Existence of Neighbouring Feasible Trajectories: Applications to Dynamic Programming for State Constrained Optimal Control Problems[1]

H. Frankowska* and R. B. Vinter**

*Centre de Recherche Viabilité, Jeux, Contrôle,
Univ. de Paris-Dauphine, 75775 Paris, Cx 16, France

**Centre for Process Systems Engineering and
Department of Electrical and Electronic Engineering,
Exhibition Road, London SW7 2BT, UK

Abstract

In this paper, the value function for an optimal control problem with endpoint and state constraints is characterized as the unique lower semicontinuous generalized solution of the Hamilton-Jacobi Equation. This is achieved under a constraint qualification (CQ) concerning the interaction of the state and dynamic constraints. The novelty of the results reported here is partly the nature of (CQ) and partly the proof techniques employed, which are based on new estimates of the distance of the set of state trajectories satisfying a state constraint from a given trajectory which violates the constraint.

Keywords: Optimal Control, State Constraints, Dynamic Programming, Hamilton Jacobi Equation.

1 Introduction

Consider the optimal control problem

$$(\mathrm{P}) \qquad \begin{cases} \text{Minimize } g(x(1)) \\ \text{over arcs } x \in W^{1,1}([0,1];R^n) \text{ satisfying} \\ \dot{x}(t) \in F(t,x(t)) \quad \text{a.e. } t \in [0,1], \\ x(t) \in A \quad \forall\, t \in [0,1], \\ x(0) = x_0, \end{cases}$$

the data for which comprise: a function $g : R^n \to R \cup \{+\infty\}$, a multifunction $F : [0,1] \times R^n \rightsquigarrow R^n$, a closed set $A \subset R^n$ and a vector $x_0 \in R^n$.

[1]This research was supported by the Human Capital and Mobility Programme of the European Commission (contract no. CHRX-CT94-0431)

Absolutely continuous arcs satisfying the differential inclusion are called F-trajectories. F-trajectories satisfying the constraints of (P), are called *feasible* arcs (for (P)).

Note that, since g is extended valued, (P) incorporates the endpoint constraint:

$$x(1) \in C$$

where $C := \mathrm{dom}\, g$.

Denote by $V : [0,1] \times A \to R \cup \{+\infty\}$ the value function for (P): for each $(t,x) \in [0,1] \times A$, $V(t,x)$ is defined to be the infimum cost for the problem

$$(P_{t,x}) \qquad \begin{cases} \text{Minimize } g(y(1)) \\ \text{over arcs } y \in W^{1,1}([t,1];R^n) \text{ satisfying} \\ \dot{y}(s) \in F(s,y(s)) \quad \text{a.e. } s \in [t,1], \\ y(s) \in A \quad \forall\, s \in [t,1], \\ y(t) = x. \end{cases}$$

Thus

$$V(t,x) = \inf(P_{t,x}).$$

(If $(\mathrm{P}_{t,x})$ has no feasible arcs, we set $V(t,x) = +\infty$.)

A major theme in Dynamic Programming is to explore the relationship between the value function and the Hamilton-Jacobi Equation:

$$(HJE) \begin{cases} \phi_t(t,x) + min_{v \in F(t,x)} \phi_x(t,x) \cdot v \;=\; 0 \\ \text{ for } (t,x) \in (0,1) \times \mathrm{int} A \\ \phi(1,x) \;=\; g(x) \text{ for } x \in R^n. \end{cases}$$

In past work aimed at characterizing V as the unique solution to (HJE) for optimal control problems with state constraints, it has been found necessary to impose some kind of constraint qualification on the dynamic constraint at boundary points of the state constraint set.

In [5], Capuzzo-Dolcetta and Lions showed that the value function is continuous and is the unique viscosity solution to (HJE) under hypotheses which include the "inward-pointing" constraint qualification:

$$\min_{v\in F(t,x)} n_x \cdot v < 0 \quad \forall x \in \text{bdy}\, A$$

where n_x denotes the unit outward normal vector at the point $x \in \text{bdy}\, A$. Hypotheses of this nature were introduced by Soner [11] to ensure continuity of the value function and to provide a characterization of the value function in terms of viscosity solutions of the relevant Hamilton-Jacobi equation, for an infinite horizon problem. Generalizations of Soner's condition, to allow for state constraint sets with nonsmooth boundaries, were used by Ishii and Koike [9], also in the context of infinite horizon problems.

