Trajectory Tracking for a Flexible One-Link Robot using a Nonlinear Noncollocated Output

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Abstract

In this paper, we introduce a method that provides a nonlinear noncollocated output for trajectory tracking of a flexible one-link robot. The link is modeled as a finite-order Lagrangian system obtained from truncated modal analysis. This noncollocated output is derived based on viewing a flexible link robot arm as an underactuated mechanical system and then applying an appropriate change of coordinates that transforms the system into a cascade nonlinear system with a minimum-phase zero dynamics. The output is the angle of rotation augmented with the (generalized) saturation of the weighted amplitudes of the deflection modes of the flexible link.

1 Introduction

Flexible-link robots are finding their way in industrial and space robotics applications due to their lighter weight and faster response time compared to rigid robots. Control of flexible robots has been studied extensively for more than a decade by several researchers [1], [2], [5], [10], [14], [13] (also, see [4] for a recent survey). Despite their applications, control of flexible link robots has proven to be rather complicated. The simplest possible flexible arm is a robot with a single flexible link. The dynamic model for a flexible link robot that has been used by almost all of the researchers is the Euler-Bernoulli model of a beam. This is a fourth order equation that leads to an originally infinite-dimensional model of a flexible link. The common collocated output for trajectory tracking of flexible one-link robots is the angle of the rotation of the link at the base. The performance of this output measurement turns out to be not satisfactory due to a weak control of the vibrations of the link [3]. This initiated finding other noncollocated output measurements like the position of the end-point of the link [2]. This leads to a nonminimum phase zero dynamics due to [11]. In [3], a noncollocated output is proposed as a linear combination of the angle of rotation and the slope of the beam at its tip. The model used in [3] is an infinite dimensional linear model of a flexible link robot. Then, an approximate finite order compensator is designed based on [13]. Here, we take a fundamentally different approach to find a noncollocated output. We use a finite-order state-space model of a flexible link derived by truncated modal analysis based on a Lagrangian formalism due to De Luca and Siciliano [5]. This model is naturally in the form of an underactuated mechanical system with $(m+1)$ degrees of freedom and a single actuator where m is the number of deformation modes of the flexible link. Based on our previous results on normal forms for underactuated mechanical systems [8], [7], we propose a noncollocated output as the angle of the rotation augmented with a generalized saturated weighted linear combination of the deformation amplitudes of the link. We prove that the zero dynamics corresponding to this output is minimum-phase. Our main contribution is to derive an output based on a unified approach that has been successful in control design for several examples of underactuated mechanical systems and prove that it leads to a minimum phase zero dynamics.

Here is an outline of the paper. First, we give the model of a flexible link in section 2. Next, we provide some background on normal forms for underactuated systems. Our main result is presented in section 4. The tracking control is given in 5. Finally, concluding remarks are presented.

2 Model of a Flexible Link

The flexible link depicted in Figure 1 is modeled as an Euler-Bernoulli beam [5] satisfying

$$EI\frac{\partial^4 w(\zeta,t)}{\partial \zeta^4} + \rho A L^4 \frac{\partial^2 w(\zeta,t)}{\partial t^2} = 0$$

where $\zeta = x/L$ is the normalized position along the link of length L and A, I, E, ρ are physical parameters of the link. By the assumption of separability in time and space, the deformation of the beam can be expressed as $W(\zeta,t) = \phi(\zeta)\delta(t)$. After performing a truncated modal analysis with m modes, the Lagrangian equations of the flexible link can be written as the following [5]

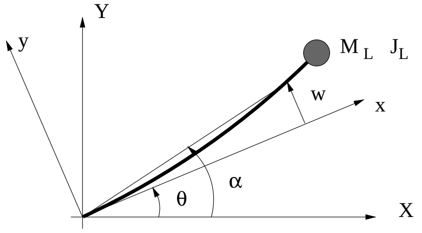

Figure 1: A flexible one-link robot arm.

