Model Predictive Control via Piecewise Constant Output Feedback for Multirate Sampled-Data Systems [1]

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Abstract: In this paper, we consider the model predictive control for continuous-time systems by multirate sampled-data approach. A output feedback multirate MPC algorithm is proposed for LTI continuous-time systems based on the so-called periodic piecewise output feedback approach. First the multirate receding horizon optimal control is derived for discrete-time systems. It is shown that the multirate receding horizon control design of the LTI continuous-time systems can be reduced to that of a LTI discrete-time systems. Then the multirate sampled-data design of the LTI continuous-time systems is developed.

Key Words: model predictive control, optimal control, multirate sampled-data, periodic systems.

1. Introduction

Model predictive control (MPC), also known as receding horizon control (RHC), is well established as the industry standard for controlling constrained multivariable processes [5]. MPC refers to a class of control algorithms in which a dynamic process model is used to predict and optimize process performance. The first MPC techniques were developed in the 1970s because conventional single-loop controllers were unable to satisfy increasingly stringent performance requirements. MPC is well suited for high performance control of constrained multivariable processes because explicit pairing of input and output variables is not required and constraints can be incorporated directly into the associated open-loop optimal control problem. However, almost all proposed MPC algorithms use the purely discrete-time (DT) model only based on the behavior at the sampling instants which ignores the intersample behavior.

During the last few years there has been an increased interest in the direct design of digital controllers using continuous-time (CT) performance measures. Such systems arise naturally when a digital controller is applied to a CT plant. A realistic model of a controlled system consists of a sampled-data (SD) system operating in continuous time. Purely CT models do not account for the practical constraint that continuous output information is rarely, if ever, available. Purely DT models completely ignore the intersample behavior of the systems. SD control systems, respectively, digital control systems, are hybrid dynamic systems which usually consider of a CT plant (which can be described by a set of first-order ordinary differential equations) and a digital controller (which can be described by a set of first-order ordinary difference equations). When data in such systems are sampled at more than one rate, such systems are called multirate SD control systems, respectively, multirate digital control systems. In contrast to standard CT or DT techniques, direct SD methods address both the intersample behavior of the process, as well as the effect the sampling frequency has on the performance.

Multirate systems arise when various measurement sampling functions and control signal hold elements operate at different rates. Optimal control of multirate systems with a quadratic cost function are well developed in [2] and the references therein. If the state of the system cannot be completely available, an output feedback controller have to be designed. In general, we have to design an observer when the state cannot be completely available. As pointed out in [6], so-called periodic output feedback can avoid the design of observer. In this paper, we will use the design method of periodic output feedback to propose a multirate MPC algorithm for LTI CT systems.

In [7], the authors presented an LMI optimization based approach for the infinite horizon constrained MPC, which is motivated from that the infinite-time optimal control can be solved by the LMI based approach. However, the authors didn't discuss the finite horizon MPC and the output feedback MPC although they are extensively used in practice. As pointed out in the above paper, there are two reasons why LMI optimization is relevant to MPC. First, LMI-based optimization problems can be efficiently solved numerically using interior-point algorithm in polynomial time, often in times comparable to that required for the evaluation of an analytical solution for a similar problem. Thus LMI optimization can be implemented on-line. Secondly, it is possible to recast much of existing robust control theory in the frame-

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work of LMIs. The implication is that we can devise an MPC scheme where, at each time instant, some LMI optimization problems (as opposed to conventional linear or quadratic programs) are solved that incorporates input and output constraints. In this paper, we will show that finite horizon output feedback optimal control can be solved by the recursive LMI optimization. An LMI-based algorithm for finite horizon MPC will be present. We will show that the finite horizon constrained optimization problem on the design of piecewise output feedback control law can be solved by a recursive LMI optimization.

2. Background

In this paper, continuous-time signals will be represented by $(\cdot)$ around an independent variable, whereas DT signals will be represented by bracket $[\cdot]$. h is the sampling period. $\mathcal{S}$ and $\mathcal{H}$ are the ideal sampler and hold operator respectively. Consider the DT linear system

$$x[k+1] = A_d x[k] + B_d u[k],\ y[k] = C_d x[k]$$

where $x[k] \in \mathcal{R}^n$ is the state vector of the plant, $u[k] \in \mathcal{R}^m$ the control vector, $y[k] \in \mathcal{R}^p$ the plant output vector. At each sampling time k, plant measurements are obtained and a model of the process is used to predict future outputs of the system. In the predictions, M control moves $u[k+i|k], i = 0, 1, \ldots, M-1$, are computed by minimizing the objective function

