An advanced algorithm based on differential algebra for disturbance decoupling of nonlinear systems

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Abstract

The behaviour of nonlinear systems can be affected by undesired inputs – the disturbances. To decrease or to decouple the influence of those disturbances this paper deals with an advanced algorithm which solves the disturbance decoupling problem. Based on the mathematical foundations of differential algebra, the algorithm determines if a system is decouplable or not and which disturbance decoupling controller can be applied. The algorithm is restricted to rational systems. To handle analytical systems as well, a system transformation is introduced in order to receive a rational substitute system. The disturbance decoupling problem can then be solved for this substitute system.

1 Introduction

Disturbances – signals or variables unwantedly affecting the outputs – occur in almost any kind of technical plant in practice. Thus, any kind of investigation diminishing this undesirable affect is worthwhile since it may dramatically simplify a controller design for these plants. One approach is to decouple the disturbances such that the abovementioned disturbance inputs no longer affect the output of the system. This result may be obtained by computing a disturbance decoupling controller, e.g. a dynamic or static state feedback controller decoupling the disturbance inputs and system outputs. This task is referred to as the disturbance decoupling problem (DDP). The plant's behaviour (the dynamics) is described most often in terms of differential equations. Since most physical models inherit ordinary but nonlinear differential equations, this contribution presents an algorithm to solve the *nonlinear* DDP.

In contrast to the differential geometric approach to nonlinear control theory [8, 10, 13] this paper presents a differential algebraic based algorithm to solve the DDP by means of a DDP-controller. Thus, this presentation starts with a brief introduction to the most important differential algebraic terms in our setting. The algorithm is restricted to rational systems and therefore, we show how a rational substitute system may be computed automatically for analytic systems. Subsequently, the disturbance decoupling algorithm is presented. In addition, a theoretical example illustrates the rational substitute system algorithm and the disturbance decoupling algorithm is presented in this paper with a simulation study.

2 Foundations of differential algebra

Differential algebra was introduced by [12] and found its way into control theory in the 1980s [5]. Classical algebra has been introduced to handle (algebraic) equations of variables with numeric coefficients. Analogously, differential algebra was introduced to handle differential equations of variables with coefficients which may be meromorphic functions in a complex region $\mathbb{C}^m$. Hence, the close connection between algebra and differential algebra is as follows: Algebraic equations may be considered differential equations in which (time) derivatives of variables do not occur i.e. the coefficients are equal to zero (c. f. [9]).

An algebraic field is constructed by a set which is closed w.r.t. two operations – namely addition and multiplication – while in differential fields the set has to be closed w.r.t. three operations: addition, multiplication and differentiation. In both kinds of fields addition and multiplication are commutative and associative and the multiplication is distributive w.r.t. addition, too. Within a differential field K a differentiation $(\cdot)'$ is defined for any $a \in K$ as a map $a \mapsto a' \in K$, with [12]

$$(a+b)' = a' + b' \qquad \text{and} \qquad (ab)' = ba' + ab'.$$

Any a for which $a' = 0$ holds is called *constant*.

Let K/k be a (differential) field extension and $L \subset K$ such that all elements of L are k-(differential)-algebraic independent. The maximum number of such elements in L is called the *(differential) transcendence degree of* K/k (diff. trgK/k) and L is called a *transcendence basis of* K/k [7].

Let k be a differential field. Then, according to [6] a *system* is a finitely generated differential field extension K/k.

Let $\boldsymbol{u}$ be the input of a dynamical system. Then, the *dynamics* of a system L/k is a finitely generated differential field extension $L/k\langle\boldsymbol{u}\rangle$ [6].

Hence, diff. trg $L/k\langle\boldsymbol{u}\rangle = 0$ for all (finitely generated) dynamics holds (c. f. [6]). Outputs of the corresponding system are finite many elements of the dynamics $y_1, \ldots, y_p \in L$. Thus, there exists a set $\{x_1, \ldots, x_n\} \subset L$, which is a (non-differential) transcendence basis of the dynamics $L/K\langle\boldsymbol{u}\rangle$. The elements of this transcendence basis of the dynamics are the states of a system.