$$\begin{bmatrix} m_{11}(\delta) & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot\theta \\ \ddot\delta \end{bmatrix} + \begin{bmatrix} c_1(\dot\theta,\delta,\dot\delta) \\ c_2(\dot\theta,\delta) \end{bmatrix} + \begin{bmatrix} 0 \\ K\delta + F\dot\delta \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \quad (1)$$

where θ is the angle of rotation, $\delta \in \mathbb{R}^m$ denotes the deformation amplitudes, $M(\delta) = \{m_{ij}\}$ is the inertia matrix that is a positive definite symmetric matrix and K, F are positive definite matrices as well. The expressions for m_{ij}'s are explicitly given in [5]. Here, we only need an explicit form of $m_{11}(\delta)$ (the rest of m_{ij}'s are constant) as

$$m_{11}(\delta) = k_1 + k_2(\Phi_e^T\delta)^2 \quad (2)$$

where $k_1 > 0$, $k_2 = M_L$ i.e. payload mass, $\Phi_e \in \mathbb{R}^m$ is a constant column vector that is determined by the m eigenfunctions obtained from the modal analysis. In addition, Coriolis and centrifugal terms can be explicitly given as

$$\begin{aligned} c_1(\dot\theta,\delta,\dot\delta) &= 2M_L\dot\theta(\Phi_e^T\delta)(\Phi_e^T\dot\delta) \\ c_2(\dot\theta,\delta) &= -M_L\dot\theta^2(\Phi_e^T\delta)\Phi_e \end{aligned}$$

Apparently, from (1), the flexible one-link robot is an *underactuated mechanical system* with $m+1$ degrees of freedom and only a single actuator. The important feature of (1) is that the inertia matrix $M(\delta)$ is *symmetric* w.r.t. θ, i.e. it is invariant under the group action $\theta \mapsto \theta + \alpha$. In the next section, we show that this symmetry property has a crucial role in derivation of a minimum phase noncollocated output for the flexible one-link robot.

3 Underactuated Mechanical Systems with Symmetry

In this section, we provide a basic result that helps us to find a noncollocated output for the dynamics of the flexible one-link robot in (1).

Theorem 3.1. *Consider an underactuated mechanical system augmented with an integrator as the following*

$$\begin{aligned} \begin{bmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{bmatrix} \begin{bmatrix} \ddot q_1 \\ \ddot q_2 \end{bmatrix} &+ \begin{bmatrix} c_1(q,\dot q) \\ c_2(q,\dot q) \end{bmatrix} \\ + \begin{bmatrix} g_1(q) \\ g_2(q) \end{bmatrix} &= \begin{bmatrix} \tau \\ 0 \end{bmatrix} \\ \dot\tau &= v \end{aligned} \quad (3)$$

The system has a configuration vector $q = (q_1, q_2) \in \mathbb{R} \times \mathbb{R}^m$ and an inertia matrix $M(q_2)$ which is symmetric w.r.t. q_1. Suppose that the following one-form is exact

$$\omega = m_{11}(q_2)^{-1} m_{12}(q_2) dq_2$$

where $q_2 = (q_2^1, \ldots, q_2^m)^T$. *Then, there exists an output*

$$y = q_1 + \gamma(q_2)$$

with $d\gamma(q_2) = \omega$ *that has global uniform relative degree* 3 *w.r.t* v *and the following change of coordinates*

$$\begin{aligned} z_1 &= q_2 \\ z_2 &= \dot{q}_2 \\ \xi_1 &= y = q_1 + \gamma(q_2) \\ \xi_2 &= \dot{y} = \dot{q}_1 + m_{11}(q_2)^{-1} m_{12}(q_2)\dot{q}_2 \\ \xi_3 &= \ddot{y} \end{aligned} \tag{4}$$

globally transforms the dynamics of the system into a partially-linear cascade nonlinear system in Byrnes-Isidori normal form ([6])

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z_1, z_2, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= u \end{aligned} \tag{5}$$

Proof. The proof is by direct calculation. See [8] or [9] for the idea behind the proof. □

To apply the result of theorem 3.1 to the dynamics of the flexible one-link robot in (1), the following one-form

$$\omega_1 = m_{11}(\delta)^{-1} m_{12} d\delta$$

must be exact. If this holds, then a nonlinear noncollocated output in the form of $y = \theta + \gamma(\delta)$ exists that transforms the system into a cascade nonlinear system. However, this is false and the one-form $m_{11}(\delta)^{-1} m_{12} d\delta$ is not exact due to the fact that $m_{12} \neq \lambda \Phi_e^T$ for any $\lambda \in \mathbb{R}$. This suggests an alternative way of constructing the desired output. By inspection of $m_{11}(\delta)$ in (2), one can observe that the following *modified* one-form

$$\omega_2 = m_{11}(\delta)^{-1} \Phi_e^T d\delta$$

is exact and thus the expression

$$\dot{\theta} + m_{11}(\delta)^{-1} \Phi_e^T \dot{\delta}$$

is integrable as an output $y = \theta + \gamma(\delta)$. Next, we show that this is indeed the output we were seeking.

