Cascade Normal Forms for Underactuated Mechanical Systems

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Abstract

In this paper, we introduce cascade normal forms for underactuated mechanical systems that are convenient for control design. These normal forms are partially linear which results from a well-known fact that underactuated systems can be partially linearized using a change of control [12]. The difficulty arises when the new control appears both in the linear and nonlinear subsystems. We introduce a method for decoupling these two subsystems by applying a change of coordinates that transforms the dynamics of the system into a cascade normal form with the property that control of the overall system reduces to control of its nonlinear subsystem. Under a symmetry condition on the inertia matrix of the system, this transformation can be obtained explicitly from the Lagrangian. This eventually leads to classification of underactuated systems. We provide several applications and two detailed examples of complex underactuated systems, namely, the Acrobot and the Rotating Pendulum.

1 Introduction

Recently, there has been extensive interest among the researchers in control of underactuated mechanical systems due to their broad applications and open nature of theoretical problems they have to offer (see [9], [11] for a survey). Many real-life control mechanical systems including aircrafts, helicopters, spacecrafts, underwater vehicles, surface vessels, mobile robots, walking robots, and flexible systems are examples of underactuated systems. Formally, underactuated mechanical systems are systems that have fewer actuators than configuration variables. This restriction of the control authority makes the control design for these systems rather complicated. Most of underactuated systems are not fully feedback linearizable. An important step towards a systematic analysis and control design for underactuated systems is to find normal forms for them that are appropriate for control design. In this paper, we focus on transformation of underactuated systems to normal forms consisting of cascade of a linear and a nonlinear subsystem, so that for broad classes, control of the original higher-order system reduces to control of its lower-order nonlinear subsystem. The justification for the linear part of this cascade normal form comes from the fact that all underactuated systems can be partially linearized using a change of control [12]. However, after applying this change of control, the new control appears both in the linear and nonlinear subsystems. This is one of the main sources of complexity of control design for underactuated systems. The contributions of this work can be summarized as follows. First, we introduce an appropriate structure for a global change of coordinates that decouples these two linear and nonlinear subsystems w.r.t. the new control and give conditions for the existence of this diffeomorphism. Secondly, under symmetry conditions on the inertia matrix of the underactuated system (which hold for broad classes including man-made systems), we obtain this transformation in closed-form from the Lagrangian of the system for both low-order and high-order underactuated systems. Thirdly, after transformation, we provide classification of the obtained normal forms and thus the original underactuated systems. The corresponding control design method for each class is also given. This classification leads to very limited number of obtained classes. These classes are cascade systems in strict feedback form [4], strict feedforward form [14], and a quadratic nontriangular form (to be defined later) [5]. We provide several examples for each class of underactuated systems and give two detailed examples: the Acrobot and the Rotating Pendulum.

Here is an outline of the paper. In section 2, a standard model for underactuated systems is given. The main normal form is presented in section 3. Results under symmetry conditions are given in section 4. In section 5, examples are provided and finally concluding remarks are made.

2 Dynamics of Underactuated Systems

In this paper, we consider underactuated (simple) Lagrangian systems with configuration vector $q \in Q$ and

Lagrangian

$$\mathcal{L}(q,\dot{q}) = K - V = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$$

where K is the kinetic energy, $V(q)$ is the potential energy, and $M(q)$ is the inertia matrix of the mechanical system. The configuration vector for an underactuated system can be decomposed as $q = (q_1, q_2)$ where q_1 and q_2 denote n unactuated and m actuated configuration variables, respectively ($d = n+m$). The Euler-Lagrange equations of motion for this system is as follows

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} &= \tau_1 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} &= \tau_2 \end{aligned} \qquad (1)$$

where $\tau_1 \equiv 0$ (because q_1 is unactuated). Equation (1), can be rewritten as

$$\begin{aligned} m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q,\dot{q}) &= 0 \\ m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q,\dot{q}) &= \tau_2 \end{aligned} \qquad (2)$$

where h_i's contain Coriolis, centrifugal, and gravity terms. In [12], a globally invertible change of control input in the following form is given

$$\tau_2 = \alpha(q)u + \beta(q,\dot{q})$$

that partially linearizes (2) as

$$\begin{aligned} \ddot{q}_1 &= -m_{11}^{-1}(q)h_1(q,\dot{q}) - m_{11}^{-1}(q)m_{12}(q)u \\ \ddot{q}_2 &= u \end{aligned} \qquad (3)$$

Denoting $p = \dot{q}$ the last equation can be expressed as

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= f_0(q,p) + g_0(q)u \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \qquad (4)$$

In (4), the control input u appears in both (q_1, p_1)-subsystem and (q_2, p_2)-subsystem. This highly complicates control design for underactuated systems. Note that defining $x = (q_1, p_1, q_2, p_2)$, (4) can be expressed as

$$\dot{x} = f(x) + g(x)u$$

with obvious definitions of f, g. Using explicit expressions of f and g, accessibility and control of a special class of underactuated mechanical systems in the form (4) is addressed in [9]. In this paper, our approach is to eliminate u from the dynamics of (q_1, p_1)-subsystem using a change of coordinates.