When the "inward pointing" constraint qualification fails, or when the terminal cost function g is chosen to take account of an endpoint constraint, we can expect that the value function will be discontinuous. In a separate development, characterizations of value functions as (possibly discontinuous) lower semicontinuous solutions to (HJE) were provided by Barron and Jensen [4] and Frankowska [6] for state constraint-free problems.

These different strands were brought together by Frankowska and Plaskacz [7] who generalized the results of [6] on the value function and lower semicontinuous solutions of (HJE) to allow for state constraints, under a constraint qualification.

[7] covers optimal control problems for which the state constraint set A is an arbitrary closed set. The constraint qualification of [7] (specialized to the case when A is the closure of an open set with smooth boundary) is the "outward pointing" condition:

$$\min_{v\in F(t,x)} n_x \cdot v > 0 \quad \forall x \in \text{bdy}\, A.$$

The "outward pointing" condition yields "backward invariance" of epigraph of solutions to Hamilton-Jacobi-Bellman equations, the tool introduced in [6] to study uniqueness of lower semicontinuous solutions for target-type problems (forward viability and backward invariance of the epigraph imply uniqueness). We underline that the "inward-pointing" condition would yield forward invariance of the hypograph of solutions, provided this last one is closed. This is the main reason why the "inward-pointing" condition is helpful to prove uniqueness of continuous solutions, but becomes helpless in the lower semicontinuous case.

In this paper we restrict attention to a special class of state constraints sets, namely a finite intersection of smooth manifolds. (Nonetheless, this is a framework which allows state constraints sets with nonsmooth boundaries, and covers state constraints encountered in most engineering applications.) In this respect, it treats a narrower range of optimal control problems than [7], who investigate the value function for problems whose state constraints are formulated in terms of a general closed set. In other respects, however, the hypotheses are weaker and the results more precise.

However the principal point of interest is the nature of the analysis employed in this paper. A key element is the application of theorems asserting the existence of an F-trajectory satisfying a state constraint, close to an F-trajectory which violates the state constraint. We refer to theorems providing such information as EFNT (Existence of Feasible Neighbouring Trajectories) theorems. An EFNT theorem was first proved by Soner [11], with a view to establishing the continuity of the value function for infinite horizon optimal control problems with state constraints. They have also recently been used by Rampazzo and Vinter [10] in the derivation of so-called nondegenerate necessary conditions of optimality. These techniques were introduced by Soner to establish continuity properties of the value function. We use them for quite different purposes, namely to establish monotonicity properties of generalized solutions to (HJE), in circumstances when the value function can be discontinuous.

Finally some notation. $B(x,r)$ denotes the closed ball, with center x and radius r, in Euclidean space. $B(0,1)$ is written simply as B. $d_C(x)$ denotes the Euclidean distance function from a point x in Euclidean space to the closed set C.

Given a lower semicontinuous function $\phi : R^k \to R \cup \{\infty\}$, a point $x \in \text{dom}\,\phi$ and a vector $u \in R^k$, $\partial_-\phi(x)$ denotes the subdifferential

$$\partial_-\phi(x) = \{\zeta \in R^k : \liminf_{x'\to x} \tfrac{\phi(x')-\phi(x)-\zeta\cdot(x'-x)}{|x'-x|} \geq 0\}$$

and $D_\uparrow\phi(x;u)$ denotes the contingent epiderivative

$$D_\uparrow\phi(x;u) = \liminf_{h\downarrow 0,\, u'\to u} \frac{\phi(x+hu')-\phi(x)}{h}.$$

In situations where ϕ is defined only on a closed subset $D \subset R^k$ and $x \in \text{dom}\,\phi \cap D$, we give meaning to these constructs by extending ϕ to all of R^k: $\phi(x) = +\infty$ for $x \notin D$.

2 The Main Results

The following theorem provides two characterizations of the value function for optimal control problems with endpoint and state constraints, in terms of lower semicontinuous solutions of the Hamilton Jacobi Equation, one in terms of epiderivative solutions and the other akin to viscosity solutions.

