DESIGN OF SLIDING-MODE OBSERVERS AND FILTERS FOR NONLINEAR DYNAMIC SYSTEMS

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Abstract

The problem of state estimation for a class of nonlinear systems with Lipschitz nonlinearities is addressed using sliding-mode estimators. Stability conditions have been found to guarantee convergence if no noise affects the system and the channel equations, and non-divergence in the presence of additive bounded disturbances. The design of such estimators is based on the solution of an algebraic Riccati equation that is difficult to solve. A method is presented in order to find a suitable solution that optimizes the performance. Successful simulations have been performed to illustrate the effectiveness of the proposed design method.

1 Introduction

The problem of designing state estimators for nonlinear systems is still difficult, even though numerous results have been reported in the literature about filters and observers. Filters for general nonlinear stochastic systems have been extensively studied since the sixties (see [1] for an introduction). Unfortunately, it is almost impossible to reach exact analytical solutions to such filtering problems, except when sufficient statistics of the underlying Markov process can be generated by a finite dimensional filter (see, among others, [2]). Thus, it is often necessary to resort to approximate filters. For example, if the noises and the state are assumed to be Gaussian processes, estimation may be accomplished by means of an extended Kalman filter (EKF). Such a filter is frequently used in practical applications and may be obtained by applying the Kalman filter to the state and channel equations linearized around the last-estimated state vector. As far as other approaches are concerned, the most recent advances in nonlinear filtering are reported in [3].

If there is no noise affecting the system and channel equations, the problem of state reconstruction can be faced using various techniques [4, 5]. One solution consists in applying a canonical state-space transformation so as to concentrate nonlinearities to a function dependent only on known input and output variables [6, 7]. This enables one to design an observer with linear error dynamics in the transformed state-space. Unfortunately, it is difficult to find this transformation for general nonlinear systems. Therefore, simpler observers are usually considered (and stability results are proved for them), like, for instance, constant-gain filters [8], which do not require such transformations. A number of theoretical results are now available for constant-gain observers [9, 10, 11]. As compared with these observers, sliding-mode estimators have both a term linearly dependent on the innovation and a variable-structure one [12], thus their potential is greater than those of other estimation techniques [13].

The works on variable-structure observers deal more with the problem of model matching compensation and with the development of viable design methods for linear systems (see, for such methods, [14] and [15]). The design of sliding-mode observers with nonlinear dynamics is faced in [12] and [16] but the focus is more on robustness to bounded nonlinearities/uncertainties. Other examples of nonlinear sliding-mode observers are reported in [13] and [17]. In [13] an approximate analysis of the noise effects is made and [17] describes an observer based on the equivalent control method. A stability analysis of a sliding-mode observer's error dynamics when the sliding conditions are satisfied is presented in [18].

In this paper, nonlinear dynamic systems with linear channels are considered and their nonlinearities are supposed to be Lipschitz. The properties of the sliding-mode observers both without noises and with noises have been studied in order to establish convergence conditions [19]. More specifically, the same conditions have been found to guarantee convergence in a purely noiseless setting and non-divergence if the noises affecting the system dynamics and the measurement channel are assumed to be bounded. These convergence conditions are similar to those obtained for constant-gain observers [9, 10]. However, the design of both such observers and the above-mentioned sliding-mode estimators turns out to be difficult. A method to choose the design parameters has been developed so that the performance of a sliding-mode observer may be optimized in a noisy case. More specifically, these parameters are chosen so as to minimize an upper bound to the asymptotic estimation error.

The paper is organized as follows. A sliding-mode observer is presented in Section 2 with the related convergence results. This estimator can be applied in a noisy case, i.e., when the state and channel equations are corrupted by additive noises. The same convergence conditions of the observer enable one to ensure non-divergence when bounded noises affect the plant, as shown in Section 3. In Section 4, the issues related to the design of such sliding-mode estimators are discussed and an optimization-based method for selecting the design parameters is described. Section 5 illustrates the results obtained by simulations and presents a performance comparison of the proposed filter with the EKF. A summary and prospects for future work are included in Section 6.

