

Solutions of Nonlinear Optimal and Robust Control Problems via a Mixed Collocation/DAE's Based Algorithm

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Abstract

A new algorithm for computing the solutions of nonlinear optimal and robust H_∞ control problems is proposed. The algorithm is based on the use of the collocation method to transform the PDE's into ODE's. Some convergence results are given and several examples are presented.

1 Introduction

The formulations of \mathcal{H}_2 optimal and \mathcal{H}_∞ robust control problems for nonlinear systems are two conceptually powerful means for the design of nonlinear control laws addressing quite sophisticated performance objectives. Theoretical results in nonlinear \mathcal{H}_2 control were developed in the late 1950's and 1960's [1, 2, 3] while nonlinear \mathcal{H}_∞ theory has been developed more recently [4, 5, 6].

The existing approaches to solve the resulting Hamilton-Jacobi-Isaacs (HJI) equations [7, 8, 9, 10] share a technical requirement that is inherently associated to the Galerkin method, namely the need to perform a high number of multi-dimensional (generally n -dimensional) quadratures.

In this paper, it is shown that these quadratures can be avoided if collocation approach is adopted provided that an adequate stabilizing procedure is used to compensate for the lack of precision generally associated to this approach.

This paper is organized as follows : First, the nonlinear optimal and/or robust control problem is briefly recalled together with some basic results from the dynamic games theory. In section 3, some functional approximation related definitions and results are presented as well as a brief presentation of the post stabilization technique's main results. The proposed algorithm is then clearly introduced in section 4 together with some convergence analysis. Finally, the proposed algorithm is applied to some examples.

2 Problem Formulation

Consider nonlinear systems given by

$$\dot{x} = f(x, u, w, t) \quad x \in \mathbb{R}^n ; u \in \mathbb{R}^m ; w \in \mathbb{R}^p \quad (1)$$

where x is the state, u is the control and w is some unknown exogenous input and/or vector of parameter uncertainties. The control u and the exogenous input w are subject to the following constraints for all $t \geq 0$

$$u(t) \in U(x(t)) \subseteq \mathbb{R}^m ; w(t) \in W(x(t)) \subseteq \mathbb{R}^p \quad (2)$$

Most of control engineering problems amount to solve the following optimization problem

Find a feedback strategy $u(t, x) \in U(x)$ such that

$$u(t, x(t)) := \text{Arg} \left\{ \min_{u(\cdot)} \max_{w(\cdot)} J_T(x(t), u(\cdot), w(\cdot), t) \right\} \quad (3)$$

for all $t \geq 0$ under (1)

where

$$J_T(x(t), u(\cdot), w(\cdot), t) := \Psi(x(T)) + \int_t^T L(x(\tau), u(\tau), w(\tau), \tau) d\tau \quad (4)$$

where $L(\cdot)$ is some penalty function expressing the control objectives. In connection with the above formulation, the Hamiltonian H is defined for all $\lambda \in \mathbb{R}^n$ by

$$H(x, \lambda, u, w, t) := \lambda^T f(x, u, w) + L(x, u, w, t) \quad (5)$$

Throughout this paper, we suppose that the following assumption holds

Assumption 1 [Existence of saddle points for H]
For all $(x, \lambda, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, there are $\hat{u}(x, \lambda, t)$ and $\hat{w}(x, \lambda, t)$ solutions of the following static game

$$\hat{H}(x, \lambda, t) := \min_{u \in U(x)} \max_{w \in W(x)} H(x, \lambda, u, w, t) \quad (6)$$

Under the above assumption, Isaacs' "Verification Theorem" may be applied [11, 12]

Theorem 1 [Isaacs verification theorem]

