Recursive Observer Design beyond the Uniform Observability

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Abstract

We propose a novel recursive design scheme of state observer for lower triangular nonlinear systems. The design begins from the bottom dynamics and propagates to upper dynamics recalling the backstepping scheme for nonlinear control. The proposed class of systems is fairly general since it includes non-uniformly observable or detectable multi-output systems. The error convergence to zero is proved assuming the boundedness of input *a posteriori*, which is preferable whereas most results in the literature assume the boundedness *a priori*. A global observer is proposed with the global Lipschitz condition of the system. However, this condition is removed via the Lipschitz extension technique when the semi-global observer is of interest.

1 Introduction

During the last two decades, the construction of nonlinear observer has been very actively studied (see e.g. [1] and the references therein). In particular, the most well-known design method for nonlinear observer is the approach of linear error dynamics [2–4] originated by [5]. Furthermore, in order to extend the class with linear error dynamics, several authors has begun to utilize the input derivatives in the designed observer (e.g. [6,7]). However, these designs have a serious drawback that one should solve some partial differential equations to find the suitable change of coordinates.

On the contrary, the approaches of high gain nonlinear observer [8–14] have an advantage that the associated change of coordinates is relatively easy because it is obtained basically from the derivatives of the output function (see [15]). In addition, their approaches are conceptually consistent with the notion of *uniform observability*, which is defined by the 'observability for *any* input' [16]. The notion owes its motivation to the well-known fact that, for the nonlinear systems with input, the observability can be destroyed by a specific input because the observability is generally defined as the distinguishability of two different initial states by *some* input [17]. It should also be mentioned that the notion has been successfully applied to the construction of output feedback controller for nonlinear systems in view of certainty equivalence [15,18–20].

The canonical form of uniform observability for single-output nonlinear systems has been proposed by Gauthier *et al.* [8] for input affine systems and by Gauthier and Kupka [9] for input non-affine systems, respectively. Roughly speaking, the canonical form admits the lower triangular structure:

$$\begin{aligned} \dot{x}_1 &= x_2 + g_1(x_1, u) \\ \dot{x}_2 &= x_3 + g_2(x_1, x_2, u) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + g_{n-1}(x_1, \cdots, x_{n-1}, u) \\ \dot{x}_n &= g_n(x, u), \qquad y = x_1 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}$. An observer for this canonical form (1) has also been presented assuming that the functions g_i, $(1 \leq i \leq n)$, are Lipschitz[1] in $(x_1, \cdots, x_i)$ uniformly in u, or Lipschitz in $(x_1, \cdots, x_i)$ and u is bounded. (See [8,9] and also [10–12].)

In this paper we extend the class of systems of lower triangular form with novel recursive design scheme. The class considered is described by

$$\begin{aligned} \dot{x}_1 &= x_2 + g_1(x_1, u) \\ \dot{x}_2 &= x_3 + g_2(x_1, x_2, u) \\ &\vdots \\ \dot{x}_{r-1} &= x_r + g_{r-1}(x_1, \cdots, x_{r-1}, u) \\ \dot{x}_r &= g_r(x_1, \cdots, x_r, \eta, u) \\ \dot{\eta} &= f(x_1, \cdots, x_r, \eta, u), \qquad y = x_1 \end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^p$, $\eta \in \mathbb{R}^l$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ such that $pr + l = n$. Assume that g_i's are Lipschitz in x, and f and g_r are Lipschitz in (x, η).

The class of systems which are diffeomorphic to (2) is a generalization of the class (1) in several aspects as follows.

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[1] A function $f(x,u)$ is said to be Lipschitz in x when there is a function $c(u)$ such that $\|f(x,u) - f(z,u)\| \leq c(u)\|x - z\|$. If, furthermore, $c(u)$ is a constant independent of u, then $f(x,u)$ is Lipschitz in x uniformly in u.