$$\min_{u[k+i|k], i=0,\ldots,M-1} J_p[k] = \sum_{i=0}^{P-1} x^T[k+i|k] Q x[k+i|k] + \sum_{i=0}^{M-1} u^T[k+i|k] R u[k+i|k]$$

where $Q > 0$ and $R > 0$ are symmetric weighting matrices, subject to constraints on the control input $u[k+i|k]$, $i = 0, 1, \ldots, M-1$, and sometimes also on the state $x[k+i|k]$, $i = 0, 1, \ldots, P-1$, or the output $y[k+i|k]$, $i = 0, 1, \ldots, P-1$. It is assumed that there is no control action after time $k+M-1$, *i.e.* $u[k+i|k] = 0$, $i \geq M$. In the receding horizon framework, only the first computed control move $u[k|k]$ is implemented. At time $k+1$, the optimization is resolved with new measurements from the plant. The purpose of taking measurements at each time step is to compensate for unmeasured disturbances and model uncertainty. This is the main feature of the receding horizon control. In particular when $P = \infty$, it is referred to as infinite horizon MPC.

3. LQ Design for Multirate SD Systems

Consider the following LTI CT system

$$\dot{x}(t) = Ax(t) + Bu(t),\quad y(t) = Cx(t), \tag{1}$$

where (A, B) and (A, C) are assumed to be stabilizable and detectable, respectively. We consider a controller which samples all the plant outputs with the same period h_0, and changes the plant input with a shorter period h (and keeps it constant over the time interval of h), where $h = h_0/N$, and N is a positive integer. We call h_0 *frame period*, and N *input multiplicity.* This means that each component is fed into the system by means of the following zero-order hold mechanism

$$u(t) = u(jh),\quad t \in [jh,\ (j+1)h) \tag{2}$$

As in [1], the control law is assumed to be piecewise constant, *i.e.* it is given by

$$u(kh_0 + jh) = F(j) y(kh_0),\quad j = 0, \cdots, N-1 \tag{3}$$

where $F(j)$ denotes constant feedback gains. This type of controller is called *multirate constant output feedback controller*, which is equivalent to the SD controller using a piecewise constant generalized hold functions. If the controller (3) is applied to the plant represented by (C, A, B), the transition matrix (for the frame period h_0) of the closed-loop system becomes $\Phi = \exp(Ah_0) + \bar{B}\bar{F}C$, where $A_d = \exp(Ah),\ B_d = \int_0^h \exp(At) B dt,\ \bar{B} = \begin{bmatrix} A_d^{N-1} B_d & \cdots & B_d \end{bmatrix}, \bar{F} = \begin{bmatrix} F(0)^T & \cdots & F(N-1)^T \end{bmatrix}^T$. It is shown in [1] that for almost every h_0, the matrix $\bar{B}\bar{F}$ can be assigned an arbitrary value by suitable choice of $\bar{F}$ if (A, B) is controllable and if $N \geq \upsilon$, where υ is a controllability index of (A, B). If (C, A) is observable, then the eigenvalues of Φ can be assigned arbitrarily for almost every h_0 by suitable choice of $\bar{F}$.

A controller designed in this way and applied to a linear system will achieve the desired closed loop behavior in the output sampling instants, but is likely to result in strong oscillation between sampling instants. An effective approach to improving intersample behavior is to compute a hold function which minimizes a CT quadratic performance index

$$\min_{u[k]} J = \int_0^{t_f} \left[x^T Q x + u^T R u \right] dt + x^T(t_f) Q_f x(t_f), \tag{4}$$

where $Q \geq 0, R > 0$ and $Q_f \geq 0$ are constant matrices. Without loss of generality, we assume that $N_f = t_f/h_0$ is a positive integer. Let (A_d, B_d, C) denote the DT system of (1) at rate $1/h$, *i.e.*, $A_d = e^{Ah},\ B_d = \int_0^h e^{As} B(s) ds$. The discretized system of (1) at rate $1/h$ is as follows

$$x[l+1] = A_d x[l] + B_d u[l],\ y[l] = Cx[l] \tag{5}$$

Also, sampling the performance index (4) at rate $1/h$ yields the equivalent DT weights $R_d = \int_0^h (\bar{H}^T(t) Q \bar{H}(t) + R) dt$, $Q_d = \int_0^h (e^{At})^T Q e^{At} dt$, and $S_d = \int_0^h (e^{At})^T Q \bar{H}(t) dt$, where $\bar{H}(t) = \int_0^t e^{As} B ds$. Then the problem becomes to find a multirate feedback control law (3) minimizing the following performance index

$$J = \sum_{l=0}^{NN_f-1} \begin{bmatrix} x[l] \\ u[l] \end{bmatrix}^T \begin{bmatrix} Q_d & S_d \\ S_d^T & R_d \end{bmatrix} \begin{bmatrix} x[l] \\ u[l] \end{bmatrix} + x^T[NN_f] Q_f x[NN_f], \tag{6}$$

with the periodic output feedback $u[l]$ with rate $1/h$. If we assume the control law is piecewise constant, then

$$u[kN+j] = F(j)y[kN], \quad j = 0, \cdots, N-1 \tag{7}$$

Define an N-periodic matrix $S(l) \in \mathcal{R}^{p\times p}$, $l = 0, 1, 2, \ldots$, as

$$S(l) = \begin{cases} 0, & l = kN \\ I, & l \neq kN \end{cases}, \; \bar{S}(l) = I - S(l)$$

