

Homoclinic Chaos in Inverted Pendula*

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Abstract

The existence of homoclinic chaos in the dynamics of two kinds of pendula with linear viscous damping, is proved via Melnikov's method. We consider the classical inverted pendulum, whose pivot can move horizontally on a cart, and the rotating inverted pendulum. Both devices are two degrees of freedom (2-DOF) underactuated systems. We analyze the case when the motion of the actuated part is periodic, with a sufficiently small amplitude.

1 Introduction

The classical inverted pendulum consists of a cart and a pendulum on it, and the control objective is to bring the pendulum from its hanging position (stable equilibrium) to its upright position (unstable equilibrium) by moving the cart horizontally. The rotating inverted pendulum is a system consisting of a pendulum connected to an arm rotating within a horizontal plane. Both devices are underactuated mechanical systems and may present a wide variety of dynamic behaviors, ranging from equilibrium points to chaotic motion. There exist many works that study the presence of bifurcations and chaos in different kinds of pendula: Alvarez and Verduzco [1], Doerner *et al* [3], Yagasaki [4, 5], and Cuerno *et al* [2]. These works consider 1-DOF systems, or include only numerical or experimental results. Here we present a formal proof of the existence of this kind of invariant sets for a class of these mechanical devices.

2 Homoclinic chaos

In this paper we consider 2-DOF underactuated mechanical systems with the following form,

$$M(q)\ddot{q} + C(q, \dot{q}) + g(q) + f(\dot{q}) = \tau, \quad (1)$$

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where $q = (q_1, q_2)$, $M(q) = \begin{pmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{pmatrix}$ with $m_{11}(q) \neq 0$ for each q , $m_{12}(q) = m_{21}(q) = \beta \cos q_2$, $m_{22}(q) = m_{22}$ constant, $C(q, \dot{q}) = (c_1(q, \dot{q}), c_2(q, \dot{q}))$, $g(q) = (g_1(q), -\alpha_2 \sin q_2)$, $\alpha_2 > 0$, $f(\dot{q}) = (f_{r_1} \dot{q}_1, f_{r_2} \dot{q}_2)$, and $\tau = (\tau_1, 0)$. We rewrite (1), obtaining

$$\begin{aligned} \tau_1 &= m_{11} \ddot{q}_1 + \beta \ddot{q}_2 \cos q_2 + c_1 + g_1 + f_{r_1} \dot{q}_1 & (2) \\ 0 &= \beta \ddot{q}_1 \cos q_2 + m_{22} \ddot{q}_2 + c_2 - \alpha_2 \sin q_2 + f_{r_2} \dot{q}_2 & (3) \end{aligned}$$

Then, if the control input is the acceleration of the actuated part, $\ddot{q}_1 = u$, system (2)-(3) reduces to

$$\ddot{q}_2 - \delta_1 \sin q_2 + \delta_2 \cos q_2 u + \bar{f}_{r_2} \dot{q}_2 + \bar{c}_2 = 0, \quad (4)$$

where $\delta_1 = \alpha_2/m_{22}$, $\delta_2 = \beta/m_{22}$, $\bar{f}_{r_2} = f_{r_2}/m_{22}$, $\bar{c}_2 = c_2/m_{22}$, and the torque τ_1 can be calculated from (2). Suppose that the actuated part acceleration is periodic,

$$u = \varepsilon \mu_1 \sin(\omega t), \quad (5)$$

factorize the friction coefficient as $\bar{f}_{r_2} = \varepsilon \mu_2$, and suppose that \bar{c}_2 is a $\mathcal{O}(\varepsilon^2)$ -function, where $\varepsilon > 0$ is small. Then Eq. (4) reduces to

$$\ddot{q}_2 - \delta_1 \sin q_2 = -\varepsilon h(q_2, \dot{q}_2, t) + \mathcal{O}(\varepsilon^2), \quad (6)$$

where $h = \mu_1 \delta_2 \cos q_2 \sin(\omega t) + \mu_2 \dot{q}_2$. We will prove that (6) exhibits chaos using Melnikov's method. Let $H(q, p) = p^2/2 + \delta_1 \cos q$ be the Hamiltonian associated to (6) for $\varepsilon = 0$, with $(q, p) = (q_2, \dot{q}_2)$. Then the perturbed Hamiltonian system associated to (6) is

$$\dot{q} = p, \quad \dot{p} = \delta_1 \sin q - \varepsilon h(q, p, t) + \mathcal{O}(\varepsilon^2),$$

where $h(q, p, t) = \mu_1 \delta_2 \cos q \sin(\omega t) + \mu_2 p$. The unperturbed system ($\varepsilon = 0$) on the cylinder has two equilibria: a hyperbolic saddle at $(0, 0) = (2\pi, 0)$ and a center at $(\pi, 0)$. The saddle is connected to itself by two homoclinic orbits given by $\gamma^\pm(t) = (q_h^\pm(t), p_h^\pm(t))$, where $q_h^\pm(t) = \pi \pm 2 \arctan(\sinh(t))$ and $p_h^\pm(t) = \pm 2\sqrt{\delta_1} \operatorname{sech}(t)$. We remark that the homoclinic trajectories can be found by solving for the level curve of

the Hamiltonian given by $H = \delta_1$. Now, the Melnikov function is given by

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} (H_p, -H_q) \wedge (0, -h)(\gamma^\pm(t+t_0))dt \\ &= -\int_{-\infty}^{\infty} (\mu_2 p^2 + \mu_1 \delta_2 \cos q \sin(\omega t) p) dt \end{aligned}$$

