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{G}_d^t & -\mathbf{B}_d^t \mathbf{G}_c^c \mathbf{K}^f \\ \mathbf{0} & (\mathbf{\Phi}^f - \mathbf{B}_d^f \mathbf{G}_c^c) \mathbf{K}^f \end{bmatrix} \tag{13}$$

then the output autocorrelation of the augmented system expressed in Equation (9) can be written conveniently as:

$$E\left\{ \begin{bmatrix} \mathbf{x}_{i+1}^t \\ \hat{\mathbf{x}}_{i+1}^{f\,-} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{i+1}^{t\,T} & \hat{\mathbf{x}}_{i+1}^{f\,T} \end{bmatrix} \right\} = \mathbf{\Xi}_{i+1} = \mathbf{T}\mathbf{\Xi}_i \mathbf{T}^T + \mathbf{\Gamma}\mathbf{Q_0}\mathbf{\Gamma}^T \tag{14}$$

Now denote the upper left partition of the resultant expectation expression as Ψ_c, and the position correlation is expressed as

$$E\{\mathbf{y}^T \mathbf{W} \mathbf{y}\} = tr[\mathbf{W}\mathbf{C}^t \Psi_c \mathbf{C}^{t\,T}] \tag{15}$$

3. Kalman Filter and Control Gain Selection Algorithm

The previous sections introduced the concept of using the position correlation (mean square regulation error) to select the best filter and controller for the control algorithm given an assumed value of the parameter vector defining the system model. Using these results, the selected filter dynamics and gain matrix and controller gain matrix do not have to be directly based on the assumed true system model in order to compute the position correlation. The first application of this approach is to address the conventional LQG controller design to determine how to incorporate filter and controller gain evaluation based on the position correlation. A second extension to the selection algorithm takes into account that the actual system model may not be a single model based on $\mathbf{a}_{\text{true}}$ but may be based on one element of a *set* of possible true $\mathbf{a}$ values. The goal is to make the filter/controller selection robust to the possible variation in the actual truth model of the plant. The potential filter/controller combinations are checked against the set of possible values of $\mathbf{a}_{\text{true}}$.

3.1 A Generalized LQG Design Approach

When one considers the design of a Luenberger observer [1] rather than a Kalman filter as the state estimator, we find that the goal is actually to speed up the dynamics of the observer relative to the truth model. This is done by choosing the observer gains to place the poles of the resultant dynamics. This is not intrinsically incorporated into the solution to the Riccati equation solution to minimize the cost function when designing the corresponding Kalman filter, although tuning of the filter (e.g., by choice of assumed dynamics noise strength) can accomplish this purpose [2]. Thus, it seems plausible to design Kalman filter for a model that is actually "faster" than the dynamics of the truth model, i.e. to use a $\mathbf{\Phi}^f$ in Equations (4)-(2) different from the truth model $\mathbf{\Phi}^t$ in Equation (8).

In Equation (9), the controller and filter gains denoted $\mathbf{G}_c^c$ and $\mathbf{K}^f$ are selected based on solutions to two Riccati equations based on assumed models not necessarily equivalent to the truth model. The object is to determine the filter dynamics and gains $\mathbf{K}^f$ and $\mathbf{G}_c^c$ which minimize the position correlation (mean squared values of regulation errors) for the particular truth model. Again, the conventional LQG approach assumes that $\mathbf{K}^f$ and $\mathbf{G}_c^c$ are determined based on models equivalent to the truth model (or reduced-order version thereof, but still based on the same parameter value $\mathbf{a}_{\text{true}}$).

Consider the assumption that the actual system states are completely and perfectly measurable. Then the problem of controller design simply becomes the classical LQR approach. It would seem that the *best* filter would need to be added if the states were not perfectly measurable, best in the sense that the *control* objective is met with the lowest cost. As previously noted with the discussion on the Luenberger observer, the indicated solution to the filter/controller gain search is to determine the full-state feedback controller and then find that *best* filter. Based on this assessment, the following is the filter controller selection algorithm for the generalized LQG control algorithm:

1) For the given system truth model, design a controller using typical LQR techniques to obtain the gain $\mathbf{G}_c^c$ (assuming that the controller design model parameter value $\mathbf{a}_{\text{controller}}$ equals $\mathbf{a}_{\text{true}}$).