If $V(t, x)$ is a function of class C^1 in t and x that satisfies the following Hamilton Jacobi Equation with boundary condition

$$V_t + \hat{H}(x, V_x, t) = 0 \quad ; \quad V(T, x) = \Psi(x) \quad (7)$$

where \hat{H} is given by (6), **then** the control strategy

$$u(t, x(t)) = \hat{u}(x(t), V_x(t, x(t)), t) \quad (8)$$

is an optimal solution for problem (3). Furthermore, the corresponding optimal value is exactly $V(t, x(t))$. ♠

Very frequently, one is interested in finding a time independent **non negative definite** function $V(x)$, solution of the stationary version of equation (7), namely

$$\hat{H}(x, V_x) = 0 \quad ; \quad V \text{ is nonnegative definite} \quad (9)$$

The aim of this paper is basically to propose a simple and efficient algorithm to solve equations (7) or (9) together with some analysis regarding the conditions under which it may converge.

3 Background

3.1 Functional Approximation Related Definitions

Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact subset over which equations (7) or (9) are to be approximately solved.

Let denote by $(\phi_i)_{i=1}^\infty$ a **complete** set of C^1 basis functions $\phi_j : \mathbb{R}^n \rightarrow \mathbb{R}$ [15]. The space of all scalar functions that are linear combinations of $(\phi_j)_{j=1}^N$ will be denoted by \mathbf{V}_N . In this paper, we assume that the compact subset under interest \mathcal{S} contains a set of regular points $(x^i)_{i=1}^N$ w.r.t $(\phi_j)_{j=1}^N$, and this, for all $N \in \mathbb{N}$. In what follows, the following notation is used

$$\Phi_N(x) := \left(\phi_1(x), \dots, \phi_N(x) \right)^T \quad (10)$$

Furthermore, when a set of interpolation points $(x^i)_{i=1}^{n_p}$ (containing a regular subset) is fixed, we define the $n_p \times N$ interpolation matrix M by

$$M := \begin{pmatrix} \Phi_N^T(x^1) \\ \dots \\ \Phi_N^T(x^{n_p}) \end{pmatrix} \in \mathbb{R}^{n_p \times N} \quad (11)$$

Note that since $(x^i)_{i=1}^{n_p}$ contains a regular subset, we necessarily have $\text{rank}(M) = N$ or, equivalently, that $M^T M$ is invertible. Furthermore, given any function $V(x)$, the least squares approximation of $V(\cdot)$ by elements of \mathbf{V}_N using the set of measures $\left((x^i), V(x^i) \right)_{i=1}^{n_p}$ is readily given by

$$V_N(x) = \left[\Phi_N^T(x) [M^T M]^{-1} M^T \right] \begin{pmatrix} V(x^1) \\ \vdots \\ V(x^{n_p}) \end{pmatrix} \quad (12)$$

Finally, given a function $F : \mathbb{R}^s \times \mathbb{R}^n \rightarrow \mathbb{R}^q : (y, x) \rightsquigarrow F(y, x)$, let us define $\bar{F} : \mathbb{R}^s \rightarrow \mathbb{R}^{q \cdot n_p}$ as follows

$$\bar{F}(y) = \left(F^T(y, x^1), \dots, F^T(y, x^{n_p}) \right)^T \in \mathbb{R}^{q \cdot n_p} \quad (13)$$

this notation will be extensively used in the the formulation of our algorithm.

3.2 Review of stabilization of ODE's with invariant sub-manifolds

Consider a nonlinear differential system given by

$$\dot{z} = f(z) \quad ; \quad f \text{ continuous} \quad (14)$$

and suppose that, theoretically, the above differential system admits the following invariant sub-manifold

$$\mathcal{M} := \left\{ z \mid Lz = 0 \right\} \quad (15)$$

Equations (14) and (15) can be viewed as an overdetermined system of **DAE**'s satisfied by the exact solution for all initial conditions $z_0 \in \mathcal{M}$. Once the system is discretized in order to solve it numerically, however, the full over-determined system may not have a solution anymore. The question becomes how to design an integration method that explicitly takes into account the above invariance property to yield a stable and consistent integration process