(i) *Multi-output System*

Deza *et al.* [13] have extended the observer design of [8] to the multi-output nonlinear systems which have the lower triangular structure like (1) for each output y_i. However, their structure did not allow any interconnection between each *channels* arisen from each outputs, and therefore is supposed to be just the multiple parallel connection of single output system (1). Their work has been extended further in [14] by incorporating some interconnection between the channels and the resultant structure recalls the block triangular form of [21].

On the other hand, the class of (2) allows another interconnection among the channels since each x_i is not a scalar but a p-valued vector. When the η-dynamics is absent, this paper also gives multi-output extension of [8]. Note that the class of (2), without η-dynamics, is uniformly observable in the sense that the value of x is completely determined by the knowledge of input, output and successive derivatives of them [18, 19].

(ii) *Non-uniformly Observable System*

Suppose the system (1) is input affine and some g_i does not satisfy the triangular structure, i.e., $\exists j > i$ such that g_i depends on the variable x_j. Then, the uniform observability does not hold any more in general (refer to the counterexample in the proof of [8, Thm. 2]), which means those designs of [8–12] cannot be applied. On the contrary, the class of (2) includes this case by taking $r = i$ and $\eta = (x_{i+1}, \cdots, x_n)^T$.

A possibility to design an observer for non-uniformly observable systems has been already presented in [22, 23]. In those articles, a conjecture has been made that, in order to design an observer for non-uniformly observable system, the injection gain would depend on the value of input u. The proposed observer in this paper also uses the information of input and its derivatives as well in the injection gain. Moreover, our viewpoint is more specific than those of [22, 23]. We regard x and η are uniformly observable and non-uniformly observable modes, respectively, and an assumption will be made only on the non-uniformly observable mode.

(iii) *Detectable System*

It is well-known that under the observability rank condition a nonlinear system has its local decomposition in which the observable state and the unobservable state are separated [17]. Analogously, suppose the given system has a global decomposition, that is, the system (2) admits the form:

$$\begin{aligned} &\vdots \\ \dot{x}_{r-1} &= x_r + g_{r-1}(x_1, \cdots, x_{r-1}, u) \\ \dot{x}_r &= g_r(x_1, \cdots, x_r, \eta_1, u) \\ \dot{\eta}_1 &= f_1(x_1, \cdots, x_r, \eta_1, u) \\ \dot{\eta}_2 &= f_2(x_1, \cdots, x_r, \eta_1, \eta_2, u), \qquad y = x_1. \end{aligned} \tag{3}$$

Then, the state x would be uniformly observable, η_1 observable but non-uniformly observable, and η_2 unobservable state. Now if η_2 is detectable in some sense, then the design of state observer may be possible. There are several notions of nonlinear detectability [23–26], but it will be seen that our proposed assumption also implies the detectability.

(iv) *Lipschitz Property*

Most of aforementioned works [8–14] assumed the Lipschitz property of vector fields in x uniformly in u, which yields the constant injection gain. For the uniformity, the boundedness of u has been often assumed, and *a priori* knowledge of the actual bound is incorporated in the design of the gain. In our case, the requirement is just the Lipschitz property in x, and the boundedness of input (and its derivatives) is assumed *a posteriori* after completing the design. This is of interest in its own right even for the uniformly observable system (1).

Another concern of the readers might be the removal of the global Lipschitz assumption since it possibly restricts the class of systems to which the proposed method is applicable. This is achieved if we consider a semi-global observer instead of global one and the system model is simply modified outside the region of interest. This modification is called as ‘Lipschitz extension’ which has been exposed in [8] and its practical methods are found in [27] (or [15]). Here by ‘semi-global observer’ we mean an observer which guarantees the error system is globally asymptotically stable at the origin as long as the state of plant remains in a compact region whose size can be arbitrarily large[2].