3 Normal Forms

Our first main result is the following theorem which provides conditions for existence and form of a global change of coordinates that decouples the two subsystems of the underactuated system in (4) w.r.t u and transforms the original system into a cascade form.

Theorem 3.1. *Consider an underactuated mechanical system with an inertia matrix $M(q) = \{m_{ij}(q)\}; i,j = 1,2$ where $q = (q_1, q_2)$ and $q_1 = \{q_1^i\} \in \mathbb{R}^n$ and $q_2 = \{q_2^j\} \in \mathbb{R}^m$ denote the unactuated and actuated configuration variables, respectively. Denote*

$$g_0(q) = -m_{11}^{-1}(q)m_{12}(q)$$

and

$$g(q) = \begin{bmatrix} g_0(q) \\ I_{m\times m} \end{bmatrix}$$

where $g_0(q) = (g_0^1(q), \ldots, g_0^m(q))$ with $g_0^j(q) \in \mathbb{R}^n$, $j = 1, \ldots, m$ and $I_{m\times m} = (e_1, \ldots, e_n)$ is the identity matrix. Define the following distribution

$$\Delta(q) = span\{columns\ of\ g(q)\}$$

that has full column rank and thus is globally nonsingular. Then, a necessary and sufficient condition for the distribution $\Delta(q)$ to be globally involutive (i.e. completely integrable) is that

$$\frac{\partial g_0^j(q)}{\partial q_1} g_0^i(q) - \frac{\partial g_0^i(q)}{\partial q_1} g_0^j(q) + \frac{\partial g_0^j(q)}{\partial q_2^i} - \frac{\partial g_0^i(q)}{\partial q_2^j} = 0 \qquad (5)$$

for all $i,j = 1, \ldots, m$. In addition, if condition (5) holds, there exists a global change of coordinates given by

$$\begin{aligned} z_1 &= \Phi(q_1, q_2) \\ z_2 &= \nabla_{q_1}\Phi \cdot p_1 + \nabla_{q_2}\Phi \cdot p_2 \\ \xi_1 &= q_2 \\ \xi_2 &= \dot{q}_2 \end{aligned}$$

that transforms the dynamics of the system into the normal form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \qquad (6)$$

Remark 3.1. Normal form (6) is a special case of the famous Byrnes-Isidori normal form [2] with a double integrator as the following

$$\begin{aligned} \dot{z} &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \qquad (7)$$

Remark 3.2. Clearly, the main advantage of the normal form (6) is that the control input of the actuated subsystem of the original system does not appear in the unactuated subsystem. This simplifies control design for underactuated systems. In fact, control design for special classes of systems in normal forms (6) and (7) has been recently addressed by the author in [5].

Proof. Note that $\Delta(q)$ globally has a full column rank of m and is therefore a globally nonsingular distribution (see [2] for definitions and notations in this proof). Calculating the Lie bracket of the ith and jth columns of $g(q)$, we get

$$\begin{aligned}[g^i(q), g^j(q)] &= \begin{bmatrix} \frac{\partial g_0^j(q)}{\partial q_1} & \frac{\partial g_0^j(q)}{\partial q_2} \\ \frac{\partial e_j}{\partial q_1} & \frac{\partial e_j}{\partial q_2} \end{bmatrix} \cdot \begin{bmatrix} g_0^i(q) \\ e_i \end{bmatrix} \\ &- \begin{bmatrix} \frac{\partial g_0^i(q)}{\partial q_1} & \frac{\partial g_0^i(q)}{\partial q_2} \\ \frac{\partial e_i}{\partial q_1} & \frac{\partial e_i}{\partial q_2} \end{bmatrix} \cdot \begin{bmatrix} g_0^j(q) \\ e_j \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} \frac{\partial g_0^j(q)}{\partial q_1} g_0^i(q) - \frac{\partial g_0^i(q)}{\partial q_1} g_0^j(q) + \frac{\partial g_0^j(q)}{\partial q_2} e_i - \frac{\partial g_0^i(q)}{\partial q_2} e_j \\ 0_{m\times 1} \end{bmatrix}$$