Using the next relations: $\int_{-\infty}^{\infty} \operatorname{sech}^2(t) dt = 2$, $\int_{-\infty}^{\infty} \operatorname{sech}(t) \cos(\omega t) dt = \pi \operatorname{sech}(\frac{\omega\pi}{2})$, and $\int_{-\infty}^{\infty} \operatorname{sech}(t)^3 \cos(\omega t) dt = \frac{\pi}{2} (1 + \omega^2) \operatorname{sech}(\frac{\omega\pi}{2})$, we obtain $M(t_0) = A \pm B \sin(\omega t_0)$, where $A = -8\mu_2 \delta_1$ and $B = 2\mu_1 \delta_2 \sqrt{\delta_1} \pi \omega^2 \operatorname{sech}(\frac{\omega\pi}{2})$. The Melnikov function has simple zeroes if

$$|B| > |A| \iff \omega^2 \operatorname{sech}\left(\frac{\omega\pi}{2}\right) > \frac{4\sqrt{\delta_1} \mu_2}{\pi |\delta_2| \mu_1}.$$

Now define $L_0 = \sup_{\omega \in \mathbb{R}} \{\omega^2 \operatorname{sech}(\omega\pi/2)\}$. Thus, if

$$\mu_1 > \frac{4\sqrt{\delta_1}}{\pi L_0 |\delta_2|} \mu_2 = \frac{4\sqrt{m_{22} \alpha_2}}{\pi L_0 |\beta|} \mu_2, \quad (7)$$

then there exists $\omega > 0$ such that $M(t_0)$ has simple zeroes. We summarize the main result.

Proposition 1 *System (2)-(3), with $\ddot{q}_1 = u$ given by (5), $\bar{f}_{r_2} = \varepsilon \mu_2$, $c_2 = \mathcal{O}(\varepsilon^2)$, will have an invariant strange set (homoclinic chaos) if ε is small enough and inequality (7) is satisfied.*

3 Examples

Classical inverted pendulum The equations of motion of the cart and pendulum system are given by

$$\begin{aligned} (m+M)\ddot{x} + ml \cos \theta \ddot{\theta} - ml \dot{\theta}^2 \sin \theta + f_{r_1} \dot{x} &= F, \\ ml \cos \theta \ddot{x} + ml^2 \ddot{\theta} - mgl \sin \theta + f_{r_2} \dot{\theta} &= 0, \end{aligned}$$

where we have assumed a massless rod and point masses, F is the external force, g is the gravity, f_{r_i} are the friction coefficients, x is the horizontal displacement of the cart and θ is the inclination of the pendulum to the vertical direction. We have $m_{22} = ml^2$, $\alpha_2 = mgl$, and $\beta = ml$; then, (7) transforms to $\mu_1 > 4\sqrt{gl} \mu_2 / (\pi L_0)$. Figure 1 shows a trajectory obtained from this system.

Rotating inverted pendulum The Lagrangian equations of motion for the rotating inverted pendulum are given by

$$\begin{aligned} F &= (\beta_1 + \beta_0 \sin^2 \theta_2) \ddot{\theta}_1 - \beta_3 \ddot{\theta}_2 \cos \theta_2 + \beta_0 \sin(2\theta_2) \dot{\theta}_1 \dot{\theta}_2 \\ &\quad + \beta_3 \dot{\theta}_2^2 \sin \theta_2 + f_{r_1} \dot{\theta}_1, \\ 0 &= -\beta_3 \ddot{\theta}_1 \cos \theta_2 + \beta_2 \ddot{\theta}_2 - \frac{1}{2} \beta_0 \sin(2\theta_2) \dot{\theta}_1^2 \dot{\theta}_2 - \alpha \sin \theta_2 \\ &\quad + f_{r_2} \dot{\theta}_2, \end{aligned}$$

where $\beta_0 = m_2 l_2^2$, $\beta_1 = (m_1 + m_2) l_1^2$, $\beta_2 = m_2 l_2^2$, $\beta_3 = m_2 l_1 l_2$, $\alpha = m_2 g l_2$, θ_1 is the inclination of the actuated arm, and θ_2 is the inclination of the pendulum to

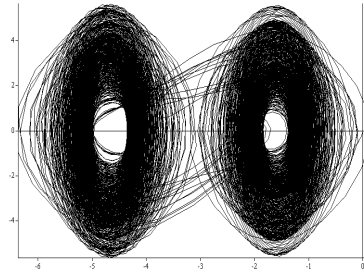


Figure 1: A chaotic trajectory of the cart-pendulum system.

the vertical direction. Now, for this case, we have that $m_{22} = m_2 l_2^2$, $\alpha_2 = m_2 g l_2$, and $\beta = -m_2 l_1 l_2$. Then, the inequality (7) transforms to $\mu_1 > 4\sqrt{gl_2} \mu_2 / (\pi l_1 L_0)$, and this system will have chaotic motion if this inequality is satisfied.

4 Conclusions

In this paper we describe some conditions that a class of underactuated mechanical devices with linear viscous damping in one of the joints must satisfy to display homoclinic chaos. Two particular examples were analyzed, the classical and the rotational pendula. The results may also be applied to other similar devices, not necessarily mechanical, *e.g.*, electronic circuits or magnetic wells.

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