Throughout this paper, the following notations are used. Let $\|\cdot\|$ denote the Euclidean norm for vectors, or the induced Euclidean norm for matrices. A function is said to be C^1 if it is continuously differentiable. For the partial derivative of f, $D_x f(x)$ is used. For notational simplicity, let $u_0 = 0$, $u_1 = u$, $u_2 = (u, \dot{u})$, $u_3 = (u, \dot{u}, \ddot{u})$ and so on. For a given function $f(x, u)$, the capital F is defined as

$$F(e; x, u) := f(e + x, u) - f(x, u). \tag{4}$$

A positive function $\psi(u)$ means that $\psi(u) > 0$ for any u. Let a function $V(x, e, u_i)$ be *quadratic in e with u_i* when there are positive functions $\psi_1(u_i)$, $\psi_2(u_i)$ and $\psi_3(u_i)$ such that

$$\begin{aligned} \psi_1(u_i)\|e\|^2 \le V(x, e, u_i) &\le \psi_2(u_i)\|e\|^2 \\ \|D_e V(x, e, u_i)\| &\le \psi_3(u_i)\|e\|. \end{aligned} \tag{5}$$

Finally, for a system $(S) : \dot{x} = f(x, u)$ and a function $V(x, u)$, $\dot{V}|_{(S)}$ implies the time derivative of V along the trajectory of (S), i.e.,

$$\dot{V}|_{(S)} = D_x V \cdot f(x, u) + D_u V \cdot \dot{u}.$$

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[2] Even when the boundedness of the state of plant is not an assumption but one of the control objectives, the concept of semi-global observer is still useful. The combination of any state-feedback control law and a semi-global observer for the purpose of output feedback stabilization is found in [15].

2 One Step Propagation

As a preliminary we derive a state observer in a generalized framework. Consider a system generally described by:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{pmatrix} = f(x, u) \\ y &= x_1 \end{aligned} \tag{6}$$

where $u \in \mathbb{R}^m$, the state $x \in \mathbb{R}^n$ is partitioned as $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^{n-p}$ according to the dimension of output $y \in \mathbb{R}^p$, and the vector field $f(x, u)$ is Lipschitz in x. We seek an observer of the form:

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2, u) + \gamma(*) = f_1(z_1, z_2, u) + v \\ \dot{z}_2 &= f_2(z_1, z_2, u) + L(u)\gamma(*) = f_2(z_1, z_2, u) + L(u)v \end{aligned} \tag{7}$$

where $z = (z_1^T, z_2^T)^T$ is the estimate of x, $L(u) \in \mathbb{R}^{(n-p)\times p}$ is a matrix-valued injection gain which is continuously differentiable with respect to u and $\gamma(*)$ is some function of known quantities such as the output of the plant, the estimate z, the input u and its derivatives. Note that γ is regarded as the virtual *control* v in the above equation.

Then, the augmented[3] error dynamics ($e := z - x$) is obtained as

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{e}_1 &= F_1(e_1, e_2; x_1, x_2, u) + v \\ \dot{e}_2 &= F_2(e_1, e_2; x_1, x_2, u) + L(u)v \\ y_a &= z_1 - y = e_1. \end{aligned} \tag{8}$$

In this description, we regard the error dynamics has the input v and the output y_a. Then, the observer construction problem becomes equivalent to finding $L(u)$ and γ with which the error e is *controlled* to be asymptotically stable by the feedback $v = \gamma(*)$.

Firstly, the condition that the gain $L(u)$ should satisfy is given as follows. (Since the argument of this section is used for the recursive design in the next section, we use the input and its derivatives (u_{j+1}) instead of u in what follows. On the first reading, just suppose $j = 0$.)

Assumption 1 *There exist a C^1 function $V(x, e_2, u_j)$ which is quadratic in e_2 with u_j where $j \geq 0$, a C^1 matrix-valued function $L(u_{j+1}) \in \mathbb{R}^{(n-p)\times p}$ and a positive function $\alpha_0(u_j)$ such that*

$$\begin{aligned} &D_x V \cdot f(x, u) + D_{e_2} V \cdot [F_2(0, e_2; x, u) \\ &\quad -L(u_{j+1})F_1(0, e_2; x, u)] + D_{u_j} V \cdot \dot{u}_j \leq -\alpha_0(u_j)\|e_2\|^2 \end{aligned}$$

for all $x \in \mathbb{R}^n$, $e_2 \in \mathbb{R}^{n-p}$ and $u, \dot{u}, \cdots, u^{(j)} \in \mathbb{R}^m$.