2) Select a filter design model that is in the neighborhood of the system truth model and design a filter using the typical LQE techniques to find $\mathbf{K}^{\mathrm{f}}$.
3) Compute the position correlation using $\mathbf{K}^{\mathrm{f}}$ and $\mathbf{G}_{\mathrm{c}}^{\mathrm{c}}$ from (1) and (2).
4) Repeat (1)-(3) for each filter model in the neighborhood of the system truth model and select the filter gain that yields the minimum position correlation.

3.2 Selection Algorithm for Robust Controller

The previous selection algorithms are based on the assumption that the $\mathbf{a}_{\mathrm{true}}$ of the system model is known exactly. However, now assume that this parameter value is not known exactly, but that it actually lies in the range given by (for a scalar parameter a):

$$\mathrm{a}_{\mathrm{true}} \in [\mathrm{a}_{\mathrm{plant}} - k\sqrt{\mathrm{P_a}}\ ,\ \mathrm{a}_{\mathrm{plant}} + k\sqrt{\mathrm{P_a}}\,] \qquad (16)$$

for some chosen scalar k, or for vector $\mathbf{a}_{\mathrm{true}}$, it is perhaps within the ellipsoid associated with scalar k that is centered at $\mathbf{a}_{\mathrm{plant}}$ and defined by the eigenvalues and eigenvectors of $\mathbf{P}_{\mathrm{a}}$. As k is made larger, as going from 0 to 1 to 2, etc., progressively larger sets of possible $\mathbf{a}_{\mathrm{true}}$ values are allowed, and thus progressively greater amounts of robustness are provided by a design that accounts for all such $\mathbf{a}_{\mathrm{true}}$ values. Since $\mathbf{a}_{\mathrm{plant}}$ that is used to select the filter and controller may not exactly match $\mathbf{a}_{\mathrm{true}}$, the selection algorithm must take into account the variation given by Equation (16).

The goal for the selection algorithm is to determine the controller/filter combination for the given $\mathbf{a}_{\mathrm{plant}}$ that can be considered robust across the whole range of possible $\mathbf{a}_{\mathrm{true}}$. The algorithm to accomplish this is:

1) Select a value for the controller model location in the range of possible $\mathbf{a}_{\mathrm{true}}$ values.
2) For the controller selected in (1), determine the filter model that yields the minimum position correlation over the range. The filter selected is determined by first computing the maximum position correlation as $\mathbf{a}_{\mathrm{true}}$ is varied over the admissible range of values, for each possible filter. Of those maximum position correlation values, choose the filter corresponding to the minimum of those computed maxima. These first 2 steps are illustrated in Figure 1a and given as:
$$\mathrm{F} \equiv \mathrm{F_i} \text{ which yields } \min_{i} (\max_{\mathbf{a}_{\mathrm{true}}} \Psi_{\mathrm{C_1,F_i}})$$
where denotes Ψ output correlation.
3) Filter selection process is repeated for each controller in the parameter space bounded by Equation (16).
4) Similar to the filter selection, the controller is determined by using the position correlation as $\mathbf{a}_{\mathrm{true}}$ is varied over the admissible range for the controller and filter combination from step (3). Of those maximum position correlation values, choose the controller and filter corresponding to the minimum of those computed maxima. These two steps (3) and (4) are illustrated in Figure 1b and given as:
$$\mathrm{C} \equiv \mathrm{C_j} \text{ which yields } \min_{\mathrm{j}} (\max_{\mathbf{a}_{\mathrm{true}}} \Psi_{\mathrm{C_j,F_i}})$$

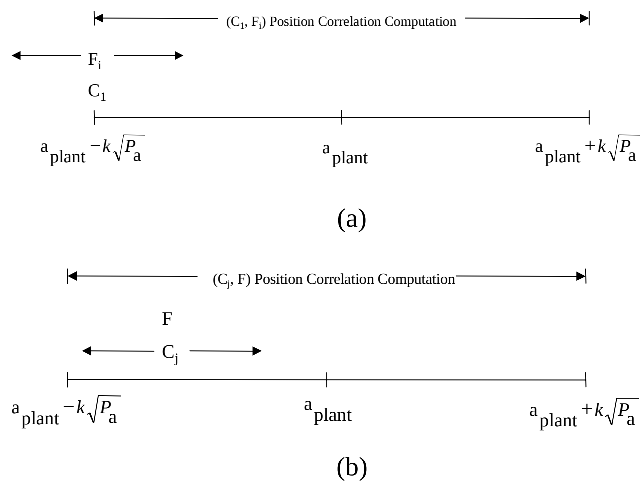

Figure 1. (a) Filter search given a controller in the parameter space. (b) Controller search and filter found for each possible controller.