Proposition 1 [Continuous Stabilization][19]

Consider the ODE (14). Apply the stabilization

$$\dot{z} = f(z) - \alpha K z \quad (16)$$

where $K = L^T Q$ and Q is a solution of the linear system $(LL^T)Q = L$. Then, **if** there is some $\alpha_0 \geq 0$ such that

$$\|Lf(z)\| \leq \alpha_0 \|Lz\| \quad \text{locally near } \mathcal{M} \quad (17)$$

then, under the dynamics (16), \mathcal{M} is a locally asymptotically stable invariant manifold for any $\alpha > \alpha_0$. ♣

The stabilized system (16) must now be discretized in order to obtain a numerical solution. It is a known fact that this discretization process may destabilize the invariant set \mathcal{M} . The so called "Post Stabilization" technique enables the precision and the stability of a given numerical scheme to be increased. Namely, if an approximate solution $z_n := z(t_n)$ of (14) is to be found at sampling instants t_n such that $t_{n+1} = t_n + k$, the following equations is used to generate $(z_n)_{n \geq 0}$:

$$\tilde{z}_{n+1} = \theta_k^p(z_n), \quad (18a)$$

$$z_{n+1} = (I - K) \tilde{z}_{n+1} \quad (18b)$$

where $\theta_k^p(z_n)$ is the numerical solution at $t_{n+1} = t_n + k$ of (16) using a p^{th} order integration scheme and starting at the initial condition $z(t_n) = z_n$.

Suppose that one is interested in finding the solution of (14)-(15) over $[0, T]$ where $T > 0$ is some fixed final time. Suppose however that only approximated knowledge of $f(z)$ is possible. More precisely, consider the modified version of equations (18a)-(18b)

$$\tilde{z}_{n+1}^\epsilon = \theta_k^p(z_n^\epsilon, \epsilon(\cdot)), \quad (19a)$$

$$z_{n+1}^\epsilon = (I - K)\tilde{z}_{n+1}^\epsilon \quad (19b)$$

where $\theta_k^p(z_n, \epsilon(\cdot))$ is the solution at instant $t_{n+1} = t_n + k$ with initial condition $z(t_n) = z_n$ of the following differential system

$$\dot{z} = f(z) + \epsilon(t) \quad (20)$$

and $\epsilon(\cdot)$ is some error profile affecting the knowledge of f .

Using some straightforward continuity arguments concerning the solutions of ODE's with continuous r.h.s over a finite time horizons [20], the following result can be easily established

Proposition 2 [Case of perturbed dynamics]

If the nominal system (14)-(15) admits a solution on $[0, T]$, then there is sufficiently small $\epsilon_0 > 0$ such that the perturbed system (20) admits a solution on $[0, T]$ for all $\epsilon(\cdot)$ satisfying $\|\epsilon\|_\infty^{[0, T]} \leq \epsilon_0$.

Furthermore, For all n s.t $t_n \in [0, T]$,

$$\lim_{\|\epsilon\|_\infty^{[0, T]} \rightarrow 0} \|z_n^\epsilon - z_n\| = 0 \quad (21)$$

♣

4 Main Results

4.1 The Proposed Algorithm

In order to progressively introduce our algorithm, we shall **temporarily** assume that the solution $V(t, x)$ of (7) is such that $V(t, \cdot)$ belongs to \mathbf{V}_N for all t [see (??)]. The above assumption on $V(t, x)$ [$V(t, \cdot) \in \mathbf{V}_N$] together with the regularity of $(x^i)_{i=1}^{n_p}$ enables to write for all t and all x

$$V(t, x) = \left[\Phi_N^T(x) [M^T M]^{-1} M^T \right] \bar{V}(t) \quad (22)$$

where $M \in \mathbb{R}^{n_p \times N}$ is given by (11) and $\bar{V}(t)$ is the function associated to $V(t, x)$ according to (13). From (22), it is possible to express $V_x(t, x)$ in terms of $\bar{V}(t)$, let $V_x(t, x) =: G(\bar{V}(t), x)$ and define $\Gamma(t, \bar{V}, x)$ as follows