Remark 1 When u_j is bounded, Assumption 1 implies that the augmented error dynamics is minimum phase with respect to e_2 in some sense, because the zero dynamics of (8), with the input v and the output y_a, is obtained as

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{e}_2 &= F_2(0, e_2; x_1, x_2, u) - L(u_{j+1})F_1(0, e_2; x_1, x_2, u). \end{aligned} \tag{9}$$

Note that the stability of the plant dynamics is not required, which is not of concern in observer problem.

Finally, the suitable $\gamma(*)$ is constructed in the following theorem.

Theorem 1 *Under Assumption 1, there are a C^1 function $W(x, e_1, e_2, u_{j+1})$ which is quadratic in (e_1, e_2) with u_{j+1}, a C^1 function $\phi(u_{j+2})$, and a positive function $\alpha_1(u_{j+1})$ such that*

$$\dot{W}|_{(S)} \leq -\alpha_1(u_{j+1})\|e\|^2$$

where (S) is the system (8) with $v = \gamma(*) = -\phi(u_{j+2})(z_1 - y)$.

Proof: By change of coordinates $\xi = T(u_{j+1})e$ where

$$T(u_{j+1}) = \begin{bmatrix} I & 0 \\ -L(u_{j+1}) & I \end{bmatrix},$$

the augmented error dynamics (8) becomes

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{\xi}_1 &= F_1(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) + v \\ \dot{\xi}_2 &= F_2(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) \\ &\quad - L(u_{j+1})F_1(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) - \tilde{L}(u_{j+2})\xi_1 \\ y_a &= \xi_1 \end{aligned} \tag{10}$$

where $\tilde{L}$ is $(n-p)\times p$ matrix-valued function whose (i, k)-th element is $D_{u_{j+1}} L_{i,k}(u_{j+1}) \cdot \dot{u}_{j+1}$. In this coordinates, it is clear that the zero dynamics is obtained as

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{\xi}_2 &= F_2(0, \xi_2; x_1, x_2, u) - L(u_{j+1})F_1(0, \xi_2; x_1, x_2, u) \\ &=: f_2^*(\xi_2, x, u_{j+1}) \end{aligned}$$

which is the same representation as (9). Using the abbreviation f_2^*, ξ_2-dynamics of (10) is rewritten as

$$\begin{aligned} \dot{\xi}_2 &= f_2^*(\xi_2, x, u_{j+1}) + F_2(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u) \\ &\quad - L(u_{j+1})F_1(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u) - \tilde{L}(u_{j+2})\xi_1 \end{aligned}$$

where the right-hand terms except f_2^* vanish when $\xi_1 = 0$.

Here, recall that f_1 and f_2 is Lipschitz in x, which yields the existence of a function $\rho(u_{j+1})$ such that

$$\begin{aligned} &\|(F_2 - L(u_{j+1})F_1)(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u)\| \\ &\qquad \leq \rho(u_{j+1})\|\xi_1\|. \end{aligned}$$

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[3] We add the term 'augmented' since x-dynamics is included in the error dynamics description.

A conservative choice of ρ would be $(c_2(u) + \|L(u_{j+1})\|c_1(u)) \cdot (1 + \|L(u_{j+1})\|)$ where c_1 and c_2 are Lipschitz coefficients of f_1 and f_2, respectively. Also define $\delta(u_{j+2}) := \|\tilde{L}(u_{j+2})\|$ and $\sigma(u_{j+1}) := \|T^{-1}(u_{j+1})\|$.