5) Steps (1) through (4) are repeated for each possible values of $\hat{\mathbf{a}}$ in the parameter space.

Upon completion of step 5, a table of controller/filter combinations is available for table look-up for design of an assumed value of $\mathbf{a}_{\mathrm{plant}}$. The associated position correlation for each point should be the worst case given the assumed variation of $\mathbf{a}_{\mathrm{true}}$. The worst-case position correlation will obviously be affected by the assumed variation of $\mathbf{a}_{\mathrm{true}}$ (i.e., the size of $k\sqrt{P_a}$ in the scalar case) that is used in the algorithm.

4. Application Evaluation

The ideal mechanical translational system is a continuous time system second order system in the form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t)$$

modeled as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} w(t) \qquad (17)$$

where x_1 is the position, x_2 is the velocity. The constants k, m and b are the spring constant, mass and damping coefficient, respectively. This true system modeled can be simplified into the more general form by assigning the undamped natural frequency as $\omega_n \equiv \sqrt{\frac{k}{m}}$ and letting the damping ratio be $\zeta \equiv 2\frac{b}{\sqrt{km}}$. This yields the stochastic truth model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} w(t) \qquad (18)$$

with

$$\mathrm{E}\{w(t)w(t+\tau)\} = Q\delta(\tau)$$

For this problem, there are potentially three unknowns upon which to base design models: the undamped natural frequency ω_n, the damping ratio ζ, and the dynamic driving noise strength Q. However, for this experiment, the damping ratio is known to be $\zeta \equiv 0.01$ and the dynamic driving noise is assigned $Q = 0.01$. The undamped natural frequency is considered the uncertain parameter for the design models. The effects of this uncertain parameter will be investigated over the range 2π rad/sec to 20π rad/sec.

For the purpose of computer simulation and control, the system model given in Equation (18) is converted to an equivalent discrete model via [2]:

$$\mathbf{x}(t_{i+1}) = \Phi(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathbf{w}_d(t_i) \quad (19)$$

where the sample period t_{i+1}-t_i = 0.01 sec, $\mathbf{\Phi}(t_{i+1},t_i)$ is the state transition matrix, and

$$\mathbf{B}_d(t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)d\tau\ . \quad (20)$$

The discrete-time white Gausian noise term $\mathbf{w}_d(t_i)$ has zero mean and covariance kernel:

$$\mathrm{E}\{\mathbf{w}_d(t_i)\mathbf{w}_d(t_j)^{\mathrm{T}}\} = \begin{cases} \mathbf{Q}_d(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \quad (21)$$

where

$$\mathbf{Q}_d(t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{G}(\tau)Q(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t_{i+1}, \tau)d\tau \quad (22)$$

Measurements are taken at each sample time and described by

$$z(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + v(t_i) \quad (23)$$

where

$$\mathbf{H} = [1 \quad 0]$$

and $v(t_i)$ is a zero-mean white discrete-time Gaussian noise process with covariance kernel

$$\mathrm{E}\{v(t_i)v(t_j)^{\mathrm{T}}\} = \begin{cases} R(t_i) & t_i = t_j \\ 0 & t_i \neq t_j \end{cases} \quad (24)$$

and

$$R(t_i) = 0.01 \quad \forall\, i$$

The output is given by

$$y(t_i) = \mathbf{C}(t_i)\mathbf{x}(t_i) \quad (25)$$

where C is set to [1 0], which corresponds to position as the desired output.

4.1 Case I: Baseline Case – Conventional LQG Design

A baseline case is used to compare the algorithms developed in the previous sections. At a given point in the parameter space, we assume that the system is completely known. Using conventional techniques, the assumed best approach is to design a LQG controller using the system model with the known parameter. The position correlation (mean squared regulation error) can be calculated for that design and system model. Now this process is repeated for each discrete point in the uncertain parameter space. A large enough number of discrete points (200 points) was chosen in order to yield a *smooth* curve.

The evaluation was accomplished using Matlab. A loop through parameter space is used to design a conventional LQG controller at each point. The position correlation is computed at each of the discrete model points. Finally, a cost is computed as in [4,5] from the position correlation as the integration of the position correlation over the parameter space normalized by the parameter space or:

$$c = J^2 \equiv \frac{\int_A E\{\mathbf{y}^{\mathrm{T}}\mathbf{W}\mathbf{y}\}da}{\int_A da} \quad (26)$$

The plot of the position correlation for each discrete point in the uncertain space is given in Figure 2 and the cost was computed as c = 0.0495. These results are considered the baseline case since it was originally thought that, if the parameter is assumed known, then the best approach is to design a conventional LQG controller at each point.