The dynamics of a system in differential algebra can also be described with the disturbances $\boldsymbol{d}$ in form of the state-space model

$$\Sigma_{\mathrm{RS}} \quad \begin{array}{ll} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})\,, & \boldsymbol{x} \in \mathbb{R}^n\,, \boldsymbol{u} \in \mathbb{R}^m\,, \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x},\boldsymbol{d}) \quad , & \boldsymbol{d} \in \mathbb{R}^q\,, \boldsymbol{y} \in \mathbb{R}^p\,. \end{array}$$

Respectively, the state-space model for the undisturbed system with $\boldsymbol{d} \equiv \boldsymbol{0}$ is as follows:

$$\Sigma_{\mathrm{RS}}^{\boldsymbol{d}\equiv 0} \quad \begin{array}{ll} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})|_{\boldsymbol{d}\equiv\boldsymbol{0}}\,, & \boldsymbol{x} \in \mathbb{R}^n\,, \boldsymbol{u} \in \mathbb{R}^m\,, \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x},\boldsymbol{d})|_{\boldsymbol{d}\equiv\boldsymbol{0}} \quad , & \boldsymbol{d} \in \mathbb{R}^q\,, \boldsymbol{y} \in \mathbb{R}^p\,. \end{array}$$

Let $\boldsymbol{d}$ ($\dim \boldsymbol{d} = q$) be the disturbances of a disturbed system $k\langle\boldsymbol{y},\boldsymbol{u},\boldsymbol{d}\rangle/k\langle\boldsymbol{u},\boldsymbol{d}\rangle$. Then, the differential output rank $\rho_{\boldsymbol{d}}^*$ of the disturbed system is defined as

$$\rho_{\boldsymbol{d}}^* := \text{diff. trg}\, k\langle\boldsymbol{y}\rangle/k,$$

with $\rho_{\boldsymbol{d}}^* \leq \min\{m+q, p\}$ [3, 5].

Let $k\langle\boldsymbol{y},\boldsymbol{u}\rangle/k\langle\boldsymbol{u}\rangle$ be any (undisturbed) system. Then, the differential output rank of the system is defined as

$$\rho^* := \text{diff. trg}\, k\langle\boldsymbol{y}\rangle/k, \text{ with } \rho^* \leq \min\{m,p\} \text{ [5].}$$

3 Algorithm for disturbance decoupling

After the introduction to the foundations of differential algebra and the disturbance decoupling problem (DDP), this section demonstrates an advanced algorithm based on differential algebra and an algorithm developed by [14], which solves the DDP. The method of differential algebra is restricted to *rational systems* (Σ_{RS}). The algorithm introduced in [1, 2] is able to handle these rational systems and to decide if a system is decouplable. If the case holds, then the control law is computed. However, in practice the behaviour of nonlinear systems can be described in many cases by *analytical systems* (Σ_{AS}). It turns out that utilizing an additional system transformation is unavoidable to receive the rational substitute system. For this system the advanced algorithm can be used again.

3.1 System transformation

In order to solve the DDP for analytical systems with the aid of the advanced algorithm which is introduced in this paper and bases on differential algebraic methodology, a *system transformation* is useful. This system transformation converts the orignal system into a rational substitute system. The rationalizability is related to two preconditions: the dynamical behaviour of the substitute system is the same as the behaviour of the analytical system and the conditions of differential algebra hold. Hence, in the original system all non-rational partial functions $r_i(x_1, \ldots, x_n)$ are substituted by new state variables x_{n+i}. By establishing the time derivative(s) for the new state variables $\mathrm{d}x_{n+i}/\mathrm{d}t$, an extended system model is obtained, which is the rational substitute system. To compute this transformation automatically, an algorithm is developed in [11] for *analytical input-affine systems* (Σ_{ALS}). To convert an Σ_{AS} into an Σ_{ALS}, the possiblity of an additional transformation exists. As this section handles Σ_{AS}, this kind of system transformation is extended from Σ_{ALS} to Σ_{AS}. This can be achieved for all technically relevant non-rational functions of the original system, which can be expanded in a Taylor series. Those functions are e.g. trigonometrical, logarithmical, exponential, square root etc. functions. The following example demonstrates the strategy:

Example: *Rational substitute system*
Consider the Σ_{AS} with inputs u_1, u_2, disturbance d and outputs y_1, y_2 given as:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} u_1\sqrt{x_1-1} + e^{x_2} \\ u_2 \sin(x_2) + d \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix}.$$

The non-rational partial functions are determined as

$$\begin{aligned} x_3 &:= r_1(x_1) = \sqrt{x_1 - 1}, \\ x_4 &:= r_2(x_2) = e^{x_2}, \\ x_5 &:= r_3(x_2) = \sin(x_2), \\ x_6 &:= r_4(x_2) = \cos(x_2). \end{aligned}$$

The time derivatives of the additional states lead to

$$\begin{aligned} \dot{x}_3 &:= \frac{\mathrm{d}}{\mathrm{d}t} r_1(x_1) = \dot{x}_1 \frac{1}{2\sqrt{x_1-1}} = \frac{\dot{x}_1}{2x_3}, \\ \dot{x}_4 &:= \frac{\mathrm{d}}{\mathrm{d}t} r_2(x_2) = \dot{x}_2 e^{x_2} = x_4 \dot{x}_2, \\ \dot{x}_5 &:= \frac{\mathrm{d}}{\mathrm{d}t} r_3(x_2) = \dot{x}_2 \cos(x_2) = x_6 \dot{x}_2, \\ \dot{x}_6 &:= \frac{\mathrm{d}}{\mathrm{d}t} r_4(x_2) = -\dot{x}_2 \sin(x_2) = -x_5 \dot{x}_2. \end{aligned}$$

Substituting $\dot{x}_1 = u_1 x_3 + x_4$ and $\dot{x}_2 = u_2 x_5 + d$ yields the rational substitute system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} u_1 x_3 + x_4 \\ u_2 x_5 + d \\ \dfrac{u_1 x_3 + x_4}{2x_3} \\ x_4(x_5 u_2 + d) \\ x_6(x_5 u_2 + d) \\ -x_5(x_5 u_2 + d) \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix}.$$

3.2 Algorithm for rational systems

The structure of the algorithm is subdivided into four steps. The first step includes all possible methods of simplification. Next, the time derivatives of the system output up to the system order n are calculated in the second step. Verifying the differential algebraic condition for disturbance decoupling $\rho_{\boldsymbol{d}}^* \leq \rho^*$ is checked in the third step. Hence, the control law is computed in the fourth and final step. The final step is an iteration step, which rewrites the time derivatives of the output $\tilde{\boldsymbol{y}}^{(l)}$ into the normal form $\boldsymbol{N}_l(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u})$. Solving this set of equations in normal form with respect to the inputs $\boldsymbol{u}$, yields the terms of inputs resp. the control variables held in the set $\mathbb{U}_l$. If the set of equations is uniquely solvable and the number of possible inputs or step n has reached, the algorithm terminates and the control law consists of the set terms.

Advanced algorithm:

Step 1:

Simplify the rational system by converting rational functions to a common denominator, factorization, expanding an expression, partial fraction and/or multidimensional division of polynomials.

Step 2:

Compute the time derivatives of the system output up to the system order n: $\dot{\boldsymbol{y}}, \ddot{\boldsymbol{y}}, \ldots, \boldsymbol{y}^{(n)}$.