Now, let $\overline{W}(x, \xi, u_j) := V(x, \xi_2, u_j) + \frac{1}{2}\xi_1^T\xi_1$. It can be shown that $\overline{W}(x, \xi, u_j)$ is quadratic in ξ with u_j from the quadraticity of $V(x, \xi_2, u_j)$. Then, by Assumption 1,

$$\begin{aligned}
\dot{\overline{W}}|_{(10)} &= D_x V f(x,u) + D_{\xi_2} V f_2^*(\xi_2, x, u) \\
&\quad + D_{\xi_2} V (F_2 - L(u_{j+1})F_1)(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u) \\
&\quad - D_{\xi_2} V \tilde{L}(u_{j+2})\xi_1 + D_{u_j} V \cdot \dot{u}_j \\
&\quad + \xi_1^T F_1(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) + \xi_1^T v \\
&\le -\alpha_0(u_j)\|\xi_2\|^2 + \big(\beta(u_j)[\rho(u_{j+1}) + \delta(u_{j+2})] \\
&\quad + c_1(u)\sigma(u_{j+1})\big)\|\xi_1\|\,\|\xi_2\| + c_1(u)\sigma(u_{j+1})\|\xi_1\|^2 + \xi_1^T v \\
&\le -\frac{3}{4}\alpha_0(u_j)\|\xi_2\|^2 + \frac{1}{\alpha_0(u_j)}\big(\beta(u_j)[\rho(u_{j+1}) + \delta(u_{j+2})] \\
&\quad + c_1(u)\sigma(u_{j+1})\big)^2\|\xi_1\|^2 + c_1(u)\sigma(u_{j+1})\|\xi_1\|^2 + \xi_1^T v
\end{aligned}$$

where $\beta(u_j)$ is such that $\|D_{\xi_2}V(x, \xi_2, u_j)\| \le \beta(u_j)\|\xi_2\|$. Finally, by choosing a C^1 function $\phi(u_{j+2})$ such that

$$\begin{aligned}
c_1(u)\sigma(u_{j+1}) + \frac{1}{\alpha_0(u_j)}&\big(\beta(u_j)[\rho(u_{j+1}) + \delta(u_{j+2})] \\
&+ c_1(u)\sigma(u_{j+1})\big)^2 + \kappa \le \phi(u_{j+2})
\end{aligned}$$

with $\kappa > 0$ and by applying $v = -\phi(u_{j+2})\xi_1$, there exists a function $\bar{\alpha}_1(u_j)$ such that

$$\dot{\overline{W}}|_{(10)} \le -\frac{3}{4}\alpha_0(u_j)\|\xi_2\|^2 - \kappa\|\xi_1\|^2 \le -\bar{\alpha}_1(u_j)\|\xi\|^2.$$

Let $W(x, e, u_{j+1}) = \overline{W}(x, \xi, u_j)|_{\xi = T(u_{j+1})e}$. Then, from the quadraticity of $\overline{W}$ it follows that $W(x, e, u_{j+1})$ is also quadratic in e with u_{j+1}. Indeed, since $\overline{W}$ is quadratic, $\exists \bar{\psi}_i(u_j)$, $1 \le i \le 3$, such that $\bar{\psi}_1(u_j)\|\xi\|^2 \le \overline{W}(x, \xi, u_j) \le \bar{\psi}_2(u_j)\|\xi\|^2$ and $\|D_\xi \overline{W}\| \le \bar{\psi}_3(u_j)\|\xi\|$, which leads to

$$\frac{\bar{\psi}_1(u_j)}{\sigma^2(u_{j+1})}\|e\|^2 \le W(x, e, u_{j+1}) \le \bar{\psi}_2(u_j)\|T(u_{j+1})\|^2\|e\|^2$$

and

$$\begin{aligned}
\|D_e W(x, e, u_{j+1})\| &\le \|D_\xi \overline{W}\|\,\|T(u_{j+1})\| \\
&\le \bar{\psi}_3(u_j)\|T(u_{j+1})\|^2\|e\|.
\end{aligned}$$

Moreover, with $v = -\phi(u_{j+2})\xi_1 = -\phi(u_{j+2})e_1$,

$$\dot{W}|_{(8)} = \dot{\overline{W}}|_{(10)} \le -\frac{\bar{\alpha}_1(u_j)}{\sigma^2(u_{j+1})}\|e\|^2,$$

which implies the existence of a positive function $\alpha_1(u_{j+1})$. ■

It should be noted that the error convergence is not yet guaranteed at this stage, because the quadratic function $W(x, e, u_{j+1})$ is not *decrescent* [28] in the sense that the function ψ_i in (5) is not upper or lower bounded uniformly in u_j.