4 The Noncollocated Output

In this section, we derive a noncollocated output and prove that its corresponding zero dynamics is minimum phase. But first we need to make some assumptions and give a lemma.

Notation. For any two vectors $x, y \in \mathbb{R}^m$ and the positive definite $m \times m$ matrix Q, define the following inner product

$$\langle x, y \rangle_Q := x^T Q y$$

Assumption 4.1. Suppose the following properties hold

i) Φ_e satisfies $\langle m_{21}, m_{21} \rangle_{m_{22}^{-1}} > \langle \Phi_e, m_{21} \rangle_{m_{22}^{-1}}$.

ii) The matrix $Q_0 = m_{21} m_{12} - m_{21} \Phi_e^T$ is positive definite.

Remark 4.1. By the specific structure of the inertia matrix $M(\delta)$ for a flexible link robot in [5], assumptions (i) and (ii) are very reasonable and can be algebraicly checked.

Lemma 4.1. *Suppose Assumption 4.1 holds. Then the following matrices are positive definite and thus invertible.*

$$\begin{aligned} Q_1 &= m_{11}(\delta) - \Phi_e^T m_{22}^{-1} m_{21} \\ Q_2 &= m_{22} - m_{21} m_{11}^{-1} \Phi_e^T \end{aligned}$$

Proof. First, note that the following matrices are symmetric and positive definite

$$\begin{aligned} Q_3 &= m_{11}(\delta) - m_{12} m_{22}^{-1} m_{21} \\ Q_4 &= m_{22} - m_{21} m_{11}^{-1}(\delta) m_{12} \end{aligned}$$

The proof for this can be found in [12]. This is true regardless of how many deformation modes are chosen as long as m_{ij}'s are elements of a symmetric positive definite inertia matrix M. Noting that Q_1 can be rewritten as

$$Q_1 = Q_3 + (m_{12} - \Phi_e^T) m_{22}^{-1} m_{21}$$

by $i)$ in Assumption 4.1 the second term is positive and Q_3 is positive as well. Thus, Q_1 is positive definite. To prove Q_2 is positive definite, notice that for any $x \in \mathbb{R}^m$ we have

$$x^T Q_2 x = x^T Q_4 x + x^T m_{21} m_{11}(\delta)^{-1} (m_{12} - \Phi_e^T) x$$

or

$$x^T Q_2 x = x^T Q_4 x + m_{11}(\delta)^{-1} x^T Q_0 x$$

and because by Assumption 4.1 Q_0 is positive definite and also $m_{11}(\delta)^{-1} > 0$, Q_2 is a positive definite matrix. □

Here is our main result:

Theorem 4.1. *The nonlinear noncollocated output*

$$y = \theta + \gamma(\delta) \tag{6}$$

where $\gamma : \mathbb{R}^m \to \mathbb{R}$ *is given by*

$$\gamma(\delta) = \frac{1}{\sqrt{k_1 k_2}} \arctan\Big(\sqrt{\frac{k_1}{k_2}} \Phi_e^T \delta\Big) \tag{7}$$

has global uniform relative degree 3 *w.r.t.* v *and determines a global change of coordinates*

$$z_1 = \delta,\ z_2 = \dot{\delta},\ \xi_1 = y,\ \xi_2 = \dot{y},\ \xi_3 = \ddot{y}$$

that transforms the dynamics of a flexible one-link robot into the partially linear cascade nonlinear system in (5). In addition, there exists an $\epsilon > 0$ *with the property* $\epsilon \propto \sqrt{\sigma_{min}(F)}$ *such that over a neighborhood* $U_\epsilon(0) = \{(z_1, z_2) : |z_1| < \epsilon,\ |z_2| < \epsilon\}$ *of* $(z_1, z_2) = 0$*, for the zero-dynamics corresponding to this output the origin is locally uniformly asymptotically stable (i.e. the noncollocated output* y *is minimum phase).*

Proof. By direct calculations, for $y = \theta + \gamma(\delta)$ in (6), we have

$$\begin{aligned} \dot{y} &= \dot{\theta} + \frac{\Phi_e^T \dot{\delta}/k_2}{k_1/k_2 + (\Phi_e^T \delta)^2} \\ &= \dot{\theta} + \frac{\Phi_e^T \dot{\delta}}{k_1 + k_2(\Phi_e^T \delta)^2} \\ &= \dot{\theta} + m_{11}(\delta)^{-1}(\Phi_e^T \dot{\delta}) \end{aligned}$$

also

$$\ddot{y} = \ddot{\theta} + m_{11}(\delta)^{-1}\Phi_e^T \ddot{\delta} - 2k_2 m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2$$

but from the second line of equation (1), $\ddot{\delta}$ can be obtained as

$$\ddot{\delta} = -m_{22}^{-1} m_{21} \ddot{\theta} - m_{22}^{-1}(c_2 + K\delta + F\dot{\delta})$$