It is assumed that the state constraint set A is expressible as

$$A \quad = \quad \cap_{j=1}^{r}\{x : h_j(x) \leq 0\}$$

for a finite family of $C^{1,1}$ functions $\{h_j : R^n \to R\}_{j=1}^r$. ($C^{1,1}$ denotes the class of C^1 functions with locally Lipschitz continuous gradients.) The "active set" of index values $I(x)$, at a point $x \in \text{bdy}\, A$, is

$$I(x) \quad := \quad \{j \in (1,\ldots,r) : h_j(x) = 0\}.$$

Recall the notations $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$ for all real numbers a, b. We write

$$h^+(x) := \left(\max_{j=1,2,\ldots,r} h_j(x)\right) \vee 0.$$

$W^{1,1}([a,b];R^n)$ denotes the space of absolutely continuous n-vector valued functions on $[a,b]$, with norm

$$||x||_{W^{1,1}} = |x(a)| + \int_a^b |\dot{x}(t)|dt.$$

Theorem 2.1 *Take a function $V : [0,1] \times A \to R \cup \{+\infty\}$. Assume that the following hypotheses are satisfied:*

(H1) F is a continuous multifunction, which takes values in the space of non-empty, closed, convex sets,

(H2) There exists $c > 0$ such that

$$F(t,x) \subset c(1+|x|)B \quad \forall\,(t,x) \in [0,1] \times R^n,$$

(H3) There exists $k_F \in L^1$ such that

$$\left\{ \begin{array}{l} F(t,x) \subset F(t,x')) + k_F(t)|x-x'|B \\ \forall\, t \in [0,1],\, x,x' \in R^n \times R^n, \end{array} \right.$$

(H4) g is lower semicontinuous.

Assume furthermore that

(CQ) For all $x \in A$ and $t \in [0,1]$ there exists $v \in F(t,x)$ such that

$$\forall j \in I(x), \quad \nabla h_j(x) \cdot v > 0.$$

Then assertions (a)-(c) below are equivalent:

(a) V is the value function for (P).

(b) V is lower semicontinuous and

(i) $\forall\,(t,x) \in ([0,1) \times A) \cap \text{dom}\, V$

$$\inf_{v \in F(t,x)} D_{\uparrow}V((t,x);(1,v)) \;\leq\; 0$$

(ii) $\forall\,(t,x) \in ((0,1] \times \text{int}\, A) \cap \text{dom}\, V$

$$\sup_{v \in F(t,x)} D_{\uparrow}V((t,x);(-1,-v)) \;\leq\; 0$$

(iii) $\forall\, x \in A$

$$\liminf_{\{(t',x')\to(1,x):t'<1,x'\in \text{int}\, A\}} V(t',x') = g(x).$$

(c) V is lower semicontinuous and

(i) $\forall\,(t,x) \in ((0,1) \times \text{int}\, A) \cap \text{dom}\, V$, $\forall\,(\eta^0,\eta^1) \in \partial_{-}V(t,x)$

$$\eta^0 + \inf_{v\in F(t,x)} \eta^1 \cdot v = 0.$$

(ii) $\forall\,(t,x) \in ((0,1) \times \text{bdy}\, A) \cap \text{dom}\, V$, $\forall\,(\eta^0,\eta^1) \in \partial_{-}V(t,x)$

$$\eta^0 + \inf_{v\in F(t,x)} \eta^1 \cdot v \leq 0$$

(iii) $\forall\, x \in A$,

$$\liminf_{\{(t',x')\to(0,x):t'>0\}} V(t',x') = V(0,x)$$

and

$$\liminf_{\{(t',x')\to(1,x):t'<1,x'\in \text{int}\, A\}} V(t',x') = V(1,x) = g(x).$$

This theorem is related to Thm.12 of [7], which treats a variant of (P), in which the dynamic constraint is modeled as a differential equation with control term and no restrictions are placed upon the nature of the closed set A (the state constraint set). In this paper, by contrast, A is required to be a finite intersection of smooth manifolds. The extra structure of A is exploited to sharpen the results of [7] in other respects.