Corollary 1 *If $\|u_{j+1}\|$ is bounded, the dynamic system*

$$\begin{aligned}
\dot{z}_1 &= f_1(z_1, z_2, u) - \phi(u_{j+2})(z_1 - y) \\
\dot{z}_2 &= f_2(z_1, z_2, u) - \phi(u_{j+2})L(u_{j+1})(z_1 - y)
\end{aligned} \tag{11}$$

where ϕ is obtained by Theorem 1 under Assumption 1, is an exponential state observer for (6).

Proof: From the boundedness, there are positive constants ψ_i such that $\psi_1\|e\|^2 \le W \le \psi_2\|e\|^2$ and

$$\dot{W}|_{(11)-(6)} \le -\psi_3\|e\|^2,$$

which shows the exponential stability of the error dynamics [28]. ■

3 Recursive Design Algorithm

Based on the one step propagation design of previous section, we now present a recursive observer design algorithm for the system (2). Consider the following observer prototype for (2);

$$\begin{aligned}
\dot{z}_1 &= z_2 + g_1(z_1, u) + l_r(*)v \\
\dot{z}_2 &= z_3 + g_2(z_1, z_2, u) + l_{r-1}(*)v \\
&\;\vdots \\
\dot{z}_{r-1} &= z_r + g_{r-1}(z_1, \cdots, z_{r-1}, u) + l_2(*)v \\
\dot{z}_r &= g_r(z, \mu, u) + l_1(*)v \\
\dot{\mu} &= f(z, \mu, u) + l_0(*)v
\end{aligned} \tag{12}$$

where z and μ are the estimates of x and η, respectively, and the terms $l_i(*)$ and v will be designed only with the available information.

Let $e := z - x$ and $\epsilon := \mu - \eta$. Then the augmented error system is written as

$$\begin{aligned}
\dot{e}_1 &= e_2 + G_1(e_1; x_1, u) + l_r(*)v \\
\dot{e}_2 &= e_3 + G_2(e_1, e_2; x_1, x_2, u) + l_{r-1}(*)v \\
&\;\vdots \\
\dot{e}_{r-1} &= e_r + G_{r-1}(e_1, \cdots, e_{r-1}; x_1, \cdots, x_{r-1}, u) + l_2(*)v \\
\dot{e}_r &= G_r(e, \epsilon; x, \eta, u) + l_1(*)v \\
\dot{\epsilon} &= F(e, \epsilon; x, \eta, u) + l_0(*)v, \qquad y_a = e_1.
\end{aligned} \tag{13}$$

where (x, η)-dynamics is omitted for simplicity. Note also that all the G_i's and F vanish when $e = 0$ and $\epsilon = 0$.

The recursive design begins by assuming the following assumption for ϵ-dynamics.

Assumption 2 *There are C^1 functions $V_0(x, \epsilon)$ and $\phi_0(u) \in \mathbb{R}^{l \times p}$ such that*

$$\begin{aligned}
\psi_1\|\epsilon\|^2 \le V_0(x, \epsilon) \le \psi_2\|\epsilon\|^2, &\quad \|D_\epsilon V_0(x, \epsilon)\| \le \psi_3\|\epsilon\| \\
D_x V_0 \cdot f(x, u) + D_\epsilon V_0 \cdot \big[&F(0, \epsilon; x, \eta, u) \\
&-\phi_0(u)G_r(0, \epsilon; x, \eta, u)\big] \le -\psi_4\|\epsilon\|^2
\end{aligned}$$

where ψ_i's are positive constants.

Remark 2 This assumption is regarded as a detectability of reduced order system which excludes the uniformly observable modes. It implies the existence of injection gain for the non-uniformly observable or/and detectable modes. The injection gain ϕ_0 generally depends on the input u, recalling the conjecture of [22, 23]. When the η-dynamics of (2) is absent (i.e. for uniformly observable systems), this assumption is unnecessary.