2 Sliding-mode observers for noiseless systems

Consider the class of continuous-time nonlinear systems described by

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + f\,(\underline{x},\underline{u}) \\ \underline{y} &= C\,\underline{x} \end{aligned} \quad , \quad t \geq 0 \tag{1}$$

where $\underline{x}(t) \in X \subset \mathbb{R}^n$ is the state vector, $\underline{u}(t) \in U \subset \mathbb{R}^p$ is the input vector, and $\underline{y}(t) \in Y \subset \mathbb{R}^m$ is the measurement vector. The function $f : X \times U \to \mathbb{R}^n$ is supposed to be locally Lipschitz in $\underline{x} \in X$, i.e., there exists $\lambda_f > 0$ such that $\| f\,(\underline{x}_1,\underline{u}) - f\,(\underline{x}_2,\underline{u})\,\| \leq \lambda_f\, \|\underline{x}_1 - \underline{x}_2\|$, $\forall\, \underline{x}_1, \underline{x}_2 \in X$, $\underline{u} \in U$. The channel is linear but, in general, not all the state variables are measurable. Here a nonlinear sliding-mode observer is proposed whose structure is the following:

$$\dot{\hat{\underline{x}}} = A\,\hat{\underline{x}} + f\,(\hat{\underline{x}},\underline{u}) + L\,(\underline{y} - C\hat{\underline{x}}) + S_{obs}\,(\hat{\underline{x}},\underline{y}) \tag{2}$$

where L is a constant gain matrix,

$$S_{obs}\,(\hat{\underline{x}},\underline{y}) \triangleq \begin{cases} \dfrac{P^{-1}\,C^T\,(\underline{y} - C\,\hat{\underline{x}})}{\|\underline{y} - C\,\hat{\underline{x}}\|}\,, & \|\underline{y} - C\,\hat{\underline{x}}\| > \varepsilon \\[2ex] \dfrac{P^{-1}\,C^T\,(\underline{y} - C\,\hat{\underline{x}})}{\varepsilon}\,, & \|\underline{y} - C\,\hat{\underline{x}}\| \leq \varepsilon \end{cases} \tag{3}$$

$\underline{e}$ is the estimation error (i.e., $\underline{e}(t) \triangleq \underline{x}(t) - \hat{\underline{x}}(t)$), and ε is the amplitude of the boundary layer [12]. The symmetric positive definite matrix P is the solution of a Riccati equation, which will be specified later on. A similar sliding-mode observer is described in [16] and the relevant results are those reported in [10] concerning an observer for systems with Lipschitz nonlinearities, as discussed in [19]. A schematic representation of the sliding-mode observer is shown in Fig. 1.

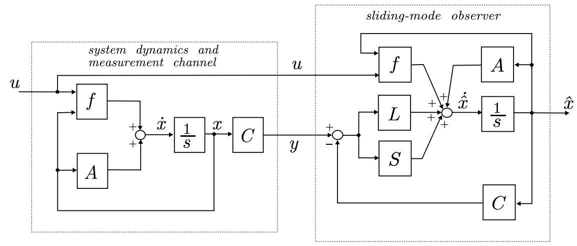

Figure 1: Scheme of a sliding-mode nonlinear observer.

A convergence result for the sliding-mode observer (2) has been established as follows (see [19]):

Theorem 1. *Given the system (1) with (A, C) observable and $f\,(\underline{x},\underline{u})$ locally Lipschitz in $\underline{x} \in X$ with the Lipschitz constant λ_f, if there exist a gain matrix L and a positive definite matrix Q such that $A - LC$ is stable and the algebraic Riccati equation*

$$(A - LC)^T\; P + P\;(A - LC) + \lambda_f^2\, P\, P + I = -Q \tag{4}$$

has a symmetric definite positive matrix P, then the estimation error dynamics of the sliding-mode observer (2) is exponentially stable. □

The proof of Theorem 1 can be found in [19].

Remark 1. The formula of the sliding-mode observer (2) presents two terms related to the innovation (i.e., $\underline{y} - C\,\underline{x}$). The first is the linear part dependent on the gain L, like most estimators for nonlinear systems (recall, for instance, the time-varying gain of the extended Kalman filter). The second is the variable-structure term, which injects a high-frequency contribution. This term may give better results in comparison with other estimation methods [13].

The sliding-mode observer (2) can also be used within a more realistic noisy framework, as it will be clarified in the next section.