$$\Gamma(t, \bar{V}, x) := -\hat{H}(x, G(\bar{V}, x), t) \quad (23)$$

then, we can use (7) to derive the dynamic equation for \bar{V} together with the associated final condition

$$\frac{d\bar{V}}{dt} = \bar{\Gamma}(t, \bar{V}) \quad ; \quad \bar{V}(T) = \bar{\Psi} \quad (24)$$

where, again, $\bar{\Gamma}$ [resp. $\bar{\Psi}$] is derived from $\Gamma(t, \bar{V}, x)$ [resp. $\Psi(x)$] using (13).

The differential system (24) with the associated final condition, when integrated backward in time with an infinite precision and using (22) gives the exact solution $V(t, x)$ we search for. The point is that the high dimension of the differential system (24) and the potentially long period over which integration is to be done may lead to a harmful error propagation phenomena. That is the reason why the following algebraic constraint is added to strength the fact that $V(t, \cdot) \in \mathbf{V}_N$

$$\left[I_{n_p \times n_p} - M (M^T M)^{-1} M^T \right] \bar{V} =: L\bar{V} = 0 \quad (25)$$

To sum up, we have to integrate the ODE's (24) under the algebraic constraint (25). To do this, we shall use the techniques recalled in section 3.2 to solve the DAE's (14)-(15). Keeping in mind the notations of section 3.2, we have the following correspondences

$$z \longleftrightarrow \bar{V} \quad ; \quad f \longleftrightarrow \bar{\Gamma} \quad ; \quad L \text{ given by (25)}$$

Based on the above discussion, we propose the following algorithm :

ALGORITHM

1. Choose a complete set of C^1 basis functions $(\phi_i(x))_{i=1}^\infty$,
2. Choose a sufficiently high $N \in \mathbb{N}$,
3. Choose a fixed set of collocation points $(x^i)_{i=1}^{n_p}$ that contains a regular set w.r.t $(\phi_i(x))_{i=1}^N$,
4. Compute
 - (a) M according to (11)
 - (b) $\Sigma := M(M^T M)^{-1} M^T$
 - (c) $L = I - \Sigma$ [see (25)]
 - (d) $K = L^T Q$ where Q is a solution of the linear system $(LL^T)Q = L$ [see proposition 1].
5. Choose sufficiently high $\alpha > 0$, a sampling period $k > 0$, a small parameter $\epsilon > 0$ and $T \in \mathbb{R}_+ \cup \{\infty\}$.
6. $n = 0$; $\bar{V}_n = \bar{\Psi}$
7. Compute $\tilde{\bar{V}}_{n+1}$ solution at k of [see (18a)]

$$\dot{\bar{V}}(\tau) = -\bar{\Gamma}(T - \tau, \bar{V}) - \alpha K \bar{V}(\tau) \quad ; \quad \bar{V}(0) = \bar{V}_n$$

8. $\bar{V}_{n+1} = (I - K)\tilde{\bar{V}}_{n+1}$ [see (18b)]
9. **If** $(nk \geq T)$ **OR** $(\|\dot{\bar{V}}_n\|_\infty \leq \varepsilon)$ **Then STOP**
Else $n = n + 1$, **GOTO Step 7**
10. The approximate solution is given by

$$V(t, x) \approx \left[\Phi_N^T(x) [M^T M]^{-1} M^T \right] \bar{V}_{n+1}(t) \quad (26)$$

4.2 Some convergence results

In this section, the convergence of the above algorithm is investigated. Let us begin by the ideal case,

Proposition 3

If there is a solution $V(t, x)$ of equations (7) such that for all $t \geq 0$ one has $V(t, \cdot) \in \mathbf{V}_N$ **then** the differential system (24) together with (22) admits an exact solution of (7).