and condition (5) implies $[g^i(q), g^j(q)] = 0$ for all i, j, q. Therefore, $\Delta(q)$ is globally involutive. To prove the converse, assume $\Delta(q)$ is globally involutive. Then, for all i, j, q, $[g^i(q), g^j(q)]$ can be expressed as a linear combination of $g^k(q)$'s. But the lower $m \times 1$ block of $[g^i(q), g^j(q)]$ is identically zero and linearly independent of e_k's. This means that $[g^i(q), g^j(q)] = 0$ and thus condition (5) holds. Now, we prove the rest of the theorem. Based on Frobenius theorem (see [2]), because $\Delta(q)$ is a globally nonsingular and involutive distribution, the following equation

$$\frac{\partial \phi}{\partial q} g(q) = 0$$

has $n = (d - m)$ linearly independent solutions $\phi_k(q_1, q_2), k = 1, \ldots, n$. Denoting $\Phi(q_1, q_2) = (\phi_1, \ldots, \phi_n)$, Φ satisfies the following property

$$\frac{\partial \Phi}{\partial q_1} g_0(q) + \frac{\partial \Phi}{\partial q_2} = 0.$$

After applying the change of coordinates

$$z_1 = \Phi(q_1, q_2),\ z_2 = \dot{z}_1$$

we get

$$z_1 = \Phi(q_1, q_2),\ z_2 = \frac{\partial \Phi}{\partial q_1} p_1 + \frac{\partial \Phi}{\partial q_2} p_2$$

and because $\frac{\partial \Phi}{\partial q_1}$ is globally nonsingular (due to the proof of Frobenius theorem), based on implicit mapping theorem, there exists a smooth function Ψ such that

$$\begin{aligned} q_1 &= \Psi(z_1, \xi_1) \\ p_1 &= \left(\frac{\partial \Phi}{\partial q_1}|_{q=(\Psi(z_1,\xi_1),\xi_1)}\right)^{-1} (z_2 - \xi_2 \cdot \frac{\partial \Phi}{\partial q_2}|_{q=(\Psi(z_1,\xi_1),\xi_1)}) \end{aligned}$$

(we drop the substitution $q = (\Psi(z_1, \xi_1), \xi_1)$ in $\partial\Phi/\partial q_i$ due to the simplicity of notation). Calculating $\dot{z}_2$ as the following

$$\begin{aligned} \dot{z}_2 &= \frac{\partial \Phi}{\partial q_1} f_0(q,p) + \frac{\partial^2 \Phi}{\partial q_1^2} (\frac{\partial \Phi}{\partial q_1})^{-2} (z_2 - \frac{\partial \Phi}{\partial q_1} \xi_2)^2 \\ &+ \frac{\partial^2 \Phi}{\partial q_2^2} \xi_2^2 + (\frac{\partial \Phi}{\partial q_1} g_0(q)) + \frac{\partial \Phi}{\partial q_2}) u \end{aligned}$$

and noting that the coefficient of u in the last equation is identically zero, we get

$$\dot{z}_2 = f(z, \xi_1, \xi_2)$$

Therefore, the dynamics of the system in new coordinates is in normal form (6). □

The following corollary is a result of condition (5).

Corollary 3.1. *All underactuated mechanical systems with a single actuator that are globally partially linearizable can be globally transformed into normal form (6) using a change of coordinates.*

Proof. In this special case, $m = 1$ and $i = j = 1$. Thus, condition (5) globally holds and the result follows. □

4 Underactuated Systems with Symmetry

In this section, we demonstrate that for rather broad classes of underactuated systems, the decoupling change of coordinates in theorem 3.1 can be found in analytically explicit form. It turns out that symmetry of the kinetic energy K of a mechanical system w.r.t. a subset of configuration variables plays a crucial role in obtaining a decoupling change of coordinates and determining the special structure of the corresponding normal form of the original underactuated system [6]. By *symmetry*, we mean that K is independent of a subset of configuration variables called *external variables.* In contrast, we call the configuration variables that appear in the inertia matrix M *shape variables.* To present our next result, we need to make the following definition.