From (7), the control action $u[kN+j]$ is related with only $y[kN]$ for $j = 0, \ldots, N-1$. If we construct a new vector $\check{y}[l]$ as the output of the following sample-and-hold mechanism

$$\begin{aligned} \hat{y}[l+1] &= S(l)\hat{y}[l] + (I - S(l))y[l] \\ \check{y}[l] &= S(l)\hat{y}[l] + (I - S(l))y[l], \end{aligned}$$

then system (5) and (7) can be rewritten as

$$\begin{aligned} \hat{x}[l+1] &= \hat{A}(l)\hat{x}[l] + \hat{B}u[l], \; \check{y}[l] = \hat{C}(l)\hat{x}[l], \\ u[l] &= F(l)\check{y}[l], \quad F(l+N) = F(l) \end{aligned} \tag{8}$$

where

$$\begin{aligned} \hat{x}[l] &= \begin{bmatrix} x[l] \\ \hat{y}[l] \end{bmatrix}, \hat{A}(l) = \begin{bmatrix} A_d & 0 \\ (I - S(l))C & S(l) \end{bmatrix} \\ \hat{B}(l) &= \begin{bmatrix} B_d \\ 0 \end{bmatrix}, \hat{C}(l) = \begin{bmatrix} (I - S(l))C & S(l) \end{bmatrix} \end{aligned} \tag{9}$$

The performance (6) can be rewritten as

$$\begin{aligned} J &= \sum_{l=0}^{NN_f-1} \begin{bmatrix} \hat{x}[l] \\ u[l] \end{bmatrix}^T \begin{matrix} \hat{Q} & \hat{S} \\ \hat{S}^T & R_d \end{matrix} \begin{bmatrix} \hat{x}[l] \\ u[l] \end{bmatrix} \\ &\quad + \hat{x}^T[NN_f]\hat{Q}_f\hat{x}[NN_f], \end{aligned} \tag{10}$$

where $\hat{Q} = \begin{bmatrix} Q_d & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{S} = \begin{bmatrix} S_d \\ 0 \end{bmatrix}$, $\hat{Q}_f = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}$. Obviously, the matrices $\hat{A}(l), \hat{C}(l)$ are N-periodic and $\hat{B}, \hat{Q}, \hat{S}$ are constant.

Sampling the system (8) at N [2], we can find that the time-varying system (8) is equivalent to the following time-invariant DT system

$$\begin{aligned} x_N[k+1] &= \tilde{A}x_N[k] + \tilde{B}u_N[k], \; y_N[k] = \tilde{C}x_N[k], \\ u_N[k] &= \tilde{F}y_N[k], \; x_N[0] = x[0], \end{aligned} \tag{11}$$

where $\tilde{F} = \mathrm{diag}(F(0), \cdots, F(N-1))$,

$$\begin{aligned} x_N[k] &= \hat{x}[kN], \quad k \geq 0 & (12) \\ u_N[k] &= [u[kN]^T, \cdots, u[kN+N-1]^T]^T, & (13) \\ y_N[k] &= [\check{y}[kN]^T, \cdots, \check{y}[kN+N-1]^T]^T, & (14) \\ \tilde{A} &= \begin{bmatrix} A_d^N & 0 \\ C & 0 \end{bmatrix}, \Psi = \begin{bmatrix} A_d & 0 \\ 0 & I \end{bmatrix}, \\ \tilde{B} &= [\Psi^{N-1}\hat{B}, \Psi^{N-2}\hat{B}, \cdots, \hat{B}], \\ \tilde{C} &= [\hat{C}^T, \hat{C}^T, \cdots, \hat{C}^T]^T, \; \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix} \end{aligned}$$

Notice that $u_N[k] = \tilde{F}y_N[k] = \tilde{F}\tilde{C}x_N[k] = \bar{F}\bar{y}_N[k]$, where $\bar{F} = \begin{bmatrix} F(0)^T & \cdots & F(N-1)^T \end{bmatrix}^T$ and

$$\bar{y}_N[k] = \hat{C}x_N[k] \tag{15}$$

is an augmented output vector.