Furthermore, $L\bar{V} = 0$ is an invariant sub-manifold w.r.t the dynamic $\bar{\Gamma}(t, \bar{V})$ and hence **steps 7 and 8** of the above algorithm implements the post stabilisation technique on (24) yielding a high precision approximation of $V(t, x)$ with low computation cost for high α 's **whenever** there is some $\alpha_0 \geq 0$ such that $\|\bar{L}\bar{\Gamma}(t, \bar{V})\| \leq \alpha_0 \|L\bar{V}\|$ locally near $L\bar{V} = 0$. ♠

In the case where there is a solution $V(t, x)$ that is not entirely contained in \mathbf{V}_N , we first need the following lemma

Lemma 1

Suppose that there is a continuously differentiable solution $V(t, x)$ of (7) over $[0, T]$. Let $S \subset \mathbb{R}^n$ be some compact subset of interest and denote by V_N the best least squares approximation of V by an element that lies in \mathbf{V}_N .

Then there is a family of functions $\delta_N(t, x)$ satisfying the following conditions

1. $\lim_{N \rightarrow \infty} \|\delta_N(t, \cdot)\|_\infty^S = 0$ for all $t \in [0, T]$,
2. $V_N(t, x)$ is the solution of the modified optimal control problem

$$\Psi(x(T)) + \int_t^T [L(x, u, w, \tau) - \delta_N(\tau, x)] d\tau \quad (27)$$

♠

It can then be readily shown that \bar{V}_N is solution of the following ODE's that is to be compared to (24) :

$$\frac{d\bar{V}}{dt} = \bar{\Gamma}_N(t, \bar{V}) := \bar{\Gamma}(t, \bar{V}) + \bar{\delta}_N(t) ; \quad \bar{V}(T) = \bar{\Psi} \quad (28)$$

Consequently, the sub-manifold $L\bar{V} = 0$ is invariant w.r.t the dynamic defined by $\frac{d\bar{V}}{dt} = \bar{\Gamma}_N(t, \bar{V})$. The above discussion enables the statement of the following main convergence result

Proposition 4 [Main convergence result]

Suppose that there is an exact C^1 solution $V_{ex}(t, x)$ of (7) on $[0, T]$. Let S be some compact set of interest and denote by V_N the best least squares approximation of V by an element of \mathbf{V}_N . **if** there exists $\alpha_0 > 0$ such that $\|\bar{L}\bar{\Gamma}_N(t, \bar{V})\| \leq \alpha_0 \|L\bar{V}\|$ locally near $L\bar{V} = 0$ **then**, The outputs $(\bar{V}_n)_{n \leq T/k}$ of the proposed algorithm are such that

$$\sup_{x \in S, n \leq T/k} \left\| \left[\Phi_N^T(x) [M^T M]^{-1} M^T \right] \bar{V}_n - V_{ex}(t_n, x) \right\| \rightarrow 0$$

As $N \rightarrow \infty$. ♣

5 Illustrative examples

Software has been developed using **digital Fortran** environment while runs have been performed on a **150MHz PC-Pentium I**. Integration have been performed using **fourth order Runge-Kutta method**. The tables that show the convergence results present the evolution with time steps of the following three key quantities: The maximum value of V over the collocation points $(x_i)_{i=1}^{n_p} : \|\bar{V}\|_\infty$, The maximum absolute value of \dot{V} over the collocation points $(x_i)_{i=1}^{n_p} : \|\dot{\bar{V}}\|_\infty$ and The maximum value of the post stabilization related correction $\|K\tilde{\bar{V}}\|_\infty$ performed in **step 8** of the algorithm.

