Uniform exponential stability for families of linear time-varying systems

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Abstract

We present sufficient conditions for uniform exponential stability of families of linear time varying (LTV) systems. That is, LTV systems characterized by certain parameter. Our conditions are in the form of classical concepts in adaptive control, such as persistency of excitation. However, our proofs are based on modern tools which can be interpreted as an "integral" version of Lyapunov theorems; rather than on the concept of uniform complete observability which is most common in the literature. Uniformity is established in both, the initial conditions of the system, and the parameter which characterizes each system of the 'family'.

Key words : Robust stability and stabilization, observability, model reference adaptive control, persistency of excitation.

Notations. In this note, $\|\cdot\|$ stands for the Euclidean norm of vectors and induced norm of matrices. $\|\cdot\|_\infty$ denotes the $\mathcal{L}_\infty$ norm of signals. Unless otherwise specified we use, in general, the letter c to denote a positive constant. For positive definite matrices we use the bounds $p_m I \leq P \leq p_M I$. The solution of a differential equation, $\dot{x} = f(t,x)$, where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz in x, uniformly in t, with initial conditions $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ with $x_\circ = x(t_\circ)$, is denoted $x(t; t_\circ, x_\circ)$ or simply, $x(t)$. $\dot{V}_{(\#)}(t,x)$ is the time derivative of the Lyapunov function $V(t,x)$ along the solutions of the differential equation (#).

1 Introduction

Stability analysis and control design for families of systems (linear or nonlinear) has been extensively studied; specifically in the context of robust control. In the terms of [6], " a feedback law robustly [asymptotically] stabilizes a parameterized family of systems, if it [asymptotically] stabilizes all the systems in the family". Other definitions of stability for families of systems have been proposed in [5, 3]. For the sake of clarity, we write below the precise definitions that we are concerned with in this note:

Definition 1 (Exponential convergence) *The system $\dot{x} = f(t,x)$ is said to be exponentially convergent, trajectory by trajectory, if, there exists $r > 0$ such that* for each *pair of initial conditions $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$ there exist γ_1 and $\gamma_2 > 0$, such that the solution $x(t, t_\circ, x_\circ)$ of the system satisfies*

$$\|x(t, t_\circ, x_\circ)\| \leq \gamma_1 \|x_\circ\| e^{-\gamma_2 (t - t_\circ)} . \qquad (1)$$

The system is said to be globally exponentially convergent if $r = \infty$. □

Definition 2 (Uniform exponential stability) *The origin of the system $\dot{x} = f(t,x)$ is said to be uniformly (locally) exponentially stable (ULES) if there exist constants γ_1, γ_2 and $r > 0$ such that for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$*

$$\|x(t, t_\circ, x_\circ)\| \leq \gamma_1 \|x_\circ\| e^{-\gamma_2 (t - t_\circ)} \qquad \forall t \geq t_\circ . \quad (2)$$

If for each $r > 0$ there exist γ_1, γ_2 such that condition (2) holds for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$, then the system is said to be uniformly semiglobally exponentially stable[1]*. If (2) holds for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ the system is uniformly globally exponentially stable (UGES).* □

The problem of establishing sufficient conditions for robust stability of families of (non)linear systems appears for instance in the context of *Model Reference Adaptive Control* (MRAC) for linear time invariant systems (LTI). This approach consists on designing a controller so that the closed loop transfer function matches a "reference model". In the case when the constant parameters of the plant are unknown, estimates of these parameters are used in the controller, and an adaptive law is added to update these estimates. What is important to remark in this context is that, even for linear plants, MRA controllers are typically nonlinear. Furthermore, due to exogenous time-varying references, the closed loop system is nonlinear time-varying. For a condensed introduction to this approach see for instance [9]. See also [18] where the problem of nonlinear adaptive control with parameter convergence is addressed for LTI plants.

In the context of robust output feedback stabilization of nonlinear systems, the question of whether a family

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[1] see, e.g. [5].

of linear time varying systems (LTV) is stable, also covers importance. For instance in [10, 8], the authors addressed such problem for feedback linearizable systems. In both of the cited references, the design approach consists of two basic steps: in the first, the authors define an adaptive state feedback controller which guarantees exponential convergence of the tracking and estimation errors; in the second, the output feedback controller is designed and, based on a converse Lyapunov function invoked for the state-feedback error dynamics, exponential convergence for the tracking and estimation errors is proved.