Step 3:

3.1: Calculate the *differential output rank* $\rho_{\boldsymbol{d}}^*$ of Σ_{RS} by computing the dimensions $\dim \overline{\mathcal{E}}_k$ of the common (non-differential) vector spaces $\overline{\mathcal{E}}_k$ via determination of the rank of Jacobian matrices $\overline{\boldsymbol{J}}_k$ ($\overline{\boldsymbol{J}}_0 = \boldsymbol{0}$; $k = 1, \ldots, n$):

$$\dim \overline{\mathcal{E}}_k = n + \operatorname{rank} \overline{\boldsymbol{J}}_k,$$

$$\overline{\boldsymbol{J}}_k = \begin{bmatrix} \dfrac{\partial \dot{\boldsymbol{y}}}{\partial \bar{\boldsymbol{d}}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \bar{\boldsymbol{d}}} & \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \dot{\bar{\boldsymbol{d}}}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{0} \\ \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \bar{\boldsymbol{d}}} & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \dot{\bar{\boldsymbol{d}}}} & \cdots & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \bar{\boldsymbol{d}}^{(k-1)}} \end{bmatrix},$$

with $\bar{\boldsymbol{d}} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{d} \end{bmatrix} \Rightarrow \rho_{\boldsymbol{d}}^* = \dim \overline{\mathcal{E}}_n - \dim \overline{\mathcal{E}}_{n-1}$.

3.2: Calculate the *differential output rank* ρ^* of $\Sigma_{\mathrm{RS}}^{\boldsymbol{d} \equiv 0}$ by computing the dimensions $\dim \mathcal{E}_k$ of the common (non-differential) vector spaces $\mathcal{E}_k$ via determination of the rank of Jacobian matrices $\boldsymbol{J}_k$ ($\boldsymbol{J}_0 = \boldsymbol{0}$; $k = 1, \ldots, n$) (see [4, 15]):

$$\dim \mathcal{E}_k = n + \operatorname{rank} \boldsymbol{J}_k,$$

$$\boldsymbol{J}_k = \begin{bmatrix} \dfrac{\partial \dot{\boldsymbol{y}}}{\partial \boldsymbol{u}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \boldsymbol{u}} & \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \dot{\boldsymbol{u}}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{0} \\ \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{u}} & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \dot{\boldsymbol{u}}} & \cdots & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{u}^{(k-1)}} \end{bmatrix},$$

$$\Rightarrow \rho^* = \dim \mathcal{E}_n - \dim \mathcal{E}_{n-1}.$$

3.3: If the condition $\rho_{\boldsymbol{d}}^* \leq \rho^*$ holds, the system is disturbance decouplable and the computation of the control law starts in step 4.

Step 4:

4.1: Rewrite the first time derivative of the output $\dot{\boldsymbol{y}}$ into the normal form

$$\boldsymbol{N}_1(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) = \dot{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) - \dot{\boldsymbol{y}} = \boldsymbol{0}.$$

Let $\mathbb{U}_0 := \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ be the initial set of all possible control variables. Therefore, the auxiliary set of control variables is defined by

$$\tilde{\mathbb{U}}_1 := \{\boldsymbol{u} \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m) | \boldsymbol{u} = \boldsymbol{r}_1(\boldsymbol{x}, \boldsymbol{d})\},$$

as the set of all roots of $\boldsymbol{u}$ for

$$\boldsymbol{N}_1(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) = \boldsymbol{0}.$$

Here, $\boldsymbol{r}_1(\boldsymbol{x}, \boldsymbol{d})$ represents the solution of the set of equations. The set of all possible control variables is further restricted to

$$\mathbb{U}_1 := \begin{cases} \mathbb{U}_0 \cap \tilde{\mathbb{U}}_1 & , \text{ for } \tilde{\mathbb{U}}_1 \neq \emptyset \\ \mathbb{U}_0 & , \text{ for } \tilde{\mathbb{U}}_1 = \emptyset. \end{cases}$$

4.2: If the number of determined control variables of $\mathbb{U}_1$ does not conform with the number of system inputs $\boldsymbol{u}$, the remaining terms for $\boldsymbol{u}$ are computed in the next step k.