5.1 Example 1

5.1.1 Problem Statement: As a first example, let us consider the following nonlinear system that has been considered in [8]

$$\dot{x} = \begin{pmatrix} -x_1^3 - x_2 \\ x_1 + x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

As it has been pointed out in [8], the above system can be stabilized using the state feedback

$$u(x) = 3x_1^5 + 3x_1^2 x_2 - x_2 + v$$

and using the coordinate transformation

$$z_1 = x_1 \quad ; \quad z_2 = x_1^3 + x_2 \quad (29)$$

the system can be written in the new coordinate z as follows

$$\dot{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

$t = nk$	$\ \bar{V}\ _\infty$	$\ \dot{V}\ _\infty$	$\ K\tilde{V}\ _\infty$
0.25	0.46	1.79	8.61E-07
1.00	1.49	0.88	7.70E-06
2.00	1.94	0.37	5.65E-06
3.00	2.08	0.10	2.45E-06
4.00	2.12	2.18E-02	8.40E-07
4.75	2.13	1.06E-02	5.29E-07

Table 1: Convergence result for example 5.1 (Execution time \approx 5 seconds)

so that the optimal solution of the LQR problem associated to the cost function

$$J = \int_0^\infty [z^T z + v^2] dt$$

is given by $z^T P z$ where P is the solution of LQR problem defined by the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Q = I_{2 \times 2}$ and $R = 1$. Calculations shows that

$$V(x) = 1.9123x_1^2 - 0.8284x_1x_2 + 1.3522x_2^2 - 0.8284x_1^4 + 2.7044x_1^3x_2 + 1.3522x_1^6 \quad (30)$$

In order to test our algorithm, **we hide the coordinate transformation (29)** and define the following nonlinear optimal control problem

$$\dot{x} = \begin{pmatrix} -x_1^3 - x_2 \\ x_1 + 3x_1^5 + 3x_1^2x_2 + v \end{pmatrix}$$

$$J = \int_0^\infty [x_1^2 + (x_1^3 + x_2)^2 + v^2] dt \quad (31)$$

which is in the form (4) with $\Psi \equiv 0$, $T = \infty$ and $L(x, u, w) = x_1^2 + (x_1^3 + x_2)^2 + u^2$. Furthermore, the system being affine in control and the cost function quadratic, assumption 1 is clearly satisfied.

5.1.2 Results: The following functional basis has been considered

$$\left(\phi_i\right)_{i=1}^{28} := \left\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, \dots, x_2^6\right\}$$

furthermore, the parameters have been taken as follows

$$n_p = 56 \quad ; \quad \varepsilon = 0.01 \quad ; \quad \alpha = 5 \quad ; \quad k = 0.05$$

the interpolation points have been randomly chosen in the hyper cube $[-1, 1] \times [-1, 1]$.

Convergence results are shown on table 1.

Neglecting all coefficients that are smaller than 10^{-3} , we get

$$V_{28}(x) = 1.9067x_1^2 - 0.8223x_1x_2 + 1.3480x_2^2 - 0.8223x_1^4 + 2.6961x_1^3x_2 + 1.3480x_1^6$$

which is to be compared to (30).

$t = nk$	$\ \bar{V}\ _\infty$	$\ \dot{V}\ _\infty$	$\ K\tilde{V}\ _\infty$
0.1	4.35E-02	0.347	2.113E-05
1.0	0.276	0.155	1.665E-05
2.0	0.351	2.666E-02	1.490E-05
3.0	0.362	4.832E-03	1.509E-05
3.5	0.363	1.946E-03	1.515E-05

Table 2: Convergence results for example 5.2 (Execution time \approx 25 seconds)

5.2 Example 2

Consider the underwater vehicle considered in [10] to which disturbances have been added :

$$\dot{x} = \begin{pmatrix} -x_1|x_1| + x_3 - x_1|x_1|w_2 + w_1 \\ x_1 \\ -5x_3 + 50u \end{pmatrix} \quad (32)$$