Also valuable for later comparison is a test of the robustness of the conventional LQG. Figure 2 shows the position correlation for each discrete point in the uncertain parameter space for the case in which the filter and controller are designed for the system model at the given point. A robustness test finds the maximum position correlation from possible system models in a specified region about the actual design model. A robustness region consisting of a radius of 10 sample points was selected arbitrarily around each point in the uncertain space. Figure 3 shows the plot of the position correlation of the robustness test overlaid on the position correlation for the conventional LQG. The resultant cost for the robustness

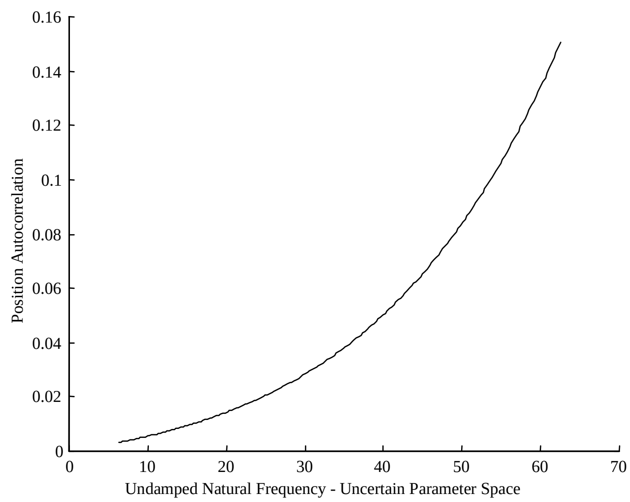

Figure 2 Position Correlation for the Baseline Case

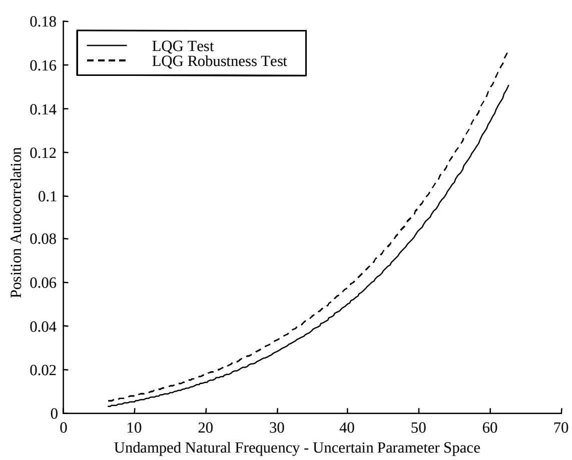

Figure 3 Position Correlation for Conventional LQG with Overlay of Robust Test Case

test was 0.0568 as compared to a cost of 0.0495 for the previous test. This test will be denoted as the conventional LQG robustness test.

4.2 Case II: Generalized LQG with Perfect Parameter Knowledge

The generalized LQG method assumes a given truth model for the controller gain design and separate filter model for the subsequent selection of a Kalman filter. This is applied to a discrete number of models in the parameter space. Controller evaluation uses the selection algorithm described previously and computes the position correlation. The cost is again computed as in Equation (26). For this evaluation, 200 discrete points evenly spaced over the parameter space were chosen. A controller was designed for each point, letting $\mathbf{a}_{\text{controller}}$ equal $\mathbf{a}_{\text{true}}$. For the filter models, 200 point designs over the parameter space also were used with an additional 40 point designs past the maximum uncertain parameter value at the same discrete sample interval. This was found to be necessary so that the upper bound of the filter selection process would not be artificially constrained to the filter model at the maximum range of the uncertain parameter space. Thus, in the selection algorithm, the point designs for the full-state feedback controller were used directly and the *best* filter was selected from the 240 filter point designs. A search algorithm was used to limit the number of filters investigated. The plot of the position correlation (mean square regulation error) for each discrete point in the uncertain space is given in Figure 4 and the cost was computed as c = 0.468. Note that this is lower than the computed cost of 0.0495 for Case I.