Step k ($k \geq 5$): Let $l := k - 3$.

k.1: Case distinction:

a) If $2 \leq l \leq n$, then no further computation of $\boldsymbol{y}^{(l)}$ is needed (c. f. step 2).

b) If $l > n$, then compute $\boldsymbol{y}^{(l)}$.

k.2: Rewrite the time derivative of the output

$$\tilde{\boldsymbol{y}}^{(l)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{y}^{(l-1)}|_{\boldsymbol{u}\in\mathbb{U}_l}\right)$$

into the normal form

$$\boldsymbol{N}_l(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u}) = \tilde{\boldsymbol{y}}^{(l)} - \boldsymbol{y}^{(l)} = \boldsymbol{0} \quad .$$

The auxiliary set of control variables for the roots of $\boldsymbol{N}_l(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u})$ w.r.t. $\boldsymbol{u}$ is defined as:

$$\tilde{\mathbb{U}}_l := \{\boldsymbol{u} \in \mathrm{C}^\infty(\mathbb{R},\mathbb{R}^m) | \boldsymbol{u} = \boldsymbol{r}_l(\boldsymbol{x},\boldsymbol{d})\}.$$

The set $\mathbb{U}_l$ is given by:

$$\mathbb{U}_l := \begin{cases} \mathbb{U}_{l-1} \cap \tilde{\mathbb{U}}_l & , \text{ for } \tilde{\mathbb{U}}_l \neq \emptyset \\ \mathbb{U}_{l-1} & , \text{ for } \tilde{\mathbb{U}}_l = \emptyset. \end{cases}$$

k.3: If the number of terms for $\boldsymbol{u}$ in $\mathbb{U}_l$ equals the number m of inputs or if step n has reached, then stop. Furthermore, the set of equations $\boldsymbol{N}_l(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u})$ has to be uniquely solvable. If this does not hold, another iteration step – step k.1 – follows.

The control variables that decouple the system from disturbances are given by all $\boldsymbol{u} \in \mathbb{U}_l$. The highest time derivatives of the output $y_i^{(j)}$ ($i = 1,\ldots,p$ and $j = 1,\ldots,l$) which occur in $\boldsymbol{u}$, have to be substituted with the new inputs v_i. All lower time derivatives of the output are replaced with additional state variables $\boldsymbol{z}$, as representatives for the control dynamics. If the algorithm yields $\rho^*_{\boldsymbol{d}} > \rho^*$, the system is not disturbance decouplable using static state feedback.

4 Application of the DDP-algorithm

Consider the Σ_{RS} with inputs u_1, u_2, disturbance d and outputs y_1, y_2 given as:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dfrac{(x_2x_3^2 - u_1x_2^2)d}{u_1x_3^2 - u_1^2x_2} + u_2 \\ x_2 - u_1u_2 \end{bmatrix}, \; \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Step 1:

Due to the transformation to a common denominator resp. the factorization of the rational function in the first state equation and cancellation of the term ${x_3}^2 - u_1x_2$, the system model can be simplified as:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dfrac{x_2d}{u_1} + u_2 \\ x_2 - u_1u_2 \end{bmatrix}, \; \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (1)$$

Step 2:

Compute the time derivatives of the system output up to the system order $n = 2$:

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \dfrac{dx_1}{u_1} + u_2 \\ x_2 - u_1u_2 \end{bmatrix} \text{ and}$$

$$\ddot{\boldsymbol{y}} = \begin{bmatrix} \dfrac{x_1\dot{d}u_1 + x_2d^2 + du_1u_2 - x_1d\dot{u}_1}{u_1^2} + \dot{u}_2 \\ x_2 - u_1u_2 - \dot{u}_1u_2 - u_1\dot{u}_2 \end{bmatrix}.$$

Step 3:

3.1: With the Jacobian matrices for $\bar{\boldsymbol{d}} = [u_1 \; u_2 \; d]^{\mathrm{T}}$

$$\overline{\boldsymbol{J}}_1 := \frac{\partial\dot{\boldsymbol{y}}}{\partial\bar{\boldsymbol{d}}} \text{ and } \overline{\boldsymbol{J}}_2 := \begin{bmatrix} \dfrac{\partial\dot{\boldsymbol{y}}}{\partial\bar{\boldsymbol{d}}} & \boldsymbol{0} \\ \dfrac{\partial\ddot{\boldsymbol{y}}}{\partial\bar{\boldsymbol{d}}} & \dfrac{\partial\ddot{\boldsymbol{y}}}{\partial\dot{\bar{\boldsymbol{d}}}} \end{bmatrix},$$