Definition 4.1. We say a system is in *strict feedback form* [4], if it has the following triangular structure

$$\begin{aligned} \dot{x} &= f(x, \xi_1), \\ \dot{\xi}_1 &= \xi_2, \\ &\cdots \\ \dot{\xi}_m &= v, \end{aligned}$$

The following theorem is generalization of the main result in [6] for higher-order underactuated systems.

Theorem 4.1. *Let $q_1 \in \mathbb{R}^n$ and $q_2 \in \mathbb{R}^m$ denote the unactuated and actuated configuration variables of an* *underactuated system and assume $M = M(q_2)$ (i.e. q_2 is the shape variable). Suppose the following one-form*

$$\omega = m_{11}^{-1}(q_2)m_{12}(q_2)dq_2 \tag{8}$$

is exact and let $\omega = d\gamma(q_2)$. Then, the following global change of coordinates obtained from the Lagrangian

$$\begin{array}{rcl} y_1 & = & q_1 + \gamma(q_2) \\ y_2 & = & m_{11}(q_2)p_1 + m_{12}(q_2)p_2 = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_1} \\ \xi_1 & = & q_2 \\ \xi_2 & = & p_2 \end{array} \tag{9}$$

transforms the dynamics of the system into a cascade nonlinear system in strict feedback form as

$$\begin{array}{rcl} \dot{y}_1 & = & m_{11}^{-1}(\xi_1)y_2 \\ \dot{y}_2 & = & g_1(y_1 - \gamma(\xi_1), \xi_1) \\ \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & u \end{array} \tag{10}$$

where $g_1(q_1, q_2) = -\partial V(q)/\partial q_1$.

Proof. By definition of $\gamma(q_2)$, the first line of (10) trivially holds. Due to the symmetry of the kinetic energy w.r.t. q_1, $\partial K/\partial q_1 = 0$, thus

$$\frac{\partial \mathcal{L}}{\partial q_1} = -\frac{\partial V(q)}{\partial q_1} =: g_1(q_1, q_2)$$

and from the first line of the Euler-Lagrange equation in (1) it follows that

$$\dot{y}_2 = g_1(q_1, q_2) = g_1(y_1 - \gamma(\xi_1), \xi_1)$$

□

Remark 4.1. The main benefit of the cascade normal form in (10) is that ξ_1 plays the role of the control input for the (y_1, y_2)-subsystem which has a lower dimension than the composite system. If a globally stabilizing smooth state feedback exists for the y-subsystem, then using standard backstepping procedure [4], a globally stabilizing state feedback can be obtained for the composite system in (10). This is the main reduction property of the global change of coordinates in (9).

Remark 4.2. It can be seen that the following additional change of coordinates

$$z_1 = y_1, \; z_2 = m_{11}^{-1} y_2$$

transforms the dynamics of the system into cascade normal form in (6).

Remark 4.3. For the special case of underactuated mechanical systems with two degrees of freedom and a single actuated shape variable, $\gamma(q_2)$ is explicitly given by

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{12}(\theta)}{m_{11}(\theta)} d\theta$$

The Acrobot [6], the Inertia Wheel Pendulum [13], the TORA example [3], and the VTOL aircraft [7] are all examples of underactuated systems with actuated shape variables and theorem 4.1 applies to them. On the other hand, the cart-pole system, the rotating pendulum, the pendubot, the beam-and-ball system, and flexible-link robot arms [8] all have unactuated shape variables. This motivated us to present the following result which applies to all the underactuated system in the second group with unactuated shape variables.

Theorem 4.2. *Let q_1 and q_2 denote the actuated and unactuated configuration variables of an underactuated mechanical system with two degrees and assume $M = M(q_2)$. Then, the following change of coordinates*

$$\begin{array}{rcl} y_1 & = & q_1 + \gamma(q_2) \\ y_2 & = & m_{21}(q_2)p_1 + m_{22}(q_2)p_2 = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_2} \\ \xi_1 & = & q_2 \\ \xi_2 & = & p_2 \end{array} \tag{11}$$

over the set $U_{q_2} = \{q_2 | m_{21}(q_2) \neq 0\}$, where

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{22}(\theta)}{m_{21}(\theta)} d\theta$$

transforms the dynamics of the system into the following cascade normal form

$$\begin{array}{rcl} \dot{y}_1 & = & m_{21}^{-1}(\xi_1)y_2 \\ \dot{y}_2 & = & g_2(y_1 - \gamma(\xi_1), \xi_1) + \Sigma(\xi_1, y_2, \xi_2) \\ \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & u \end{array} \tag{12}$$

where $g_2(q_1, q_2) = -\partial V(q)/\partial q_2$ and Σ is a quadratic form in (y_2, z_2) as