In this paper we will present sufficient conditions for uniform exponential stability of families of linear time-varying systems. Our results are stated in terms of the well known concept of *persistency of excitation* and can be applied in the stability analysis of MRAC systems. For clarity of exposition let us briefly discuss the nonlinear system encountered in the MRAC approach (see e.g. [9, pp. 635]):

$$\begin{bmatrix} \dot e \\ \dot\theta \end{bmatrix} = \begin{bmatrix} A & B\phi(t,z)^\top \\ -\phi(t,z)C^\top & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix},\ z := \begin{bmatrix} e \\ \theta \end{bmatrix} \tag{3}$$

where $e \in \mathbb{R}^n$ represents a tracking error, $\theta \in \mathbb{R}^m$ a parameter estimation error and $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^m$ is a bounded smooth function, and the triple (A, B, C) is strictly positive real, i.e., satisfies the Kalman-Yakubovich-Popov (KYP) lemma (see [9, p.407]). As it is nicely summarized for example in [17, 1, 7], if the regressor ϕ depends only on time, the system (3) is uniformly (globally) exponentially stable if $\phi(t)$ is bounded, absolutely continuous and *persistently exciting* (PE), i.e., if there exist positive constants μ and T such that

$$\mu I \leq \int_t^{t+T} \phi(\tau)\phi(\tau)^\top d\tau\,, \qquad \forall t \geq 0\ . \tag{4}$$

However, when the function ϕ depends also on the state, this result is not applicable straightforwardly. A solution often taken in the literature (see for instance [14, 4, 7, 9, 18, 8]), is to construct an LTV system for each trajectory of (3). Then, under the assumption that the triple (A, B, C) is strictly positive real, exponential convergence of the state, is guaranteed by assuming that each trajectory $z(\cdot, t_\circ, z_\circ)$ produces a function

$$\tilde\phi(t) := \phi(t + t_\circ, z(t + t_\circ, t_\circ, z_\circ)) \tag{5}$$

satisfying (4). In this case, by defining the parameter $\lambda := (t_\circ, z_\circ)$ one can associate a parameterized LTV system to *each* pair of i.c. of the system (3) (equivalently, to each trajectory):

$$\begin{bmatrix} \dot e \\ \dot\theta \end{bmatrix} = \begin{bmatrix} A & B\tilde\phi(t,\lambda)^\top \\ -\tilde\phi(t,\lambda)C^\top & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix},\ z := \begin{bmatrix} e \\ \theta \end{bmatrix} \tag{6}$$

with initial conditions $(t_\star, z_\star)$, can be associated. Notice that in general, $(t_\circ, z_\circ) \neq (t_\star, z_\star)$, however, whenever the initial conditions of both systems (3) and (6) coincide, so do their solutions. The important implication of this is that, if *for each* λ the LTV system (6) is exponentially convergent, each trajectory of (3) converges exponentially fast to zero. If moreover, (6) is exponentially stable, uniformly in the parameter λ and in $(t_\star, z_\star)$ then (3) is also uniformly exponentially stable.

In this paper we show that some results on exponential stability and observability for linear systems, included in the classical books on adaptive control previously cited, can be extended for families of LTV systems. Even though the extension is conceptually straightforward, it is stymied by the fact that all the stability conditions, put in terms of Lyapunov functions, observability conditions and properties of PE regressors, depend on the parameters which characterize the family. Thus we have to restudy certain results to provide sufficient conditions for uniform (hence) robust stability. Besides, we have not been able to locate in the literature, precise arguments to establish uniform exponential stability of parameterized families of linear time-varying systems (6) in terms of persistency of excitation.

2 Preliminaries

Even though our conditions are stated in the form of persistency of excitation, our proofs involve modern tools, which can be interpreted as an "integral" version of Lyapunov theorems. This is in contrast to the lengthy calculations typically done when using the classical concept of uniform complete observability, as we may find in adaptive control books (see for instance [1, 14, 17, 7]). We present in this section the main tools and definitions that we will use. We start with the following stability definition which is the precise property that we will prove for a family of LTV systems including (6).

Definition 3 (λ-UGES) *The origin of the system $\dot x = f(t, \lambda, x)$ is said to be λ-UGES if for any $\lambda \in \mathcal{D}$, there exist two functions $k, \gamma : \mathcal{D} \to \mathbb{R}_{>0}$ such that for all $t \geq 0$*

$$\|x(t, \lambda, t_\circ, x_\circ)\| \leq k(\lambda)\, \|x_\circ\|\, e^{-\gamma(\lambda)(t - t_\circ)},$$

moreover, if $\mathcal{D}$ is non-compact then the bound above holds with

$$k_\lambda := \sup_{\lambda \in \mathcal{D}} k(\lambda) < \infty, \quad \gamma_\lambda := \inf_{\lambda \in \mathcal{D}} \gamma(\lambda) > 0.$$

□

We present next two general lemmas on λ-UGES. Lemma 2 was proven in [15] for nonlinear systems $\dot x = f(t, x)$, however, it can be shown along the same lines that the result holds for families of systems $\dot x = f(t, \lambda, x)$. Lemma 1 can be proven along the same lines as the proof of [9, Theorem 3.12].