A notable feature of Thm. 2.1, like Thm. 12 of [7], is that it provides a characterization of the value function as a unique solution of (HJE) in some situations where the value function is discontinuous. The following example illustrates this point.

Example 2.2 Consider

$$\left\{ \begin{array}{l} \text{Minimize } g(x(1)) \\ \dot{x}(t) \in F(t,x(t)) \\ x(t) \in A \\ x(0) = x_0, \end{array} \right.$$

in which $n = 1$, $g(x) = x$, $F(t,x) = \{1\}$, $A = \{x : x \leq 0\}$, $x_0 = 0$.

By inspection

$$V(t,x) = \left\{ \begin{array}{ll} +\infty & \text{if } x > -(1-t) \\ x + (1-t) & \text{if } x \leq -(1-t) \end{array} \right.$$

The hypotheses for application of Thm. 2.1 are satisfied, including the outward-pointing constraint qualification (CQ). Thm. 2.1 therefore tells us that V is the unique solution of (HJE) (in the sense specified).

Notice that $V(t,x) = +\infty$ at some points in $[0,1] \times A$, despite the fact that g is everywhere finite valued (no endpoint constraints).

3 A Neighbouring Feasible Trajectories Theorem

A key role in the proof of Thm. 2.1 is played by an estimate governing the distance of the set of F-trajectories satisfying a given state constraint from a given F-trajectory which violates the constraint. This estimate is provided by the following Existence of Feasible Neighbouring Trajectories (EFNT) Theorem, which can be regarded as a kind of refined viability theorem, in which the information that a 'viable' F-trajectory exists whenever viability condition holds true is supplemented by information about where it is located, in relation to a given F-trajectory when a 'strict' viability condition holds true.

As before, we limit attention to state constraint sets A associated with a family of functional inequalities:

$$A = \cap_{j=1}^{r} \{x : h_j(x) \leq 0\},$$

in which the h_j's are given $C^{1,1}$ functions.

Theorem 3.1 *Fix $r_0 > 0$. Assume that for some $c > 0$, $\alpha > 0$ and $k_F(.) \in L^1$, the following hypotheses are satisfied:*

(i) F takes values in the space of non-empty, closed sets and $F(.,x)$ is measurable for each $x \in R^n$.

(ii) $F(t,x) \subset c(1+|x|)B \quad \forall\,(t,x) \in [0,1] \times R^n$.

(iii) $F(t,x) \subset F(t,x') + k_F(t)|x-x'|B$ $\forall\, t \in [0,1],\ x, x' \in R^n$.

Assume furthermore that there exists some $\alpha > 0$ such that

(CQ)' $\quad \min_{v\in F(t,x)} \max_{j\in I(x)} \nabla h_j(x)\cdot v < -\alpha \quad ,$
$\forall\, x \in B(0, e^c(r_0+c)) \cap \operatorname{bdy} A,\ t \in [0,1]$.

Then there exists a constant K (which depends on r_0, c, α and $k_F \in L^1$) with the following property: given any $t_0 \in [0,1]$ and any F-trajectory $\hat{x}$ on $[t_0,1]$ such that $\hat{x}(t_0) \in B(0,r_0) \cap A$, an F-trajectory x can be found such that $x(t_0) = \hat{x}(t_0)$,

$$x(t) \in A \quad \forall\, t \in [t_0,1]$$

and

$$||x - \hat{x}||_{W^{1,1}([t_0,1];R^n)} \leq K \max_{t\in[t_0,1]} h^+(\hat{x}(t)).$$

Thm. 3.1, whose hypotheses do not require F to be continuous or convex valued and which supplies a tighter $W^{1,1}$ estimate in place of the customary L^∞ estimate, goes beyond what is strictly required for the Dynamic Programming applications of this paper. These features are included however, as a reference for other applications. Related results appear in [8].