$$\Sigma(\xi_1, y_2, \xi_2) = (y_2, z_2)\sigma(\xi_1)(y_2, z_2)^T$$

Proof. The proof is by direct calculation. Clearly, $\dot{y}_1 = y_2/m_{21}(q_2)$. From the second line of (1), it follows that

$$\dot{y}_2 = \frac{1}{2}\frac{dm_{11}}{dq_2}p_1^2 + \frac{dm_{21}}{dq_2}p_1p_2 + \frac{1}{2}\frac{dm_{22}}{dq_2}p_2^2 + g_2(q_1, q_2)$$

or

$$\begin{array}{rcl} \dot{y}_1 & = & y_2/m_{21}(q_2) \\ \dot{y}_2 & = & g_2(q_1, q_2) + \dfrac{1}{2m_{21}^2}\dfrac{dm_{11}}{dq_2}y_2^2 \\ & + & \left\{ \dfrac{1}{m_{21}}\dfrac{dm_{21}}{dq_2} - \dfrac{m_{22}}{m_{21}^2}\dfrac{dm_{11}}{dq_2} \right\} y_2 p_2 \\ & + & \left\{ \dfrac{m_{22}^2}{2m_{21}^2}\dfrac{dm_{11}}{dq_2} - \dfrac{m_{22}}{m_{21}}\dfrac{dm_{21}}{dq_2} + \dfrac{1}{2}\dfrac{dm_{22}}{dq_2} \right\} p_2^2 \\ \dot{q}_2 & = & p_2 \\ \dot{p}_2 & = & u \end{array}$$

that is explicit expression of normal form (12). □

We call (12) a *quadratic nontriangular normal form.*

Remark 4.4. Due to the limitation of space, the case of theorem 4.2 for underactuated systems with more than two degrees of freedom is not treated here but is available (e.g. [8]).

Proposition 4.1. *Assume all the conditions in theorem 4.2 hold. In addition, suppose i)* $g_2(q_1, q_2)$ *is independent of* q_1*, i.e.* $D_{q_1} D_{q_2} V(q) = 0$*, ii)* m_{11} *is constant, and iii)* $\psi(q_2) = g_2(q_2)/m_{21}(q_2)$ *satisfies* $\psi'(0) \neq 0$*. Then, applying the change of coordiantes*

$$z_1 = y_1,\ z_2 = y_2/m_{21}(q_2)$$

transforms the system in (12) into a cascade nonlinear system in feedforward form. Moreover, the origin for this feedforward system can be globally asymptotically stabilized using nested saturations (see [14] for the definitions).

Proof. Based on the proof of theorem 4.2, we have

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \psi(q_2) + \left\{ \frac{1}{2m_{21}} \frac{dm_{22}}{dq_2} - \frac{m_{22}}{m_{21}^2} \frac{dm_{21}}{dq_2} \right\} p_2^2 \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned}$$

which is in feedforward form. The stabilization using nested saturations follows from condition *iii*) and [14]. □

The rotating pendulum satisfies all the conditions of corollary 4.1 except for condition *ii*) because $m_{11}(q_2)$ is not constant. Therefore, the normal form (12) associated with rotating pendulum is neither in strict feedback form, nor in feedforward form.

5 Examples

In this section, we give two examples of complex underactuated mechanical systems that are of high interest in the current literature. Namely, the Acrobot and the Rotating Pendulum. For examples of higher-order underactuated systems, we refer the reader to [7], [8].

Example 5.1. (Acrobot) Consider the *Acrobot* [10] that is a two-link planar robot with revolute joints and one actuator at the elbow as shown in Figure 1. The inertia matrix for the Acrobot is given by

$$\begin{aligned} m_{11} &= m_1 l_1^2 + m_2(L_1^2 + l_2^2 + 2L_1 l_2 \cos(q_2)) + I_1 + I_2 \\ m_{12} &= m_{21}(q_2) = m_2(l_2^2 + L_1 l_2 \cos(q_2)) + I_2 \\ m_{22} &= m_2 l_2^2 + I_2 \end{aligned}$$

where q_i, m_i, L_i, l_i, and I_i denote angles, masses, lengths, lengths of center of masses, and inertia, respectively. The potential energy for the Acrobot is