We aim to solve a nonlinear H_∞ disturbance attenuation problem. For, we seek for a stationary solution to the min max optimization problem for $\gamma > 0$ as small as possible

$$\min_u \max_w J = \int_0^\infty \left[\|x(\tau)\|^2 + 0.01u^2(\tau) - \gamma^2 \|w(\tau)\|^2 \right] d\tau$$

The above problem has been successfully solved over the cube $[-0.6, 0.6]^3$ and $\gamma = 0.1$ using a quadratic polynomial in the unknown x_1, x_2 and x_3 and taking the following parameters

$$n_p = 30 \quad ; \quad \varepsilon = 0.001 \quad ; \quad \alpha = 1 \quad ; \quad k = 0.001$$

to yield the following approximate solution over the hyper cube $[-0.6, 0.6]^3$:

$$V(x) \approx 0.0496x_1^2 + 0.0983x_1x_2 + 0.0046x_1x_3 + 1.0473x_2^2 + 0.0046x_2x_3$$

Convergence results are presented on table 2.

5.2.1 Simulation results: Figure 1 shows the closed loop behaviour in both the nominal case and in the case where $w_2 = 1$. Initial state was taken equal to $x_0 = (0.5, 0.15, -0.6)^T$.

Figure 2 shows the closed loop behaviour under the external disturbance w_1 for the same initial state. As one can expect with regard to the small value of the attenuation factor $\gamma = 0.1$, the closed loop system presents a very good disturbance rejection behaviour. This is strengthened in figure 3 where comparison is made between $\int_0^t z^T z d\tau$ and $\gamma^2 \int_0^t w^T w d\tau$ attesting that the H_∞ basic inequality is respected.

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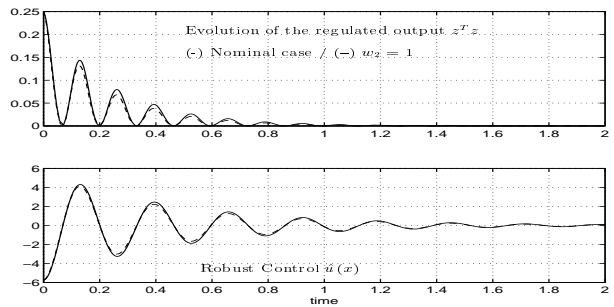


Figure 1: Closed loop behaviour under parameter uncertainties for example 5.2

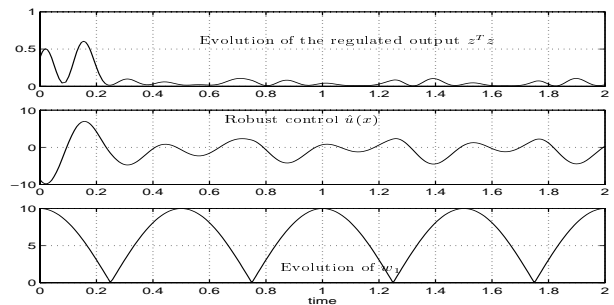


Figure 2: Closed loop behaviour under external disturbances w_1 for example 5.2 - Initial state $x_0 = (0.5, 0.15, -0.6)^T$

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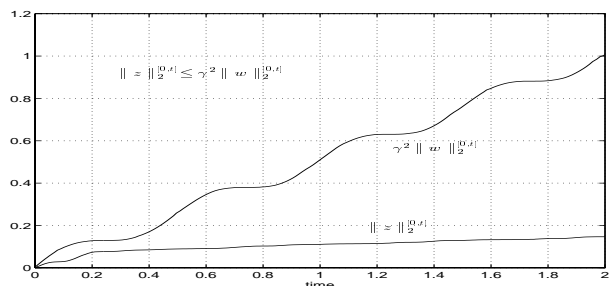


Figure 3: Test of the H_∞ attenuation inequality for example 5.2 - Initial state $x_0 = (0.5, 0.15, -0.6)^T$