The LTI system (11) can be seen as a state-sampled representation of system (8), feeding by an augmented input vector (13) and producing an augmented output vector (15). System (11) is stable if and only if system (8) is stable [2]. Obviously for $0 < j < N$,

$$\hat{x}[kN+j] = \Phi_{\hat{A}}(j,0)x_N[k] + \Xi(j)u_N[k]$$

where $\Xi(j) := \begin{bmatrix} \Phi_{\hat{A}}(j,1)\hat{B}(0) & \cdots & \hat{B}(j-1) & 0 & \cdots & 0 \end{bmatrix}$. Define $\Xi(0) := \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$. So, with initial condition $x_N[0] = \hat{x}[0]$, (10) can be rewritten as

$$\begin{aligned} J &= \sum_{k=0}^{N_f-1} \begin{bmatrix} x_N[k] \\ u_N[k] \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x_N[k] \\ u_N[k] \end{bmatrix} \\ &\quad + x_N^T[N_f]\hat{Q}_f x_N[N_f] \end{aligned} \tag{16}$$

where $\tilde{Q}, \tilde{S}$ and $\tilde{R}$ are time-invariant matrices

$$\tilde{Q} = \sum_{j=0}^{N-1} \Phi_{\hat{A}}(j,0)^T \hat{Q}(j) \Phi_{\hat{A}}(j,0) \tag{17}$$

$$\begin{aligned} \tilde{R} &= \sum_{j=1}^{N-1} \Xi(j)^T \hat{Q}(j) \Xi(j) + \tilde{R}_s + \tilde{R}_s^T \\ &\quad + \mathrm{diag}\left[\hat{R}_c(0), \hat{R}_c(1), \cdots, \hat{R}_c(N-1)\right] \end{aligned} \tag{18}$$

$$\begin{aligned} \tilde{S} &= \sum_{j=1}^{N-1} \Phi_{\hat{A}}(j,0)^T \hat{Q}(j) \Xi(j) + \Big[\hat{S}(0), \Phi_{\hat{A}}(1,0)^T \hat{S}(1), \\ &\quad + \cdots, \Phi_{\hat{A}}(N-1,0)^T \hat{S}(N-1)\Big] \end{aligned} \tag{19}$$

where $\tilde{R}_s = \left[\Xi(0)^T \hat{S}(0), \cdots, \Xi(N-1)^T \hat{S}(N-1)\right]$. Clearly, $\tilde{Q}, \tilde{S}$ and $\tilde{R}$ are constant. It is well known that the optimal state feedback control that attains the minimum of (16) is [2]

$$\begin{aligned} u_N[k] &= K(k)x_N[k] & (20) \\ K(k) &= -(\tilde{R} + \tilde{B}^T \tilde{P}(k+1)\tilde{B})^{-1}(\tilde{B}^T \tilde{P}(k+1)\tilde{A} + \tilde{S}^T) \end{aligned}$$

where $\tilde{P}(k)$ is the backward solution of the following DT Riccati equation

$$\begin{aligned} \tilde{P}(k) &= \tilde{Q} + \tilde{A}^T \tilde{P}(k+1)\tilde{A} - (\tilde{B}^T \tilde{P}(k+1)\tilde{A} + \tilde{S}^T)^T \\ &\quad (\tilde{R} + \tilde{B}^T \tilde{P}(k+1)\tilde{B})^{-1}(\tilde{B}^T \tilde{P}(k+1)\tilde{A} + \tilde{S}^T) \end{aligned}$$

with final condition $\tilde{P}(N_f) = \hat{Q}_f$.

The discrete time-invariant system (11) has the state $x_N[k] = \hat{x}[kN]$. However, as the lower part of the state $\hat{x}[\cdot]$ consists of augmented vector $\hat{y}[\cdot]$ as given in (9), which are known to be only related to $x[kN]$, any state feedback control $u_N[k] = K(k)x_N[k]$ can be written as

$$u[kN+i] = K_{i+1,1}(k)x[kN] + K_{i+1,2}(k)\hat{y}[kN],$$

for $i = 0, \ldots, N-1$. The output feedback control law will be derived based on the dynamic programming.