3 Sliding-mode filters for noisy systems

An extension of the class of nonlinear systems (1) can be described as

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + f\,(\underline{x},\underline{u}) + \underline{w} \\ \underline{y} &= C\,\underline{x} + \underline{v} \end{aligned} \quad , \quad t \geq 0 \tag{5}$$

where $\underline{x}(t) \in X \subset \mathbb{R}^n$ is the state vector, $\underline{u}(t) \in U \subset \mathbb{R}^p$ is the input vector, $\underline{w}(t) \in W \subset \mathbb{R}^n$ is the system disturbance vector, $\underline{y}(t) \in Y \subset \mathbb{R}^m$ is the measurement vector, and $\underline{v}(t) \in V \subset \mathbb{R}^m$ is the channel noise. The vectors $\underline{w}(t)$ and $\underline{v}(t)$ are bounded noises with unknown statistics: the only knowledge of these disturbances concerns their norm upper bounds $\bar{w} > 0$ and $\bar{v} > 0$ such that $\|\underline{w}(t)\| \leq \bar{w}$ and $\|\underline{v}(t)\| \leq \bar{v}$, $\forall t \geq 0$.

In this case, the variable-structure term is

$$S_{fil}\,(\hat{\underline{x}}, \underline{y}) \triangleq \begin{cases} \dfrac{P^{-1}\,R\,C^T\,\left(\underline{y} - C\,\hat{\underline{x}}\right)}{\|\underline{y} - C\,\hat{\underline{x}}\|}\,, & \|\underline{y} - C\,\hat{\underline{x}}\| > \varepsilon \\ \dfrac{P^{-1}\,R\,C^T\,\left(\underline{y} - C\,\hat{\underline{x}}\right)}{\varepsilon}\,, & \|\underline{y} - C\,\hat{\underline{x}}\| \leq \varepsilon \end{cases} \tag{6}$$

where R is a symmetric matrix of appropriate dimension such that $R\,C^T\,C$ is positive semi-definite. Thus, the sliding-mode filter turns out to be

$$\dot{\hat{\underline{x}}} = A\,\hat{\underline{x}} + f\,(\hat{\underline{x}}, \underline{u}) + L\,\left(\underline{y} - C\hat{\underline{x}}\right) + S_{fil}\,(\hat{\underline{x}}, \underline{y}) \tag{7}$$

Remark 2. The variable-structure term (6) differs from (3) in the presence of measurement noises and in the matrix R. This matrix is very useful in designing the filter as it provides more degrees of freedom. As a consequence, the introduction of such matrix allows a more effective filter design in order to optimize the filter performance, as will be clarified in Section 4. Of course, there exists a basic requirement, i.e., that the sliding-mode filter may perform as an observer if the process and measurement noises are identically zero. This is ensured if R is chosen such that $R\,C^T\,C$ is positive semi-definite, which implies the exponential convergence of the estimation error in a noiseless setting. If R is taken equal to the identity matrix, the resulting estimator is identical to that presented in [19].

The noises $\underline{w}$ and $\underline{v}$ prevent one from attaining convergence results for the sliding-mode observer but the same conditions as in Theorem 1 enable one to guarantee non-divergence. In other words, we can state the following

Theorem 2. *Given the system (5) with (A, C) observable, $f\,(\underline{x}, \underline{u})$ locally Lipschitz in $x \in X$ with the Lipschitz constant λ_f, and the bounded noises $\underline{w}(t)$ and $\underline{v}(t)$ (with norm upper bounds $\bar{w}$ and $\bar{v}$, respectively), if there exist a gain matrix L and a positive definite matrix Q such that $A - LC$ is stable and the algebraic Riccati equation*

$$(A - LC)^T\;P + P\;(A - LC) + \lambda_f^2\,P\,P + I = -Q \tag{8}$$

has a symmetric definite positive matrix P, then the sliding-mode filter (7) has a uniformly ultimately bounded estimation error, i.e., there exist a positive constant $\bar{e}$ and a Lyapunov function $V = \underline{e}^T\,P\,\underline{e}$ such that $\dot{V} < 0$ for $\|\underline{e}\| > \bar{e}$. □

The proof of Theorem 2 is given in the Appendix.