The results presented in Figure 4 indicate that the generalized LQG approach yields potential improvement over the conventional LQG design. Notice that the improvement becomes more significant as the natural frequency of the system increases. Since it is the selection of the filters that differs, it is worth examining the filters selected at each point in the parameter space, as presented in Figure 5. Clearly, even at the lowest frequencies, a filter with "faster" dynamics than the truth model (equal to the controller design model) is selected. Also notable is that the separation between the controller model and the filter model increases as the frequency increases. This information, along with the position correlation plots, indicates that at the lower frequency, the improvement is not as significant as at the higher frequency.

Figure 5 indicates which filter model should be used for the corresponding controller and system truth model. For simplification, the selection algorithm could be employed at a single point rather than a whole uncertain parameter space. This result could possibly impact the approach to conventional LQG design in a very substantive way.

4.3 Case III: Robust Controller Evaluation

For the previous evaluation, the true plant model was assumed perfectly known. However, for some design problems this is not the case and it cannot be assumed that the controller and filter can be designed based on a single assumed truth model. For this case, the controller and filter are selected based on the position correlation using the algorithm outlined in Section 3.2.

A difference of note between this algorithm and the algorithm for perfectly known parameter value is that the controller is *not* generally designed for the parameter given by $\mathbf{a}_{\text{plant}}$. The controller and the filter are both selected based on the combination of the two that yields the best performance over the range of possible $\mathbf{a}_{\text{true}}$'s. Thus, the truth model may not match the filter model or the controller model. However, for the actual implementation of the control algorithm, the assumed $\mathbf{a}_{\text{plant}}$ is used to index into a table that lists the filter/controller combinations.

Again, for this evaluation, 200 discrete points evenly spaced over the parameter space were chosen. A controller and filter were designed for each point. The number of discrete points in the uncertain parameter space was selected arbitrarily, but it was found that additional models did not significantly affect the computed cost. A search algorithm was used to limit the number of filters

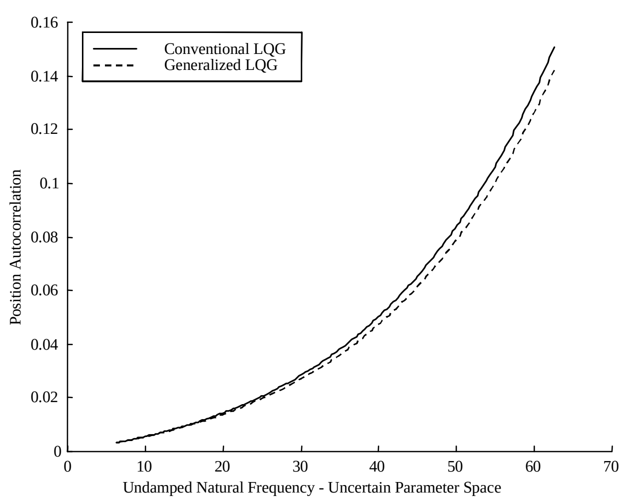

Figure 4 Position Correlation for the Generalized LQG and Conventional LQG Control

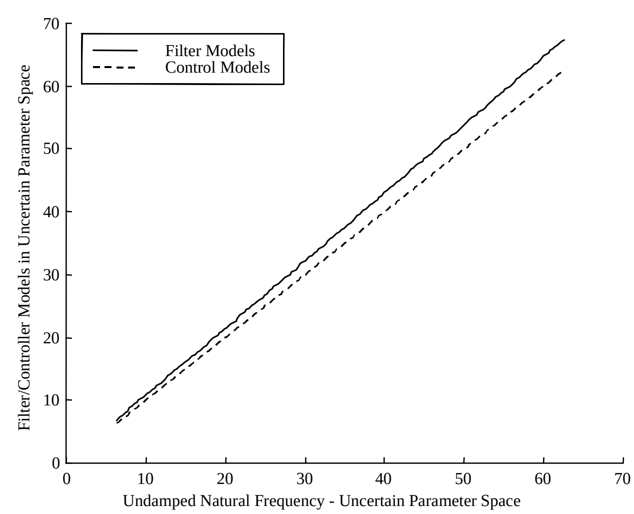

Figure 5 Filter Models Selected for Each Point in the Uncertain Parameter Space.

investigated, and the range of possible $\mathbf{a}_{trues}$'s limited the number of controllers investigated. The possible range of $\mathbf{a}_{true}$'s was arbitrarily chosen to be a radius of 10 discrete samples around the assumed value of $\mathbf{a}_{plant} = \mathbf{a}_{true}$. The plot of the position correlation for each discrete point in the uncertain space is given in Figure 6 and the cost was computed as c = 0.0501. This cost is less than computed cost of 0.0568 for the conventional LQG robustness test and only slightly greater than 0.0495 for the baseline from Case I and a cost of 0.0468 for Case II where robustness was not tested.