We now use the optimality principle to derive an recursive optimization approach for the above DT output feedback problem corresponding to the performance index (16) and the dynamic system (11). Let

$$J_k = \begin{bmatrix} x_N^T[k] & u_N^T[k] \end{bmatrix} \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x_N[k] \\ u_N[k] \end{bmatrix}.$$

Then the output feedback optimal control problem is to design control law $u_N[k] = \bar{F}(k)\bar{y}_N[k]$ minimizing

$$J = \sum_{k=0}^{N_f-1} J_k + x_N^T[N_f]\hat{Q}_f x_N[N_f].$$

Let $J^*(x_N[k], k)$ denote the minimal value of the performance starting at time k. Then the optimality principle states that any input that is optimal over the interval (k, N_f) must necessarily be optimal over the interval $(k+1, N_f)$ so that the following recursive relation must hold true

$$J^*(x_N[k], k) = \min_{u_N[k]} \left(J_k + J^*(x_N[k+1], k+1) \right), \quad (21)$$

for $k = N_f - 1, N_f - 2, \cdots$, where

$$x_N[k+1] = \tilde{A}x_N[k] + \tilde{B}u_N[k], \; \bar{y}_N[k] = \hat{C}x_N[k].$$

Equation (21) is the optimization equation we are seeking. The above optimization equation can be solved by starting at the end time N_f and optimizing backward in time step by step. Thus, e.g., at time $k = N_f$ if the state $x_N[N_f]$ is known, we have

$$J^*(x_N[N_f], N_f) = x_N^T[N_f]\hat{Q}_f x_N[N_f]$$

Now we move back one step in time and assume $x_N[N_f - 1]$ is known. Then

$$\begin{aligned} & J^*(x_N[N_f-1], N_f-1) \quad (22) \\ = & \min_{u_N[N_f-1]} \left(J_{N_f-1} + J^*(x_N[N_f], N_f) \right) \end{aligned}$$

The value of $u_N[N_f - 1]$ that minimizes the right-hand side of (22) may be computed by taking the gradient with respect to $u_N[N_f - 1]$ and setting it equal to zero. Output feedback control law may be computed by using a gradient-search-based numerical optimization approach [10]. In the following we will give an LMI-based approach to compute the optimal output feedback gains. Let the optimal output feedback control is $u_N^*[N_f - 1] = \bar{F}_{N_f-1}\bar{y}_N[N_f - 1]$. Then

$$\begin{aligned} & J^*(x_N[N_f-1], N_f-1) \\ = & \min_{\bar{F}_{N_f-1}} \left(x_N^T[N_f-1] P_{N_f-1} x_N[N_f-1] \right), \end{aligned}$$

where $P_{N_f} = \hat{Q}_f, \;\; \bar{Q}_i = \tilde{Q} + \tilde{S}\bar{F}_i\hat{C} + (\tilde{S}\bar{F}_i\hat{C})^T$,

$$\begin{aligned} P_{N_f-1} &= (\tilde{A} + \tilde{B}\bar{F}_{N_f-1}\hat{C})^T P_{N_f} (\tilde{A} + \tilde{B}\bar{F}_{N_f-1}\hat{C}) \\ &\quad + \bar{Q}_{N_f-1} + (\bar{F}_{N_f-1}\hat{C})^T \tilde{R} \bar{F}_{N_f-1}\hat{C}. \quad (23) \end{aligned}$$

This optimization problem can be solved by the following optimization

$$\begin{aligned} & \min_{\bar{F}_i} (\mathrm{trace}(P_i)) \quad \text{s.t.} \\ & P_i \geq (\tilde{A} + \tilde{B}\bar{F}_i\hat{C})^T P_{i+1} (\tilde{A} + \tilde{B}\bar{F}_i\hat{C}) + \bar{Q}_i + (\bar{F}_i\hat{C})^T \tilde{R}\bar{F}_i\hat{C}, \end{aligned}$$

with $P_{i+1} = P_{N_f}$, which is an LMI optimization problem

$$\begin{aligned} & \min_{\bar{F}} (\mathrm{trace}(P_i)) \quad \text{s.t.} \\ & \begin{bmatrix} \bar{Q}_i - P_i & (\tilde{A} + \tilde{B}\bar{F}_i\hat{C})^T & (\bar{F}_i\hat{C})^T \\ \tilde{A} + \tilde{B}\bar{F}_i\hat{C} & -P_{i+1}^{-1} & 0 \\ \bar{F}_i\hat{C} & 0 & -\tilde{R}^{-1} \end{bmatrix} \leq 0. \quad (24) \end{aligned}$$

Denote the solution is $\bar{F}_{N_f-1}$ and define P_{N_f-1} as in (23). Then, recursively, the feedback gains $\bar{F}_{N_f-2}, \cdots, \bar{F}_0$ can be computed by the above LMI optimization with P_{i+1} substituted by $P_{N_f} - 1, \cdots, P_1$, respectively. This means that finite-horizon output feedback optimal control problem can be recursively solved by an LMI-based algorithm. In every step, an LMI optimization problem is required to be solved.

Algorithm 1: LMI Algorithm for Finite-Horizon Output Feedback Optimal Control

Step 1. Set $i = N_f - 1$ and $P(i+1) = \hat{Q}_f$.