4 Design of sliding-mode estimators for noisy nonlinear systems

In the previous section, the Lyapunov stability theory has been used to derive a bound to the estimation error. As can be deduced from the proof of Theorem 2 in the Appendix, the bound $\bar{e}$ depends on $\bar{e}_1$ and $\bar{e}_2$, which are functions of constant parameters, like $\bar{v}$, $\bar{w}$, λ_f, A, C (all given by the dynamical and channel equations) and L, P, Q, R, ε. Thus, the design parameters are L, P, Q, R and ε, which can be suitably found. ε can be chosen on the basis of the amplitudes of the noises in standard conditions by taking, for instance, a value of ε close to $\bar{v}$. The choices of L, P, Q, and R are more difficult.

One possible solution is to apply the algorithm presented in [10]. This algorithm is based on the idea of finding L such that $\dfrac{\lambda_{min}\,(A - LC)}{\kappa_2(T)}$ (T is the Jordan matrix of $A - LC$) may be minimized [$\kappa_2(D) \triangleq \|D\|\,\|D^{-1}\|$, $\lambda_{min}(D)$, and $\lambda_{min}(D)$, are the condition number (if D is invertible), maximum eigenvalue, and minimum eigenvalue of a general square matrix D]. Then it is possible to apply Theorem 2 and Theorem 5 of [10]: the condition $\dfrac{-\lambda_{min}\,(A - LC)}{\kappa_2(T)} > \lambda_f$ ensures that the algebraic Riccati equation (8) will be satisfied for certain P and Q. In order to design the sliding-mode observer, the algorithm suggests a procedure of the type:

1) find L such that $\dfrac{-\lambda_{min}\,(A - LC)}{\kappa_2(T)} > \lambda_f$;
2) solve the Riccati equation (8) by computing P for a given Q;
3) find $R = R^T$ such that $R\,C^T\,C \geq 0$; (9)
4) compute $\bar{e}_1$ and $\bar{e}_2$;
5) if the values of $\bar{e}_1$ and $\bar{e}_2$ are acceptable, then exit; otherwise go to 1).

This algorithm exhibits two considerable drawbacks. One is the solution of the Riccati equation (8), which must be computed symbolically, thus it is not easily solvable with many state variables. The second drawback is the fact that the solution depends on the values of the design parameters, difficult to select in some suitable way. Similar design procedures are presented in [15, 16].

The above-written disadvantages may be overcome by using the different algorithm proposed here for the computation of the design parameters. It has been demonstrated in Theorem 2 that there exists a Lyapunov function such that its derivative becomes negative if $\|\underline{e}\| > \bar{e} = \max(\bar{e}_1, \bar{e}_2)$ (see the proof of Theorem 2 in the Appendix). As both $\bar{e}_1$ and $\bar{e}_2$ depend on the matrices of the Lyapunov design, one way of solving the problem lies in finding L, P, Q, and R such that the norm upper bounds may be minimized. A possible method for accomplishing this task relies on the application of minimax optimization; such optimization can be expressed as follows:

$$\min_{L,P,R} \{\max[\bar{e}_1\ (L,P,Q,R)\ ,\ \bar{e}_2\ (L,P,Q,R,\varepsilon)]\} \qquad (10)$$

such that

$$\begin{array}{l} Q = -\left[(A-LC)^T\ P + P\ (A-LC) + \lambda_f^2\, P\, P + I\right] \\ A - LC < 0 \\ P > 0 \quad , \quad P = P^T \\ Q > 0 \\ R\,C^T\,C \geq 0 \quad , \quad R = R^T \end{array}$$

In order to perform the minimax optimization, the Matlab function *minimax* of the Optimization Toolbox has been used in the examples. The constrained optimization allows one to lower the upper bounds as much as possible, provided that the stability conditions are fulfilled.

Remark 3. The minimax optimization for the choice of the design parameters may suffer from the nuisance of local-minima trapping. However, the initial choice of the parameters is improved by the proposed method. Moreover, it is worth noting that this algorithm does not require that (8) be solved as the Riccati equation is used to compute the value of Q as a function of P and L by means of (11). Standard minimax routines are available in widespread mathematical programs to perform the optimization successfully.