Then, from the function V_0 and the gain ϕ_0 in Assumption 2, the design propagates step by step yielding a new function V_i and accumulated injection gains $l_0, \cdots, l_i$ as follows.

Step 1:
Consider a system which is obtained from the last two equations of (13) by letting $e_1 = e_2 = \cdots = e_{r-1} = 0$, $l_0 = \phi_0$, $l_1 = I$ and $y_a = e_r$:

$$\begin{aligned} \dot{e}_r &= G_r(0, \cdots, 0, e_r, \epsilon; x, \eta, u) + v \\ \dot{\epsilon} &= F(0, \cdots, 0, e_r, \epsilon; x, \eta, u) + \phi_0(u_1)v. \end{aligned}$$

By applying Theorem 1 with Assumption 2, we obtain $\phi_1(u_2)$ and $V_1(x, e_r, \epsilon, u_1)$ which is quadratic in (e_r, ϵ) with u_1 such that

$$\dot{V}_1|_{(S1)} \leq -\alpha_1(u_1)\|(e_r, \epsilon)\|^2$$

where $\alpha_1(u_1)$ is a positive function, and

$$(S1): \left\{ \begin{array}{lcl} \dot{e}_r &=& G_r(0, \cdots, 0, e_r, \epsilon; x, \eta, u) - \phi_1(u_2)e_r \\ \dot{\epsilon} &=& F(0, \cdots, 0, e_r, \epsilon; x, \eta, u) - \phi_1(u_2)\phi_0(u_1)e_r. \end{array} \right.$$

Step 2:
Now consider the system from the last three equations of (13) and let $e_1 = e_2 = \cdots = e_{r-2} = 0$, $l_0 = \phi_1\phi_0$, $l_1 = \phi_1$, $l_2 = I$ and $y_a = e_{r-1}$:

$$\begin{aligned} \dot{e}_{r-1} &= e_r + G_{r-1}(0, \cdots, 0, e_{r-1}; x, u) + v \\ \dot{e}_r &= G_r(0, \cdots, 0, e_{r-1}, e_r, \epsilon; x, \eta, u) + \phi_1(u_2)v \\ \dot{\epsilon} &= F(0, \cdots, 0, e_{r-1}, e_r, \epsilon; x, \eta, u) + \phi_1(u_2)\phi_0(u_1)v. \end{aligned} \tag{14}$$

The result of Step 1, i.e. the existence of ϕ_1 and V_1, guarantees Assumption 1 for (14), because the zero dynamics of (14) is the same as $(S1)$. Then, Theorem 1 again gives $\phi_2(u_3)$ and $V_2(x, e_{r-1}, e_r, \epsilon, u_2)$ which is quadratic in (e_{r-1}, e_r, ϵ) with u_2 such that

$$\dot{V}_2|_{(S2)} \leq -\alpha_2(u_2)\|(e_{r-1}, e_r, \epsilon)\|^2$$

where $\alpha_2(u_2)$ is a positive function, and

$$(S2): \left\{ \begin{array}{lcl} \dot{e}_{r-1} &=& e_r + G_{r-1}(0, \cdots, 0, e_{r-1}; x, u) \\ && -\phi_2(u_3)e_{r-1} \\ \dot{e}_r &=& G_r(0, \cdots, 0, e_{r-1}, e_r, \epsilon; x, \eta, u) \\ && -\phi_2(u_3)\phi_1(u_2)e_{r-1} \\ \dot{\epsilon} &=& F(0, \cdots, 0, e_{r-1}, e_r, \epsilon; x, \eta, u) \\ && -\phi_2(u_3)\phi_1(u_2)\phi_0(u_1)e_{r-1}. \end{array} \right. \tag{15}$$