Thus

$$\begin{aligned} \ddot{y} &= m_{11}(\delta)^{-1}(m_{11}(\delta) - \Phi_e^T m_{22}^{-1} m_{21})\ddot{\theta} \\ &- m_{11}(\delta)^{-1}\Phi_e^T m_{22}^{-1}(c_2 + K\delta + F\dot{\delta}) \\ &- 2k_2 m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2 \end{aligned}$$

on the other hand, after cancelling $\ddot{\delta}$ from the first and second lines of (1), one obtains

$$\begin{aligned} (m_{11}(\delta) - m_{12} m_{22}^{-1} m_{21})\ddot{\theta} &+ c_1 \\ -m_{12} m_{22}^{-1}(c_2 + K\delta + F\dot{\delta}) &= \tau \end{aligned}$$

Hence, taking

$$\tilde{m}_{11}(\delta) = (m_{11}(\delta) - m_{12} m_{22}^{-1} m_{21}) > 0$$

we have

$$\begin{aligned} \ddot{y} &= m_{11}(\delta)(m_{11}(\delta) - \Phi_e^T m_{22}^{-1} m_{21})\tilde{m}_{11}(\delta)^{-1} \\ &\cdot (\tau - c_1 + m_{12} m_{22}^{-1}(c_2 + K\delta + F\dot{\delta})) \\ &- m_{11}(\delta)^{-1}\Phi_e^T m_{22}^{-1}(c_2 + K\delta + F\dot{\delta}) \\ &- 2k_2 m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2 \end{aligned}$$

this means that applying a change control

$$\begin{aligned} w &= (m_{11}(\delta) - m_{12} m_{22}^{-1} m_{21}) \\ &\cdot (m_{11}(\delta) - \Phi_e^T m_{22}^{-1} m_{21})^{-1} m_{11}(\delta)\tau \\ &+ \beta(\dot{\theta}, \delta, \dot{\delta}) \end{aligned}$$

(where β can be explicitly calculated from the last equation of $\ddot{y}$) partially linearizes the dynamics of the system as

$$\ddot{y} = w$$

and therefore the output y has global relative degree 2 w.r.t. τ and global relative degree 3 w.r.t. v. To determine the stability of the zero dynamics corresponding to y, let $z_1 = \delta, z_2 = \dot{\delta}$ and set $y \equiv 0$ (thus, $\dot{y}, \ddot{y} \equiv 0$). From the second equation in (1), the zero dynamics system can be determined as

$$\begin{aligned} (m_{22} - m_{21} m_{11}(\delta)^{-1}\Phi_e^T)\ddot{\delta} &+ \\ 2k_2 m_{21} m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2 &+ \\ c_1 + K\delta + F\dot{\delta} &= 0 \end{aligned}$$

but $c_1 = -k_2 \dot{\theta}^2 (\Phi_e^T \delta)\Phi_e$, or

$$c_1 = -k_2 m_{11}(\delta)^{-2}(\Phi_e^T \dot{\delta})^2 (\Phi_e^T \delta)\Phi_e$$

Thus, the equations of the zero dynamics can be expressed as the following

$$\begin{aligned} (m_{22} - m_{21} m_{11}(\delta)^{-1}\Phi_e^T)\ddot{\delta} &+ \\ k_2 m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2 (2m_{21} - \Phi_e) &+ \\ K\delta + F\dot{\delta} &= 0 \end{aligned} \tag{8}$$

Denoting

$$Q(\delta) = (m_{22} - m_{21} m_{11}(\delta)^{-1} \Phi_e^T)$$

and noting that $Q(\delta)$ is a positive definite matrix, the following Lyapunov function for the zero dynamics system can be proposed

$$V(\delta, \dot{\delta}) = \frac{1}{2} \dot{\delta}^T Q(\delta) \dot{\delta} + \frac{1}{2} \delta^T K \delta$$