5 A numerical example

Consider the nonlinear dynamic system

$$\begin{array}{l} \dot{x}_1 = -a\,x_1 - b\,x_1\ |x_1| + c\,u \\ \dot{x}_2 = x_1 \end{array} \qquad (11)$$

with measures $\underline{y} = (x_1, x_2)^T$: it represents a system with a quadratic drag, like, for instance, a rigid body rotating in water. x_1 and x_2 describe the angular speed and the heading, respectively. The parameters are $a = 0.2 \cdot 1/sec$, $b = 2.0 \cdot 1/rad$, and $c = 0.02 \cdot 1/(Kgm^2)$. Pseudorandom inputs with maximum torques equal to $50\,N\,m$ have been considered and the Lipschitz constant has been taken equal to $\lambda_f = 0.3060$.

The solution of the Riccati equation for $P \triangleq \begin{pmatrix} 2.4819 & 0.2051 \\ 0.2051 & 2.3461 \end{pmatrix}$, with $L \triangleq \begin{pmatrix} 2.3969 & 0.7215 \\ 0.7428 & 2.8626 \end{pmatrix}$, $Q \triangleq \begin{pmatrix} 11.2045 & 2.2143 \\ 2.2143 & 12.2087 \end{pmatrix}$, and $R \triangleq 10^{-3} \begin{pmatrix} 0.2359 & -0.2355 \\ -0.2355 & 0.4159 \end{pmatrix}$, has been obtained by applying (10) with initial guesses of the identity matrices for P, L, and R, $\varepsilon \triangleq 0.1$, $\bar{v} \triangleq 0.0001$, and $\bar{w} \triangleq 0.0001$. For the sake of comparison, it is interesting to recall a result obtained using (9) for the above-described system (see [19]), that is, $P \triangleq \begin{pmatrix} 1.2374 & -0.4473 \\ -0.4473 & 0.7710 \end{pmatrix}$, $L \triangleq \begin{pmatrix} 3.0964 & -2.9344 \\ 1.2432 & 2.7036 \end{pmatrix}$, and $Q \triangleq \begin{pmatrix} 6.7782 & -6.0429 \\ -6.0429 & 5.7193 \end{pmatrix}$ (R has been taken equal to the identity matrix), $\varepsilon \triangleq 0.1$, $\bar{v} \triangleq 0.0001$, and $\bar{w} \triangleq 0.0001$.

Simulations have been carried out in a stochastic setting by applying a simple Euler's discretization for a sample time of $0.1\,s$ to (11). A Gaussian white noise with standard deviation equal to $0.5\,deg/s$ was assumed to act on the first equation of (11). The measures of x_1 and x_2 were corrupted by additive Gaussian white noises with standard deviations equal to $1\,deg/s$ and $1\,deg$, respectively. The initial states are distributed around the zero vector according to a Gaussian distribution with standard deviations equal to $5\,deg/s$ and $5\,deg$, respectively. An EKF has been designed using the dynamics model (11) and its covariances have been initialized with the aforesaid Gaussian distributions. The RMS errors on 1000 simulation runs are shown in Figs. 2 and 3 for both the sliding-mode filter and the EKF. The plots of the estimates of x_1 are in Fig. 4. The computations of $\bar{e}_1$ and $\bar{e}_2$ obtained by means of (10) for $\varepsilon \triangleq 0.1$, $\bar{v} \triangleq 0.0831$, and $\bar{w} \triangleq 0.0410$ give the values of 0.0903 and 0.1803, respectively. The same evaluation based on (9) provides the values of 0.3110 and 0.1172 for $\bar{e}_1$ and $\bar{e}_2$, respectively. The plots of the Euclidean norm of the estimation error for a randomly chosen simulation run and of the upper bounds (i.e., $\bar{e} \triangleq \max(e_1, e_2)$) obtained by the two methods are shown in Fig. 5.

The design of the sliding-mode filter has been accomplished well, as the filter performances are similar to those of the EKF. The particular system chosen here is well suited for the estimation with the EKF, as the linearization of the quadratic term in (11) does not cause any troubles. The EKF performs like the sliding-mode observer in the estimation of

x_1, whereas its performance is a little higher in the estimation of x_2 (see Figs. 2 and 3).

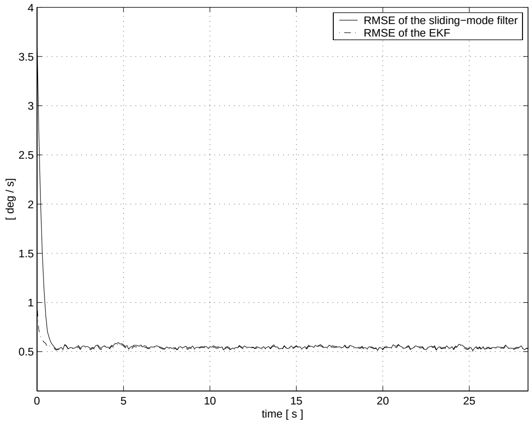

Figure 2: RMS errors on the x_1 estimates.