Step 2. Solve the LMI optimization (24). Denote $\bar{F}(i)$ as the solution.

Step 3. Set

$$\begin{aligned} P_i &= (\tilde{A} + \tilde{B}\bar{F}_i\hat{C})^T P_{i+1} (\tilde{A} + \tilde{B}\bar{F}_i\hat{C}) \\ &\quad + \bar{Q}_i + (\bar{F}_i\hat{C})^T \tilde{R}\bar{F}_i\hat{C} \quad (25) \end{aligned}$$

Step 4. If $i = 0$ stop, else set $i = i - 1$ and goto Step 2.

Note that $\hat{C} = \begin{bmatrix} C & 0 \end{bmatrix}$ and denote

$$\bar{F}_i = \begin{bmatrix} F_i(0)^T & \cdots & F_i(N-1)^T \end{bmatrix}^T.$$

Then the piecewise output feedback control law can be rewritten as

$$\begin{bmatrix} u[kN] \\ \cdots \\ u[kN+N-1] \end{bmatrix} = \begin{bmatrix} F_k(0) \\ \cdots \\ F_k(N-1) \end{bmatrix} Cx[kN]. \quad (26)$$

Obviously, the above control law is very simple.

Remark 1 *In general, we cannot obtain a constant gain,* i.e., $\bar{F}_{N_f-1}, \cdots, \bar{F}_0 = \bar{F}$ *for the output feedback optimal control problem. Hence the control law* $u[kN+j] = F_k(j)Cx[kN]$, $j = 0, \cdots, N-1$, *may not be periodic. For the finite-horizon state feedback optimal control, it is known that one has to solve a recursive Riccati equation, where the computation is simple. However, in the case of output feedback, it is required to solve an LMI optimization problem in every step, which is obviously more complicated than in the case of state feedback.*

Remark 2 *Equation (25) is a standard recursive discrete Lyapunov equation for the case of output feedback. In the case of state feedback, the recursive discrete Lyapunov equation can be solved using the corresponding recursive Riccati equations.*

In applications the design of feedback controllers is often complicated by the presence of physical constraints: saturating actuators, temperatures and pressures within safety margins, working space limited by constructive restrictions, etc. This stimulated substantial theoretical advancements in the field of feedback control of dynamic systems subject to input/state constraints. Handling of hard constraints is in fact one of the potential benefits of predictive control [5, 8]. Here, we consider componentwise peak bounds on the input $u[kN+j]$, given as

$$|u_l[kN+j]| \le u_{l,\max},\ j = 0, \cdots, N-1, \tag{27}$$

for $l = 1, \cdots, n_u$. Note that the control law to be designed is $u_N[i] = \bar{F}_i \bar{y}_N[i]$, *i.e.*, $u[iN+j] = F_i(j)Cx[iN],\ i = 0, \cdots, M-1$. Define a polyhedral set as

$$\Omega(\alpha) = \{x[k] \in \mathcal{R}^n |\, |x_i[k]| \le \alpha_i,\ \alpha_i > 0,\ i = 1, \cdots, n\}$$

which denotes discrete state constraints. Then inequalities (27) hold if

$$\sum_{m=1}^{n} \left| f_{lm}^{ij} \right| \alpha_m \le u_{l,\max},\ \left\{ f_{lm}^{ij} \right\} = F_i(j)C. \tag{28}$$

Note that $x_N[i+1] = \begin{bmatrix} A_d^N + \bar{B}\bar{F}_i C & 0 \\ 0 & 0 \end{bmatrix} x_N[i]$. This means that $x[(i+1)N] = (A_d^N + \bar{B}\bar{F}_i C)x[iN]$. Then for any $x[kN] \in \Omega(\alpha)$, $x[(k+1)N] \in \Omega(\alpha)$ if and only if

$$\sum_{m=1}^{n} \left| \bar{a}_{lm}^{i} \right| \alpha_m \le \alpha_l,\ \left\{ \bar{a}_{lm}^{i} \right\} = A_d^N + \bar{B}\bar{F}_i C \tag{29}$$

which is the discrete state invariance condition. Obviously, (28) and (29) are two sets of LMIs on $\bar{F}_i$ if $\alpha = \{\alpha_i\}$ is given. Hence, we can add the two set of inequalities into the LMI optimization for the input constraint problem.