$\rho^*_{\boldsymbol{d}}$ can be calculated as:

$$\begin{aligned} &\dim \overline{\mathcal{E}}_1 = n + \operatorname{rank} \overline{\boldsymbol{J}}_1 = 2 + 2 = 4 \;, \\ &\dim \overline{\mathcal{E}}_2 = n + \operatorname{rank} \overline{\boldsymbol{J}}_2 = 2 + 4 = 6 \;, \\ &\Rightarrow \rho^*_{\boldsymbol{d}} = \dim \overline{\mathcal{E}}_2 - \dim \overline{\mathcal{E}}_1 = 6 - 4 = 2. \end{aligned}$$

3.2: With the Jacobian matrices for $\boldsymbol{u} = [u_1 \; u_2]^{\mathrm{T}}$

$$\boldsymbol{J}_1 := \frac{\partial\dot{\boldsymbol{y}}}{\partial\boldsymbol{u}} \text{ and } \boldsymbol{J}_2 := \begin{bmatrix} \dfrac{\partial\dot{\boldsymbol{y}}}{\partial\boldsymbol{u}} & \boldsymbol{0} \\ \dfrac{\partial\ddot{\boldsymbol{y}}}{\partial\boldsymbol{u}} & \dfrac{\partial\ddot{\boldsymbol{y}}}{\partial\dot{\boldsymbol{u}}} \end{bmatrix},$$

ρ^* can be calculated as:

$$\begin{aligned} &\dim \mathcal{E}_1 = n + \operatorname{rang} \boldsymbol{J}_1 = 2 + 2 = 4 \;, \\ &\dim \mathcal{E}_2 = n + \operatorname{rang} \boldsymbol{J}_2 = 2 + 4 = 6 \;, \\ &\Rightarrow \rho^* = \dim \mathcal{E}_2 - \dim \mathcal{E}_1 = 6 - 4 = 2. \end{aligned}$$

3.3: Consequently, the condition $\rho^*_{\boldsymbol{d}} \leq \rho^*$ holds and the control law for the disturbance decoupling can be worked out as follows in step 4.

Step 4:

4.1: The normal form $\boldsymbol{N}_1(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u}) = \boldsymbol{0}$ is given by

$$\boldsymbol{N}_1(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u}) = \begin{bmatrix} \dfrac{dx_1}{u_1} + u_2 - \dot{y}_1 \\ x_2 - u_1u_2 - \dot{y}_2 \end{bmatrix}.$$

The solution of this set of equations w.r.t. the inputs $\boldsymbol{u}$ is given by

$$\mathbb{U}_1 = \left\{ \boldsymbol{u} \middle| u_1 = \frac{x_2d + x_2 - \dot{y}_2}{\dot{y}_1}, \; u_2 = \frac{(x_2 - \dot{y}_2)\,\dot{y}_1}{dx_2 + x_2 - \dot{y}_2} \right\},$$

since $\mathbb{U}_0 := \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m)$,

$$\tilde{\mathbb{U}}_1 := \{\boldsymbol{u} \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m) | \boldsymbol{u} = \boldsymbol{r}_1(\boldsymbol{x}, \boldsymbol{d})\}$$

and hence, $\mathbb{U}_1 := \mathbb{U}_0 \cap \tilde{\mathbb{U}}_1$.

4.2: The number of controls in $\mathbb{U}_1$ complies exactly with the number of inputs $m = 2$ and furthermore, the set of equations is uniquely solvable. Thus, the algorithm terminates.