Step 3:
Similarly, consider last four equations from (13) with $e_1 = e_2 = \cdots = e_{r-3} = 0$, $l_0 = \phi_2\phi_1\phi_0$, $l_1 = \phi_2\phi_1$, $l_2 = \phi_2$, $l_3 = I$ and $y_a = e_{r-2}$:

$$\begin{aligned} \dot{e}_{r-2} &= e_{r-1} + G_{r-2}(0, \cdots, 0, e_{r-2}; x, u) + v \\ \dot{e}_{r-1} &= e_r + G_{r-1}(0, \cdots, 0, e_{r-2}, e_{r-1}; x, u) + \phi_2 v \\ \dot{e}_r &= G_r(0, \cdots, 0, e_{r-2}, e_{r-1}, e_r, \epsilon; x, \eta, u) + \phi_2\phi_1 v \\ \dot{\epsilon} &= F(0, \cdots, 0, e_{r-2}, e_{r-1}, e_r, \epsilon; x, \eta, u) + \phi_2\phi_1\phi_0 v. \end{aligned} \tag{16}$$

The previous step guarantees Assumption 1 for this system, and Theorem 1 gives $\phi_3(u_4)$ and the quadratic $V_3(x, e_{r-2}, e_{r-1}, e_r, \epsilon, u_3)$ such that, with a positive function α_3,

$$\dot{V}_3|_{(S3)} \leq -\alpha_3(u_3)\|(e_{r-2}, e_{r-1}, e_r, \epsilon)\|^2$$

where

$$(S3): \left\{ \begin{array}{lcl} \dot{e}_{r-2} &=& e_{r-1} + G_{r-2}(0, \cdots, 0, e_{r-2}; x, u) \\ && -\phi_3 e_{r-2} \\ \dot{e}_{r-1} &=& e_r + G_{r-1}(0, \cdots, 0, e_{r-2}, e_{r-1}; x, u) \\ && -\phi_3\phi_2 e_{r-2} \\ \dot{e}_r &=& G_r(0, \cdots, 0, e_{r-2}, e_{r-1}, e_r, \epsilon; x, \eta, u) \\ && -\phi_3\phi_2\phi_1 e_{r-2} \\ \dot{\epsilon} &=& F(0, \cdots, 0, e_{r-2}, e_{r-1}, e_r, \epsilon; x, \eta, u) \\ && -\phi_3\phi_2\phi_1\phi_0 e_{r-2}. \end{array} \right. \tag{17}$$

In this way, the suitable l_i's and v can be found step by step. At the last step r, we finally get $V_r(x, e, \epsilon, u_r)$ which is quadratic in (e, ϵ) with u_r such that

$$\dot{V}_r|_{(Sr)} \leq -\alpha_r(u_r)\|(e, \epsilon)\|^2$$

where (Sr) is the system (13) with $l_k = \phi_r \cdots \phi_k$ $(0 \leq k \leq r)$ and $v = -e_1$. Now, under *a posteriori* assumption that the norm of u_r is bounded, the obtained system (12) becomes the exponential observer for (2) by Corollary 1.

4 Conclusions

We have presented a new recursive observer design scheme for an extended class of nonlinear systems. The class contains multi-output systems and non-uniformly observable or detectable systems. The injection gain depends on the input, which coincides with the conjecture of [22, 23] that the non-uniformly observable system should have the input-dependent injection gain. Our scheme enables to assume the boundedness of the input *a posteriori*, which is beneficial even for the class of [8] since there is no need to redesign the gain when the bound of input is changed.

The proposed recursive procedure resembles the well-known control method 'backstepping' in that the Lyapunov function is constructed with the virtual control (the output injection in our case) at each step. The proposed scheme considers the observer only, compared to the observer backstepping method in [29, 30] whose concern is the feedback stabilization with the constructed observer. Therefore, the proposed scheme is more likely the dual concept of the backstepping control.

Assumption 1 has been motivated by the minimum phase condition for the passivity analysis of error dynamics in [31]. In fact, the proposed observer falls within the class of passivity-based state observer proposed in [31].

Finally, we have to mention about the possible restriction of the proposed scheme; Lipschitz condition and the design complexity. The Lipschitz assumption can be removed when the semi-global observer is considered via the Lipschitz extension technique [8, 15, 27]. On the other hand, the complexity, which arises mainly due to the derivatives of the input, disappears when the bound of input is known *a priori*. That is, with the value of input bound, constant injection gains can be obtained at each step, which consequently yields the constant gain observer as in [8].

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