4. Multirate MPC for SD Systems

Let (A, B, C) be a given CT system

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) \tag{30}$$

with $x(0) = x_0$ and h_0 be the sampling period. At each sampling time kh_0, assume that exact measurement of the state of the system is available, *i.e.*, $x(kh_0|kh_0) = x(kh_0)$, and Mh_0 is the control horizon, *i.e.* $u(kh_0 + t|kh_0) = 0,\ \forall t > Mh_0$. The Singlerate MPC problem[4] is to design the control input $u(kh_0 + t|kh_0), 0 \le t < Mh_0$, which minimize the following objective function

$$\min_{u(kh_0+t|kh_0), 0 \le t < Mh_0} J = \int_{kh}^{\infty} \left(x^T Q x + u^T R u \right) dt \tag{31}$$

where $Q > 0$ and $R > 0$. Using the zero order hold, the objective function can be rewritten as

$$\min_{u[k+i|k], i=0,\ldots,M-1} J = \int_{kh}^{\infty} \left(x^T Q x + u^T R u \right) dt dt \tag{32}$$

Note that this is a hybrid optimization problem since both the CT and DT signals are involved in (32).

In this paper, we will use the multirate control approach of last section, *i.e.* changing the plant input with shorter period h and keeping it constant over the time interval of h, where $h = h_0/N$. M control moves in $[kh_0, (k+M)h_0)$ will be computed by minimizing the objective function (31) and only the computed control moves of the first period h_0, *i.e.* $[kh_0, (k+1)h_0)$, are implemented. At time $(k+1)h_0$, the optimization is resolved with the new measurements from the plant, *i.e.* $x((k+1)h_0)$, (or $y((k+1)h_0)$. The above control law is called *multirate MPC* (MMPC). According to the derivation of last section, the optimization problem in (31) can be transformed to a N-periodic DT LQ problem.

First we will discuss the case that A is stable. The objective function (31) can be written as $J = \tilde{J} + x^T(k+M|k)Q_M x(k+M|k)$, where

$$\begin{aligned} \tilde{J} &= \int_{kh}^{(k+M)h} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \\ Q_M &= \int_{Mh}^{\infty} e^{A^T(t-Mh)} Q e^{A(t-Mh)} dt = \int_0^{\infty} e^{A^T t} Q e^{At} dt \end{aligned}$$

From the derivation of last section, the hybrid optimization problem is reduced to a DT one

$$\begin{aligned} J &= \sum_{i=0}^{M-1} \begin{bmatrix} x_N[k+i|k] \\ u_N[k+i|k] \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x_N[k+i|k] \\ u_N[k+i|k] \end{bmatrix} \\ &\quad + x_N^T[k+M|k]\hat{Q}_M x_N[k+M|k] \end{aligned} \tag{33}$$

where $\hat{Q}_M = \begin{bmatrix} Q_M & 0 \\ 0 & 0 \end{bmatrix}$, subject to

$$x_N[k+1] = \tilde{A}x_N[k] + \tilde{B}u_N[k],\ \bar{y}_N[k] = \hat{C}x_N[k] \tag{34}$$

Since A is stable, Q_M can be obtained by the following CT Lyapunov equation [4]

$$A^T Q_M + Q_M A + Q = 0$$

Now we are ready to obtain the following theorem.

Theorem 1 *Consider the stable system (A, B, C) and the objective function (31). The optimal discrete output feedback controller can be obtained as follows*

$$u_N[k+i|k] = \bar{F}_i \bar{y}_N[k+i|k],\ i = 0, \cdots, M-1 \tag{35}$$

where $\bar{F}_i$ is the solution of the following recursive optimization problems

$$\begin{aligned} &\min_{\bar{F}_i} (trace(P_i)) \quad \text{s.t.} \\ &\begin{bmatrix} \tilde{Q}_i - P_i & (\tilde{A} + \tilde{B}\bar{F}_i\hat{C})^T & (\bar{F}_i\hat{C})^T \\ \tilde{A} + \tilde{B}\bar{F}_i\hat{C} & -P_{i+1}^{-1} & 0 \\ \bar{F}_i\hat{C} & 0 & -\tilde{R}^{-1} \end{bmatrix} \le 0 \end{aligned} \tag{36}$$

with $P_M = \hat{Q}_M$, *and for* $i = 0, \ldots, M-1$,

$$P_i = (\tilde{A} + \tilde{B}\bar{F}_i\hat{C})^T P_{i+1}(\tilde{A} + \tilde{B}\bar{F}_i\hat{C}) + \bar{Q}_i + (\bar{F}_i\hat{C})^T \tilde{R}\bar{F}_i\hat{C} \quad (37)$$

And hence the implemented multirate MPC control law in the interval $[kh_0, (k+1)h_0)$ *can be constructed as*

$$u[kN + j|kN] = F_0(j)Cx[kN],\ j = 0, \ldots, N-1. \quad (38)$$

The above theorem can be easily proven based on the dynamic programming. The theorem is the extension of the single-rate result [4]. However, the recursive Riccati equation is substituted by the LMI optimization (36). From (33) and the result of [4], we can find that the equivalent DT objective function for SD system includes a crossover which differs from the discrete MPC case.