Substituting the time derivatives of the outputs $\dot{y}_1$, $\dot{y}_2$ with the new inputs v_1, v_2 yields the control law

$$\boldsymbol{u}(\boldsymbol{x}, d, \boldsymbol{v}) = \begin{bmatrix} \dfrac{x_2 d + x_2 - v_2}{v_1} \\ \dfrac{(x_2 - v_2)\, v_1}{d x_2 + x_2 - v_2} \end{bmatrix}, \qquad (2)$$

$$\forall\ (v_1 \neq 0) \wedge (d x_2 + x_2 - v_2 \neq 0).$$

Finally, we obtain the linearized and disturbance decoupled substitute system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \ \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The fact that the substitute system is disturbance decoupled, can be proved easily. By computing the time derivatives of the substitute system output $\boldsymbol{y}$ the condition $\dfrac{\partial \boldsymbol{y}^{(k)}}{\partial d} = \boldsymbol{0}$ holds for any $k \in \mathbb{N}$.

The plant and control law are implemented in the simulation program MATLAB®/SIMULINK®. The initial state variables are chosen to $x_{1,0} = 1$, $x_{2,0} = 3$. For this system implementation a simulation study with test signals as the new inputs and a constant disturbance signal are realized. The system inputs of the uncontrolled system (1) are sine curves:

$$u_1 := 5\sin(4t + 0.5) + 10, \quad u_2 := 2\sin(5t) + 4.25.$$

If no DDP-controller is applied, the system output y_1 is dependent of any constant disturbance $d \in \{0; 5; 10; 50\}$ as shown in figure 1a and y_2 is independent of d (see figure 1b).

Applying the DDP-controller with the new inputs

$$v_1 := 5\sin(4t + 0.5) + 10, \quad v_2 := 2\sin(5t) + 4.25,$$

the control variables u_1, u_2 are visualized in figures 2a/b and show the mode of operation and the excellent performance of the controller for the time-varying controller inputs v_1 and v_2. For constant disturbances d the outputs y_1, y_2 are made independent by using the DDP-controller as presented in figure 3. The system ouput errors

$$\tilde{e}_1 = y_1^{d \not\equiv 0} - y_1^{d \equiv 0} \quad \text{and}$$
$$\tilde{e}_2 = y_2^{d \not\equiv 0} - y_2^{d \equiv 0}$$

between the disturbed and undisturbed controlled system are zero and shown in figure 4.

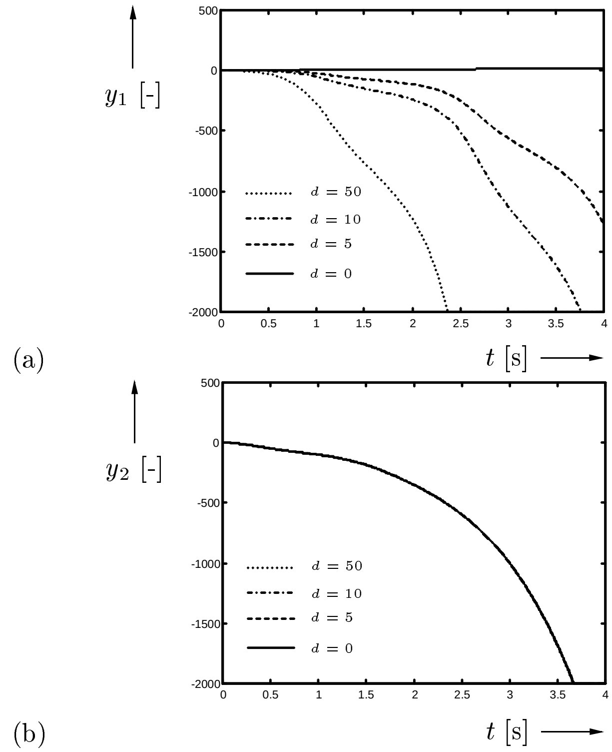

Figure 1: Uncontrolled system outputs: (a) y_1 (b) y_2 ($u_1 := 5\sin(4t{+}0.5){+}10$, $u_2 := 2\sin(5t){+}4.25$)