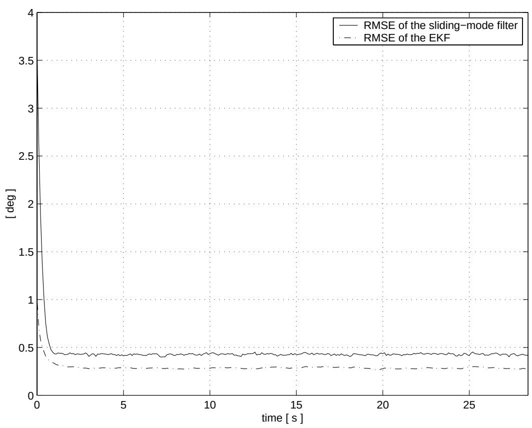

Figure 3: RMS errors on the x_2 estimates.

6 Conclusions

A sliding-mode estimation method for a class of nonlinear systems has been considered. The stability of the error dynamics has been studied both in a purely noiseless setting (i.e., with uncertainty due only to the initial state) and in a noisy case (i.e., assuming additive bounded noises to affect the dynamic and channel equations). A suitable method for selecting the design parameters has been presented that aims to optimize the performance of the proposed estimator. Satisfactory simulation results have been obtained. Moreover, the proposed method has been successfully applied to fault detection in underwater vehicles [20].

Future work will concern improvements in the design method and the extension of the estimator to a more general class of systems, including the possibility of a variable-structure vector thresholding.

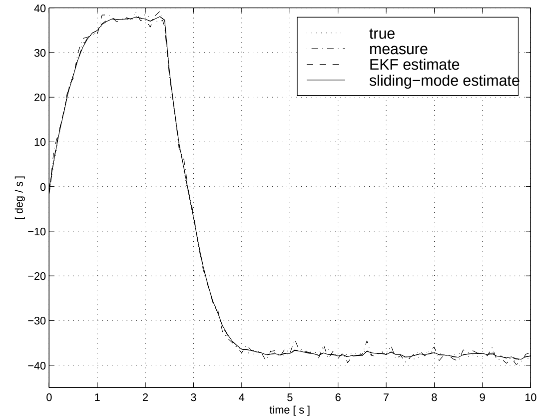

Figure 4: Plots of the true value, estimates, and measure of x_1.

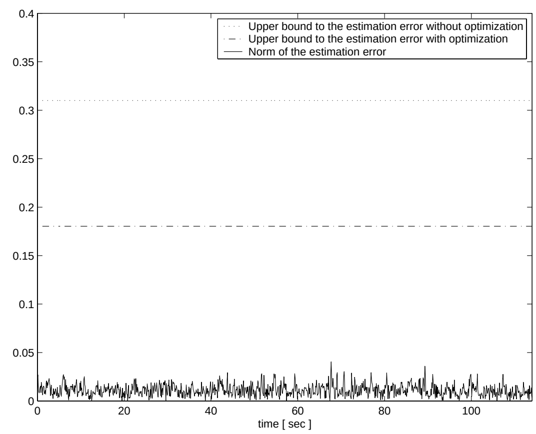

Figure 5: Plots of the upper bounds to the sliding-mode estimation error and of the Euclidean norm of this error.

Appendix

Proof of Theorem 2. Let us prove the result stated in Theorem 2. The norm of a general matrix B is defined as $\| B \| \triangleq \sqrt{\lambda_{max}\,(B^T B)}$. Moreover, $f(\underline{x})$ is replaced $f(\underline{x}, \underline{u})$, for the sake of compactness. By using equations (5) and (2), it is possible to compute the error dynamics:

$$\dot{\underline{e}} = (A - LC)\,\underline{e} + f(\underline{x}) - f(\hat{\underline{x}}) - S_{fil}(\hat{\underline{x}}, \underline{y}) + \underline{d}$$

where $\underline{d} \triangleq \underline{w} - L\underline{v}$ (once the gain L has been chosen). Consider the Lyapunov function $V = \underline{e}^T P\,\underline{e}$ (where, P is a positive definite matrix) and compute its derivative

$$\dot{V} = \underline{e}^T \left[(A - LC)^T P + P\,(A - LC)\right] \underline{e} + 2[f(\underline{x}) - f(\hat{\underline{x}})]^T P\,\underline{e} - 2\,\underline{e}^T P\,S_{fil}(\hat{\underline{x}}, \underline{y}) + 2\,\underline{e}^T P \underline{d}$$