The need to introduce into (CQ) $'$ the positive parameter α arised because it is not hypothesized in Thm. 3.1 that F is a continuous multifunction. In the case when F is continuous, then (CQ) $'$ is implied by the condition

$$\min_{v\in F(t,x)} \max_{j\in I(x)} \nabla h_j(x)\cdot v < 0$$
$$\forall\, x \in B(0, e^c(r_0+c)) \cap \operatorname{bdy} A,\ t \in [0,1].$$

4 Proof of Thm. 2.1

We isolate in the following lemma the steps in the proof of Thm. 2.1 requiring the constraint qualification (CQ). Clearly, the assertions of Thm. 2.1 remain valid if (CQ) is replaced by any alternative hypotheses for which the assertions of the lemma remain true.

Lemma 4.1 *(i) Take any point $x_1 \in A$. Then there exists $\delta \in (0,1)$ and an F-trajectory $y : [1-\delta, 1] \to R^n$ such that $y(1) = x_1$ and*

$$y(t) \in \operatorname{int} A \quad \forall\, t \in [1-\delta, 1).$$

(ii) Take any $t_0 \in [0,1)$ and any F-trajectory $x : [t_0,1] \to R^n$. Take also a sequence of points $\{(\tau_i,\xi_i)\}$ in $[t_0,1) \times \operatorname{int} A$ such that $(\tau_i,\xi_i) \to (1, x(1))$. Then there exists a sequence of F-trajectories $\{x_i : [t_0,\tau_i] \to R^n\}$ such that $x_i(\tau_i) = \xi_i$

$$x_i(t) \in \operatorname{int} A \quad \forall\, t \in [t_0,\tau_i], i = 1,2,\ldots$$

and

$$||x_i - x||_{L^\infty([t_0,\tau_i];R^n)} \to 0 \text{ as } i \to \infty.$$

The proof is similar to the one from [7] and is omitted.

In the next lemma, reference is made to the δ-tube about $\bar{x} : [t_0,t_1] \to R^n$:

$$T_\delta(\bar{x}) \ := \ \{(t,x) \in [t_0,t_1] \times R^n : |x - \bar{x}(t)| \leq \delta\}.$$

Lemma 4.2 *Take $[t_0,t_1) \subset [0,1]$ such that $t_0 < t_1$, an F-trajectory $\bar{x} : [t_0,t_1] \to R^n$, $\delta > 0$ and a lower semicontinuous function $V : [t_0,t_1] \times R^n \to R \cup \{+\infty\}$ such that*

$$\forall\,(t,x) \in T_\delta(\bar{x}), (\eta^0,\eta^1) \in \partial_- V(t,x)$$
$$\eta^0 + \inf_{v\in F(t,x)} \eta^1 \cdot v = 0. \qquad (4.1)$$

Then, for any subinterval $[t',t''] \subset [t_0,t_1)$,

$$V(t',\bar{x}(t')) \leq V(t'',\bar{x}(t'')).$$

Proof. Since V is lower semicontinuous, taking value nowhere $-\infty$, it is bounded below on compact sets. Let M be a number such that

$$V(t,x) > M \quad \forall\,(t,x) \in T_\delta(\bar{x}).$$

Now define

$$\bar{V}(t,x) \;=\; \begin{cases} V(t,x) & \text{for } (t,x) \in T_\delta(\bar{x}) \\ M & \text{otherwise.} \end{cases}$$

$\bar{V}$ is lower semicontinuous and, as can be easily checked, satisfies condition (4.1) for all $(t,x) \in [t_0,t_1] \times R^n$, (and not merely for all $(t,x) \in T_\delta(\bar{x})$). $\bar{V}$ satisfies the "global" hypotheses for application of results from [6]. We deduce from ([6], Thm. 3.3 and Lemmas 4.3 and 4.4) that

$$\bar{V}(t',\bar{x}(t')) \le \bar{V}(t'',\bar{x}(t'')).$$

The fact that $t'' < t_1$ (strict inequality) is important here, since no regularity hypotheses have been imposed on $t \to V(t,.)$ at $t = t_1$. The assertions of the lemma now follow from the fact that $\bar{V}$ and V coincide on $T_\delta(\bar{x})$.

Proof of Thm.2.1

(a) $\Rightarrow$ (b). Suppose that V is the value function. Standard closure and continuous dependence results concerning the solution sets of 'convex' differential inclusions ensure that V is lower semicontinuous.