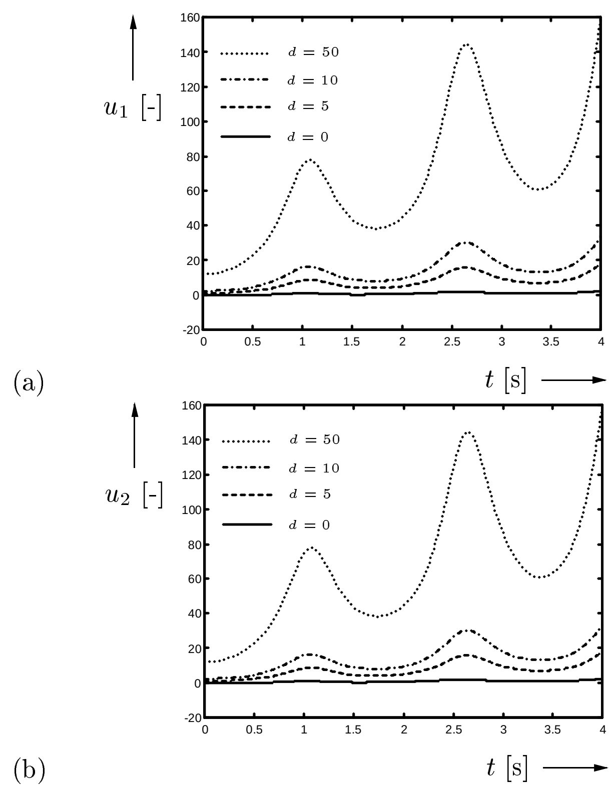

Figure 2: Control variables with DDP-controller: (a) u_1 (b) u_2

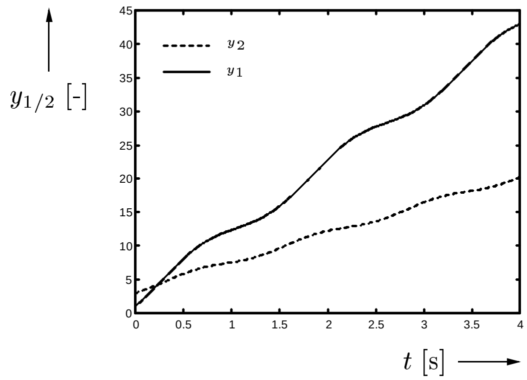

Figure 3: System outputs y_1 and y_2 with DDP-controller ($v_1 := 5\sin(4t+0.5)+10$, $v_2 := 2\sin(5t)+4.25$, $d \in \{0; 5; 10; 50\}$)

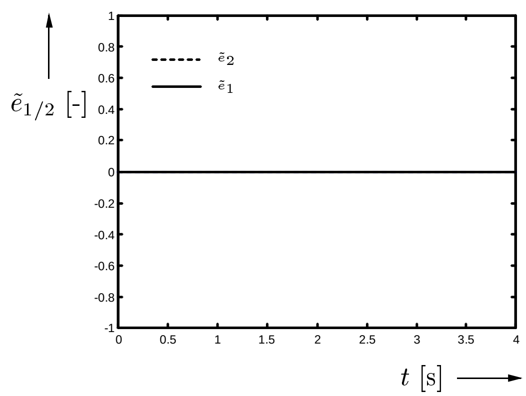

Figure 4: System ouput errors $\tilde{e}_1$, $\tilde{e}_2$ with DDP-controller

5 Conclusion

This paper presents an algorithm to solve the disturbance decoupling problem. The algorithm is based on differential algebra and subsequently, the system must be described by rational differential equations. However, using a transformation of the system shown here, the disturbance decoupling can be solved for most analytical systems. The algorithm checks if the analysed system can be disturbance decoupled or not by a separate condition. In case of disturbance decouplability, the computation of the control law follows. For a MIMO-system the disturbance decoupling problem is solved with the presented algorithm. To illustrate the function of the control law simulations for several disturbances and new inputs are included. It is proven that the disturbances have no more influence on the outputs. Further investigations are required to reduce disturbances in systems which are not disturbance decouplable.

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