We obtain:

$$\dot{V} \leq -\underline{e}^T Q \underline{e} - 2\,\underline{e}^T P\, S_{fil}(\hat{\underline{x}}, \underline{y}) + 2\,\underline{e}^T P \underline{d} \qquad (12)$$

By a simple algebra, we can compute an upper bound to $2\,\underline{e}^T P \underline{d}$:

$$2\,\underline{e}^T P\,\underline{d} \leq 2\,\lambda_{max}(P)\,\|\underline{e}\|\,\|\underline{d}\| \leq 2\,\lambda_{max}(P)\,\bar{d}\,\|\underline{e}\|$$

where $\bar{d} \triangleq \bar{w} + \|L\|\,\bar{v}$. Now, two cases have to be considered, i.e., $\|\underline{y} - C\,\hat{\underline{x}}\| > \varepsilon$ (Case 1) and $\|\underline{y} - C\,\hat{\underline{x}}\| \leq \varepsilon$ (Case 2).

Case 1. Suppose $\|\underline{y} - C\,\hat{\underline{x}}\| > \varepsilon$; from (12), we derive

$$\begin{aligned}\dot{V} &\leq -\underline{e}^T Q\,\underline{e} + 2\,\|R\,C\|\,\|\underline{e}\| + 2\,\lambda_{max}(P)\,\bar{d}\,\|\underline{e}\| \\ &\leq -a_1\|\,\underline{e}\,\|^2 + b_1\,\|\underline{e}\|\end{aligned}$$

where $a_1 \triangleq \lambda_{min}(Q)$ and $b_1 \triangleq 2\,\|\,R\,C\,\| + 2\,\lambda_{max}(P)\,\bar{d}$. Thus $\dot{V} < 0$, if $\|\,\underline{e}\,\| > \bar{e}_1 \triangleq \dfrac{b_1}{a_1}$.

Case 2. In this case, upper bounds can be found in two possible ways, depending on the use of condition $\|\underline{y} - C\,\hat{\underline{x}}\| \leq \varepsilon$. If this condition is satisfied, it is easy to obtain the same result of Case 1, $\dot{V}$ becomes negative if $\|\,\underline{e}\,\| > \bar{e}_1 \triangleq \dfrac{b_1}{a_1}$, as in Case 1. Otherwise, another way of obtaining an upper bound is

$$\begin{aligned}\dot{V} \quad &< -\underline{e}^T Q\,\underline{e} - \frac{2\,\underline{e}^T R\,C^T C\,\underline{e}}{\varepsilon} - \frac{2\,\underline{e}^T R\,C^T\,\underline{v}}{\varepsilon} \\ &+2\,\underline{e}^T P \underline{d} \leq -\underline{e}^T \left(Q + \frac{2\,R\,C^T\,C}{\varepsilon}\right)\underline{e} \\ &+ \left| -\frac{2\,\underline{e}^T\,R\,C^T\,\underline{v}}{\varepsilon} \right| + 2\,\lambda_{max}(P)\,\bar{d}\,\|\underline{e}\| < \\ &< -\underline{e}^T \left(Q + \frac{2\,R\,C^T\,C}{\varepsilon}\right)\underline{e} + \frac{2\,\|R\,C\,\underline{e}\|\,\|\,\underline{v}\,\|}{\varepsilon} \\ &+2\,\lambda_{max}(P)\,\bar{d}\,\|\underline{e}\| \leq -a_2\,\|\underline{e}\|^2 + b_2\,\|\underline{e}\|\end{aligned}$$

where $a_2 \triangleq \lambda_{min}\left(Q + \dfrac{2\,R\,C^T\,C}{\varepsilon}\right)$ and $b_2 \triangleq \dfrac{2\,\bar{v}\,\|R\,C\|}{\varepsilon} + 2\,\lambda_{max}(P)\,\bar{d}$. Thus, $\dot{V} < 0$ if $\|\underline{e}\| > \bar{e}_2 \triangleq \dfrac{b_2}{a_2}$.