$$V(q_1, q_2) = (m_1 l_1 + m_2 L_1) g \sin(q_1) + m_2 l_2 g \sin(q_1 + q_2)$$

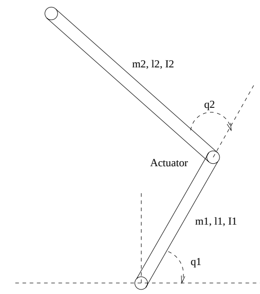

Figure 1: The Acrobot

Clearly, the inertia matrix of the Acrobot only depends on q_2. Thus, q_2 is an actuated shape variable for the Acrobot and based on theorem 4.1 using the following change of coordinates

$$\begin{aligned} z_1 &= q_1 + \gamma(q_2) \\ z_2 &= m_{11}(q_2)p_1 + m_{12}(q_2)p_2 \end{aligned}$$

where

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{12}(\theta)}{m_{11}(\theta)} d\theta =$$

the dynamics of the Acrobot can be transformed into a strict-feedback system as the following

$$\begin{aligned} \dot{z}_1 &= z_2/m_{11}(q_2) \\ \dot{z}_2 &= -(m_1 l_1 + m_2 L_1) g \cos(z_1 - \gamma(q_2)) \\ &- m_2 l_2 g \cos(z_1 - \gamma(q_2) + q_2) \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \tag{13}$$

q_2 plays the role of the control input for the z-subsystem. Due to the fact that q_2 appears in a highly nonlinear way in the dynamics of the z-subsystem control design for the Acrobot is challenging. In [6], it is shown that the Acrobot can be globally asymptotically stabilized around its upright equilibrium point using a state feedback and related simulation results are provided.

Example 5.2. (Rotating Pendulum) Consider the *Rotating Pendulum* introduced in [1] (see Figure 2). The elements of the inertia matrix for the Rotating Pendulum are given by

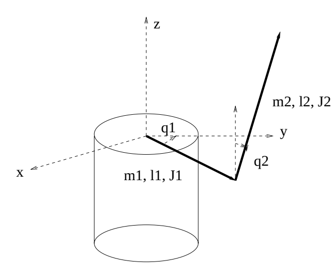

Figure 2: The Rotating Pendulum

$$\begin{aligned} m_{11} &= I_1 + m_1 l_1^2 + m_2(L_1^2 + l_2^2 \sin^2(q_2)) \\ m_{12} &= m_{21} = m_2 L_1 l_2 \cos(q_2) \\ m_{22} &= I_2 + m_2 l_2^2 \end{aligned}$$

The potential energy for this system is

$$V(q_1, q_2) = m_2 g l_2 \cos(q_2)$$

and thus

$$g_2(q_1, q_2) = -m_2 g l_2 \sin(q_2).$$

The inertia matrix of the Rotating Pendulum only depends on q_2. Since q_2 is an unactuated shape variable, based on theorem 4.2, the dynamics of the Rotating Pendulum can be transformed into the quadratic non-triangular form in (12) after applying the change of coordinates

$$y_1 = q_1 + \gamma(q_2),\ y_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2}$$

where $\gamma(q_2)$ is defined over $U_{q_2} = (-\pi/2, \pi/2)$. Using a second change of coordinates as

$$z_1 = y_1,\ z_2 = y_2/m_{21}(q_2)$$

the overall dynamics of the Rotating Pendulum can be transformed into normal form (6) as the following

$$\begin{array}{rcl} \dot{z}_1 &=& z_2 \\ \dot{z}_2 &=& \dfrac{g}{L_1}\tan(q_2) + \dfrac{l_2}{L_1}\sin(q_2)(z_2 - \dfrac{m_{22}}{m_{21}(q_2)}p_2)^2 \\ &+& \dfrac{m_{22}}{m_{21}(q_2)}\tan(q_2)p_2^2 \\ \dot{q}_2 &=& p_2 \\ \dot{p}_2 &=& u \end{array} \tag{14}$$

See [5] for further details on stabilization of the Rotating Pendulum to its upright equilibrium point and related simulation results.

6 Conclusions

We introduced new cascade normal forms for underactuated mechanical systems. These normal forms can be obtained under certain conditions on integrability of a distribution depending on the inertia matrix of the mechanical system. Under further symmetry conditions that hold for broad classes of underactuated system, the change of coordinates that transforms the original system into its normal form can be obtained in explicit form. This provided a way for classification of underactuated systems based on their associated normal forms. The main benefit of this transformation is to reduce control of the overall system to control of a lower-order nonlinear subsystem of the obtained normal form. We gave several examples for underactuated in each class and provided two detailed examples: the Acrobot and the Rotating Pendulum.

7 Acknowledgment

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