Under the hypotheses, $(t,x) \in \operatorname{dom} V$ implies that $(\mathrm{P}_{t,x})$ has a solution. It is a straightforward matter to show that, if y is a minimizer for $(\mathrm{P}_{t,x})$, then $s \to V(s,y(s))$ is constant on $[t,1]$; b(i) can be deduced from this property.

It can also be shown that, if $y : [t,1] \to R^n$ is an F-trajectory satisfying the constraints of $(\mathrm{P}_{t,x})$, then $s \to V(s,y(s))$ is non-decreasing on $[t,1]$; b(ii) can be deduced from this latter property.

Since V is lower semicontinuous, it remains only to verify

$$\liminf_{\{(t',x')\to(1,x):t'<1,x'\in \operatorname{int} A\}} V(t',x') \le V(1,x) \quad \forall\, x \in A.$$

Lemma 4.1 tells us that there exists $\delta \in (0,1)$ and an F-trajectory $y : [1-\delta,1] \to R^n$ such that $y(1) = x$ and

$$y(t) \in \operatorname{int} A \quad \forall\, t \in [1-\delta,1).$$

But $V(t,y(t)) \le V(1,x)$, a basic monotonicity property of the value function. Since y is continuous,

$$\liminf_{\{(t',x')\to(1,x):t'<1,x'\in \operatorname{int} A\}} V(t',x') \le \limsup_{t\uparrow 1} V(t,y(t)) \le V(1,x).$$

as required.

(b) $\Rightarrow$ (c). This implication is a consequence of well-known duality relationships between $\partial_- V$ and $D_\uparrow V$. (See [3].)

(c) $\Rightarrow$ (a). Assume that V satisfies (c). Take any $x_0 \in A$ and $t_0 \in [0,1]$.

Step 1: We show that

$$V(t_0,x_0) \;\ge\; \inf(P_{t_0,x_0}). \tag{4.2}$$

This inequality is automatically satisfied if $V(t_0,x_0) = +\infty$. So we assume that $V(t_0,x_0) < +\infty$.

Notice that, since $\operatorname{dom} V \subset A$, conditions c(i) and c(ii) imply

$$\begin{array}{l} \forall (t,x) \in ((0,1) \times R^n) \cap \operatorname{dom} V, \\ \forall\,(\eta^0,\eta^1) \in \partial_- V(t,x) \\ \eta^0 + \inf_{v\in F(t,x)} \eta^1 \cdot v \le 0 \end{array}$$

and

$$\liminf_{\{(t',x')\to(0,x):t'>0\}} V(t',x') = V(0,x) \quad \forall\, x \in R^n,$$

(We here regard V as a function on $[0,1] \times R^n$ which takes value $+\infty$ at points $(t,x) \notin [0,1] \times A$.) But then known results concerning the state constraint-free problem ([6], Thm.3.3, Lemma 4.3 and Thm.5.1) permit us to deduce the existence of an F-trajectory $x : [t_0,1] \to R^n$ such that $x(t_0) = x_0$ and

$$V(t_0,x_0) \ge V(t,x(t)) \quad \forall\, t \in [t_0,1].$$

This inequality implies that $V(t,x(t)) < +\infty$ for all $t \in [t_0,1]$. Since $\operatorname{dom} V \subset A$, we conclude that $x(.)$ satisfies the state constraint. It also implies that

$$V(t_0,x_0) \ge V(1,x(1)) = g(x(1)) \ge \inf(\mathrm{P}_{t,x}).$$

This is the required inequality.

Step 2: We show that

$$V(t_0,x_0) \;\le\; \inf(P_{t_0,x_0}). \tag{4.3}$$

This will complete the proof, since (4.3) combines with (4.2) to give $V(t_0,x_0) = \inf(P_{t_0,x_0})$.

(4.3) is automatically satisfied if $\inf(P_{t_0,x_0}) = +\infty$. So we assume that it is finite. In this case, $\inf(P_{t_0,x_0})$ is the infimum of $g(x(1))$ over all arcs satisfying the constraints of (P_{t_0,x_0}). It therefore suffices to show that

$$V(t_0,x_0) \le g(\bar{x}(1)),$$

where $\bar{x} \in W^{1,1}([t_0,1];R^n)$ is an arbitrary arc satisfying the constraints of problem (P_{t_0,x_0}).