Calculating $\dot{V}$ along the solutions of the zero dynamics (8), we obtain

$$\dot{V} = \dot{\delta}^T Q(\delta) \ddot{\delta} + \dot{\delta}^T K \delta + \frac{1}{2} \dot{\delta}^T \dot{Q} \dot{\delta}$$

where

$$\dot{Q} = \frac{\partial Q}{\partial \delta} \dot{\delta} = 2 k_2 m_{21} m_{11}(\delta)^{-2} (\Phi_e^T \delta)(\Phi_e^T \dot{\delta}) \Phi_e^T$$

and

$$\frac{1}{2} \dot{\delta}^T \dot{Q} \dot{\delta} = k_2 m_{11}(\delta)^{-2} (m_{12} \dot{\delta})(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2$$

Hence

$$\begin{array}{rcl} \dot{V} & = & -\dot{\delta}^T F \dot{\delta} - k_2 m_{11}(\delta)^{-2} (\Phi_e^T \dot{\delta})^2 (\Phi_e^T \delta) \\ & \cdot & ((m_{12} \dot{\delta}) - (\Phi_e^T \dot{\delta})) \end{array}$$

and because $m_{11}(\delta)^{-2} \leq k_1^{-2}$, we get

$$\dot{V} \leq -\sigma_{min}(F) |\dot{\delta}|^2 + \frac{k_2}{k_1^2} |\Phi_e|^3 (|m_{21} - \Phi_e|) |\dot{\delta}|^3 |\delta|$$

where $\sigma_{min}(F)$ is the smallest singular value of F. Denoting

$$k_3 = \frac{k_2}{k_1^2} |\Phi_e|^3 |m_{21} - \Phi_e|$$

and defining

$$\epsilon = \sqrt{\frac{\sigma_{min}(F)}{k_3}}$$

for $|\delta|, |\dot{\delta}| < \epsilon$ (i.e. over $U_\epsilon(0)$), $\dot{V} \leq 0$. But the largest invariant set in $z_2 = \dot{\delta} = 0$ (i.e. $\dot{V} = 0$) for (8) is $(z_1, z_2) = (0, 0)$. Therefore, from LaSalle's invariance principle the origin $(\delta, \dot{\delta}) = (z_1, z_2) = 0$ is uniformly locally asymptotically stable for the zero dynamics system, i.e. the output y is (locally) minimum phase. □

Remark 4.2. Viewing $\sigma_{min}(F)$ as the *strength of damping* for the flexible link, the result of the preceding theorem can be interpreted as follows. The higher the strength of damping, the larger the region of attraction of the origin for the zero dynamics.

Remark 4.3. For a sufficiently small deformation amplitudes δ. the output y in (6) can be expressed as a linear combination of the angle of rotation and weighted deformation amplitudes as

$$y = \theta + \frac{1}{M_L} (\Phi_e^T \delta)$$

which depends on the payload mass M_L. This is in agreement with the fact that intuitively one expects the payload mass matters in trajectory tracking control design for a flexible arm.

5 Tracking Control

The control for trajectory tracking can be given as the following. Let $y_d(t)$ denote the desired trajectory and define the error functions

$$e_1 = y - y_d(t),\ e_2 = \dot{y} - \dot{y}_d,\ e_3 = \ddot{y} - \ddot{y}_d$$

Then

$$\begin{array}{rcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & e_3 \\ \dot{e}_3 & = & u - y_d^{(3)} \end{array}$$

where $y_d^{(k)} = \dfrac{d^k y_d(t)}{dt^k}$. Thus, setting

$$u = y_d^{(3)} + c_p(y - y_d) + c_d(\dot{y} - \dot{y}_d) + c_a(\ddot{y} - \ddot{y}_d)$$

with $c_p, c_d, c_a < 0$, or applying the partial state feedback

$$u = y_d^{(3)} + c_p(\xi_1 - y_d) + c_d(\xi_2 - \dot{y}_d) + c_a(\xi_3 - \ddot{y}_d) \tag{9}$$

guarantees that $(e_1, e_2, e_3) \rightarrow 0$ as $t \rightarrow \infty$ and for sufficiently small initial conditions $(\delta(0), \dot{\delta}(0))$ asymptotic output tracking can be achieved [6].

6 Conclusion

In this paper, we considered trajectory tracking for a flexible one-link robot. A finite-order nonlinear state-space model of a flexible link obtained from truncated modal analysis of the Euler-Bernoulli beam has been used. This model is obtained from a Lagrangian analysis and is in the form of an underactuated mechanical system. Based on our previous results on normal forms for underactuated systems, we derived a noncollocated output that is equal to the angle of the rotation augmented with the (generalized) saturation of weighted deformation amplitudes and showed that the corresponding zero-dynamics of this output is uniformly locally minimum phase over a region of attraction that increases by the strength of damping of the link.

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