To sum up, Case 1 and Case 2 lead to find $\bar{e}_1$ and $\bar{e}_2$ such that $\dot{V}$ becomes negative if $\|\underline{e}\| > \bar{e} \triangleq \max(\bar{e}_1, \bar{e}_2)$.

■

References

[1] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, New York, 1970.

[2] J. Levine and G. Pignie, "Exact finite dimensional filters for a class of discrete-time systems", *Stochastics*, vol. 18, pp. 97–132, 1986.

[3] S. Haykin, P. Yee, and E. Derbez, "Optimum nonlinear filtering", *IEEE Trans. on Signal Processing*, vol. 45, no. 11, pp. 2774–2786, November 1997.

[4] B. L. Walcott, M. J. Corless, and S. H. Zak, "Comparative study of nonlinear state-observation techniques", *Int. J. of Control*, vol. 45, no. 6, pp. 2109–2132, 1987.

[5] E. A. Misawa and J. K. Hedrick, "Nonlinear observers – A state-of-the-art survey", *J. of Dynamic Systems, Measurement, and Control*, vol. 111, pp. 344–352, September 1989.

[6] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers", *Systems & Control Letters*, vol. 3, pp. 47–52, 1983.

[7] A. J. Krener and W. Respondek, "Nonlinear observer with linearizable error dynamics", *SIAM J. Control and Optimization*, vol. 23, no. 2, pp. 197–216, 1985.

[8] T.-J. Tarn and Y. Rasis, "Observers for nonlinear stochastic systems", *IEEE Trans. on Automatic Control*, vol. AC-21, no. 4, pp. 441–448, August 1976.

[9] S. Raghavan and J. K. Hedrick, "Observer design for a class of nonlinear systems", *Int. J. of Control*, vol. 59, no. 2, pp. 515–528, 1994.

[10] R. Rajamani, "Observers for Lipschitz nonlinear systems", *IEEE Trans. on Automatic Control*, vol. 43, no. 3, pp. 397–401, March 1998.

[11] R. Rajamani and Y. M. Cho, "Existence and design of observers for nonlinear systems: relation to distance to unobservability", *Int. J. of Control*, vol. 69, no. 5, pp. 717–731, 1998.

[12] B. L. Walcott and S. H. Zak, "State observation of nonlinear uncertain dynamical systems", *IEEE Trans. on Automatic Control*, vol. 32, no. 2, pp. 166–170, February 1987.

[13] J.-J. Slotine, J. K. Hedrick, and E. A. Misawa, "On sliding observers for nonlinear systems", *J. of Dynamic Systems, Measurement, and Control*, vol. 109, pp. 245–252, September 1987.

[14] I. Haskara, Ü. Özgüner, and V. Utkin, "On sliding mode observers via equivalent control approach", *Int. J. of Control*, vol. 71, no. 6, pp. 1051–1067, 1998.

[15] C. Edwards and S. Spurgeon, "On the development of discontinuous observers", *Int. J. of Control*, vol. 59, no. 5, pp. 1211–1229, 1994.

[16] E. Yaz and A. Azemi, "Variable structure observer with a boundary-layer for correlated noise/disturbance models and disturbance minimization", *Int. J. of Control*, vol. 57, no. 5, pp. 1191–1206, 1993.

[17] S. V. Drakunov, "Sliding-mode observers based on equivalent control method", in *Proceedings of the 31st Conference on Decision and Control*, Tucson, Arizona, 1992, pp. 2368–2369.

[18] J.-H. Choi, E. A. Misawa, and G. E. Young, "A study on sliding mode estimation", *J. of Dynamic Systems, Measurement, and Control*, vol. 121, pp. 255–260, June 1999.

[19] A. Alessandri, "Sliding-mode estimators for a class of nonlinear systems", in *3rd Symposium on Robust Control Design, ROCOND 2000*, June 2000.

[20] A. Alessandri, G. Bruzzone, M. Caccia, P. Coletta, and G. Veruggio, "Fault detection through dynamics monitoring for unmanned underwater vehicles", in *Safeprocess 2000*, June 2000, vol. 2, pp. 973–978.