The MMPC algorithm for SD systems is then obtained from Theorem 1.

Algorithm 2: MMPC Algorithm for Stable CT Systems

Step 1 Compute $\tilde{Q}, \tilde{R}, \tilde{S}$, and $\hat{Q}_M$.

Step 2 Get the state measurement $x(kh_0)$ at the sampling time kh_0, compute the optimal output feedback control $\bar{F}_i$ by LMI optimization (36) for $i = 0, \cdots M-1$. Implement the multirate control (38) in the interval $[kh_0, (k+1)h_0)$.

Step 3 Return Step 2 at the next sampling time $(k+1)h$.

For the unstable plant, as suggested in [9], rather than assuming the control moves to be zero after the end of the control horizon one can introduce a stabilizing local controller

$$u_N[k+i|k] = \bar{F}_N\bar{y}_N[k+i|k], \text{ for } i \geq M, \quad (39)$$

(as opposed to $u_N[k+i] = 0$). The idea is essentially the same, but the terminal cost need to be defined with respect the system $x_N[k+1] = (\tilde{A} + \tilde{B}\bar{F}_N\hat{C})x_N[k]$ rather than $x_N[k+1] = \tilde{A}x_N[k]$. The local feedback $\bar{F}_N$ can be obtained off-line by solving the following Lyapunov equation

$$\hat{Q}_M = (\tilde{A}+\tilde{B}\bar{F}_N\hat{C})^T\hat{Q}_M(\tilde{A}+\tilde{B}\bar{F}_N\hat{C}) + \bar{Q}_N + \bar{F}_N\hat{C}^T\tilde{R}\bar{F}_N\hat{C} \quad (40)$$

Notice that $\bar{F}_N = \begin{bmatrix} F_N(0)^T & \cdots & F_N(N-1)^T \end{bmatrix}^T$ is constant and hence the resulting output feedback

$$u[\bar{k}N + j|kN] = F_N(j)Cx[\bar{k}N|kN], \quad (41)$$

where $\bar{k} = k+i$, $i \geq M$, $j = 0, \cdots, N-1$, which is a periodic constant output feedback. It was shown in [1] that, if a system is controllable and observable, then for almost all output sampling rates, any self-conjugate pole configuration can be assigned to the discretized closed-loop system by periodic piecewise constant output feedback, provided the number of gain changes is not less than the systems controllability index. By using the iterative LMI algorithm of [3], we can always find a feasible solution of Lyapunov equation (40) off-line.

As in [4], we can show that the above control law is stabilizing just like in the DT case.

Theorem 2 *For stable* A *and* $M \geq 1$, *assume that the sampling period* h *is not pathological. Then the MPC control law (35)-(37) is stabilizing.*

5. Conclusion

In this paper, an output feedback multirate MPC algorithm is proposed for LMI CT systems through multirate SD approach. The finite horizon output feedbackoptimal control problem is solved by a recursive LMI optimization.

References

[1] M. Araki and T. Hagiwara. Pole assignment by multirate sampled-data output feedback. *Int. J. Control*, 44:1661–1673, 1986.

[2] S. Bittanti, P. Colaneri, and G.D. Nicolao. The periodic Riccati equation. *The Riccati Equation, (eds. S. Bittanti,A.J.Laub,J.C. Willems)*, pages 127–162, 1991.

[3] Y.-Y. Cao, J. Lam, and P.M. Frank. Simultaneous LQ optimal control design via periodic constant output feedback. *European J. of Control*, submitted, 1999.

[4] L. Chai, Y.-Y. Cao, Y.-X. Sun, and L. Qiu. Model predictive control for sampled-data systems. *14th IFAC World Congress*, D:177–182, Beijing, China, July 1999.

[5] D.W. Clarke. *Advances in model-based predictive control.* Oxford: Oxford University Press, 1994.

[6] G. L. Hyslop, H. Schattler, and T.-J. Tarn. Descent algorithms for optimal periodic output feedback control. *IEEE Trans. Automat. Control*, 37(12):1893–1904, 1992.

[7] M.V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32:1361–1379, 1996.

[8] J. Rawlings and K. Muske. The stability of constrained receding-horizon control. *IEEE Trans. Automat. Control*, 38:1512–1516, 1993.

[9] P. O. M. Scokaert and J. B. Rawlings. Constrained linear quadratic regulation. *IEEE Trans. Automat. Control*, 43(8):1163–1169, 1998.

[10] A. Varga and S. Pieters. Gradient-based approach to solve optimal periodic output feedback control problems. *Automatica*, 34(4):477–481, 1998.