Obviously, the additional quality of robustness came at an *insignificant* increase in cost over the baseline case. In terms of non-adaptive single controller design, this could impact future design approaches. This algorithm allows the designer to specify the amount of robustness for given parameter(s) and can determine how the robustness affects the position correlation (mean square regulation error as an indicator of performance).

Similar to the previous algorithm, the filter and controller were not selected based on the same model, as shown in Figure 7. Also evident is that the controller design model is not necessarily selected to be equivalent the truth model, as was the case in the previous section. In Figure 7, the solid lines are the assumed value of the truth model ($\mathbf{a}_{plant}$ in Equation (16)) and the range within which $\mathbf{a}_{true}$'s can exist. At the lower frequencies, the controller and filter models were both biased higher than $\mathbf{a}_{plant}$. At the higher frequencies, two different phenomena occurred. As in the previous algorithm, the filter is shifted higher than the controller, but the controller is shifted down from the truth model. Also, the bias between the filter and controller locations increases as the frequency increases. The bias between the filter and controller models is consistent with the previous discussions on *faster* filter than controller. However, the shift down of the controller from the truth model is a new phenomenon, resulting from the optimization allowing $\mathbf{a}_{controller}$ to be different from $\mathbf{a}_{plant}$.

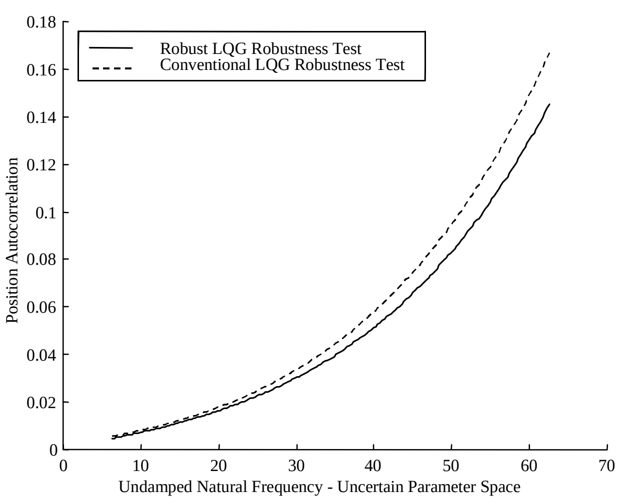

Figure 6 Position Correlation of the Robust LQG and Convential Algorithm Implementions

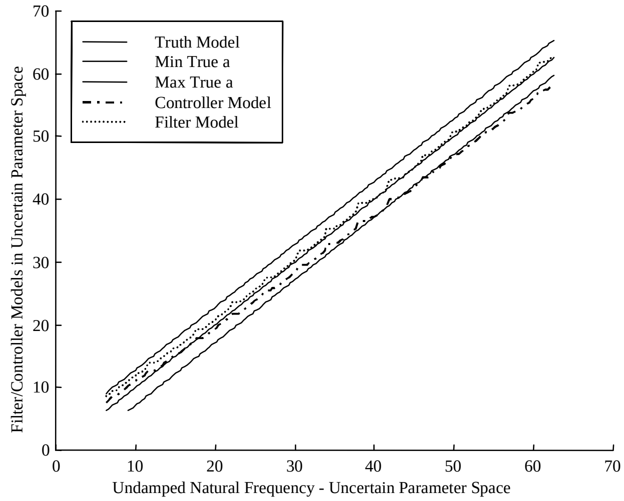

Figure 7 Filter and Controller Models Selected Over the Uncertain Parameter Space

5. Summary

In this paper we have demonstrated a new approach to the conventional LQG design. This approach focuses on the output correlation (mean squared regulation error) as the performance measure for the design of the best controller. This measure takes into account the three potentially different models: filter design model, full-state feedback controller design model and assumed truth. The results of the optimization for an example application are that the conventional LQG controller (synthesized by basing both filter design model and full-state-feedback controller design model on the best model of the true system) is outperformed by an LQG controller in which the filter design model assumes faster dynamics than does the controller design model. When neither of these models is forced equal to the best model of the true system and robustness is sought against a set of possible real-world systems, the full-state-feedback controller design model may assume slower dynamics than the best (average) model of the real-world system, and again the filter assumes a faster dynamics model than does the LQ regulator design.

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