By hypothesis,

$$g(\bar{x}(1)) = \liminf_{\{(\tau,\xi)\to(1,\bar{x}(1)):\tau<1,\xi\in \text{int}\, A\}} V(\tau,\xi).$$

There exists, therefore, a sequence $\{(\tau_i,\xi_i)\}$ in $[t_0,1)\times$ int A such that $\xi_i \to \bar{x}(1)$ and

$$V(\tau_i,\xi_i) \to g(\bar{x}(1)). \tag{4.4}$$

Lemma 4.1(ii) asserts the existence of a sequence of F-trajectories $x_i : [t_0,\tau_i] \to R^n$ such that $x_i(\tau_i) = \xi_i$,

$$x_i(t) \in \text{int}\, A \quad \forall\, t \in [t_0,\tau_i]$$

and

$$||x_i - \bar{x}||_{L^\infty([t_0,\tau_i];R^n)} \to 0 \quad \text{as } i \to \infty. \tag{4.5}$$

Filippov's Existence Theorem tells us that x_i can be extended to all of $[t_0,1]$ (we write the extension also x_i) as an F-trajectory. Choose $\sigma_i \in (\tau_i,1)$ and $\epsilon_i > 0$ such that

$$x_i(t) + \epsilon_i B \subset \text{int}\, A \quad \forall\, t \in [t_0,\sigma_i].$$

Now apply Lemma 4.2 with $\sigma_i = t_1$ and $\bar{x} = x_i$. Since $t_0 \le \tau_i < \sigma_i$, we conclude that

$$V(t_0,x_i(t_0)) \le V(\tau_i,\xi_i).$$

It follows from (4.4), (4.5) and the lower semicontinuity of V that

$$V(t_0,x_0) = V(t_0,\bar{x}(t_0)) \le \liminf_i\, V(t_0,x_i(t_0)) \le \lim_i V(\tau_i,\xi_i) = g(\bar{x}(1))$$

as required.

References

[1] J.-P. Aubin, "Viability Theory", *Birkhäuser*, Boston, Basel, Berlin, 1991.

[2] J.-P. Aubin and A. Cellina, "Differential Inclusions", *Springer Verlag*, Berlin, 1984.

[3] J.-P. Aubin and H. Frankowska, "Set-Valued Analysis", *Birkhäuser*, Boston, Basel, Berlin, 1990.

[4] E. N. Barron and R. Jensen, "Semicontinuous Viscosity Solutions for Hamilton-Jacobi Equations with Convex Hamiltonian", *Comm. Partial Diff. Equations*, 15 (1990), pp. 1713-1742.

[5] I. Capuzzo-dolcetta and P.-L. Lions, "Hamilton-Jacobi Equations with State Constraints", *Trans. Am. Math. Soc.*, 318 (1990), pp.643-685.

[6] H. Frankowska, "Semicontinuous Solutions of the Hamilton Jacobi Equation", SIAM J. Control and Optimization, 31 (1993), pp.257-272 (see also Proceedings of 30th CDC Conference, IEEE, Brighton, December 11-13, 1991).

[7] H. Frankowska and S. Plaskacz, "Semicontinuous Solutions of Hamilton Jacobi Equations with State Constraints", *Differential Inclusions and Optimal Control, Lecture Notes in Nonlinear Analysis*, Vol. 2, J. Schauder Center for Nonlinear Studies, Ed. J. Andres, L. Gorniewicz and P. Nistri, (1998), pp. 145-161.

[8] H. Frankowska and F. Rampazzo (2000), "Filippov's and Filippov-Ważewski's Theorems on Closed Domains", J. Diff. Eqs., 161, 449-478

[9] H. Ishii and S. Koike, "A New Formulation of State Constraints Problems for First Order PDE's", *SIAM J. Control and Optim.*, 34 (1996), pp. 554-571.

[10] F. Rampazzo and R. B. Vinter,"Degenerate Optimal Control Problems with State Constraints", *submitted.*

[11] H.M. Soner, "Optimal Control with State-Space Constraints", *SIAM J. Control and Optimization* **24** (1986), pp. 552-561.