Lemma 1 *Let $\lambda \in \mathcal{D} \subset \mathbb{R}^p$ and $\mathcal{A} : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{q\times q}$ be continuous and uniformly bounded for all $\lambda \in \mathcal{D}$ and almost all $t \geq 0$. If the system $\dot{x} = \mathcal{A}(t,\lambda)x$ is λ-UGES then there exist constants c_1, c_2, $c_3 > 0$ and for each $\lambda \in \mathcal{D}$, there exists $P_\circ(t,\lambda)$, such that defining*

$$V_\circ(t,\lambda,x) := x^\top P_\circ(t,\lambda)x \tag{7}$$

we have that

$$c_1 \|x\|^2 \leq V_\circ(t,\lambda,x) \leq c_2 \|x\|^2 \tag{8}$$

$$\dot{V}_\circ(t,\lambda,x) \leq -c_3 \|x\|^2 \,. \tag{9}$$

□

Lemma 2 *Assume that there exist constants r, $c_1, c_2 > 0$ such that all existing solutions $x(\cdot,\lambda,t_\circ,x_\circ)$ of $\dot{x} = f(t,\lambda,x)$ satisfy*

$$\|x\|_\infty \leq c_1 \|x_\circ\| \tag{10}$$

$$\|x\|_p \leq c_2 \|x_\circ\| \tag{11}$$

for all $x_\circ \in B_r$, all $t_\circ \geq 0$ and some $p > 0$. Then the system is λ-ULES with $k_\lambda := c_1$ and $\gamma_\lambda := [c_1 c_2 \mathrm{e}]^{-p}$, where $\mathrm{e} = 2.71\ldots$ Moreover, if c_1 and $c_2 > 0$ exist for all $x_\circ \in R^n$, the system λ-UGES. □

Proof. Let $\Delta = 1/\gamma_\lambda$. We claim that for all $x_\circ \in B_r, t_\circ \geq 0$, all $t \geq t_\circ$ and any solution, we have that

$$\|x(t+\Delta,\lambda,t_\circ,x_\circ)\| \leq \mathrm{e}^{-1} \|x(t,\lambda,t_\circ,x_\circ)\| \,. \tag{12}$$

In view of (10) it is sufficient to prove that for any $t \geq t_\circ$ there exists $t' \in [t, t+\Delta]$ such that

$$\|x(t',\lambda,t_\circ,x_\circ)\| \leq \frac{1}{c_1 \mathrm{e}} \|x(t,\lambda,t_\circ,x_\circ)\| \,.$$

Assume the opposite, that is assume that there exists $t \geq t_\circ$ such that $\|x(\tau,\lambda,t_\circ,x_\circ)\| > \dfrac{1}{c_1 \mathrm{e}} \|x(t,\lambda,t_\circ,x_\circ)\|$ for all $\tau \in [t, t+\Delta]$. Then, defining $c_2 := \frac{1}{c_1 \mathrm{e}} \Delta^{1/p}$ we have that

$$\int_t^{t+\Delta} \|x(\tau,\lambda,t_\circ,x_\circ)\|^p \, d\tau > c_2^p \|x(t,\lambda,t_\circ,x_\circ)\|^p \,. \tag{13}$$

On the other hand, using (11) and with an abuse of notation, we obtain that

$$\begin{aligned} \int_t^{t+\Delta} \|x(\tau,\lambda,t_\circ,x_\circ)\|^p \, d\tau \quad = & \\ & \int_t^{t+\Delta} \|x(\tau,\lambda,t,x(t,\lambda,t_\circ,x_\circ))\|^p \, d\tau \\ & \leq \int_t^{\infty} \|x(\tau,\lambda,t,x(t,\lambda,t_\circ,x_\circ))\|^p \, d\tau \\ & \leq c_2^p \|x(t,\lambda,t_\circ,x_\circ)\|^p \end{aligned}$$

which contradicts (13), hence the claim is proved.

Next, to prove λ- ULES, we proceed similar to the proof of [9, Theorem 4.5]. For any $t \geq t_\circ$ let $N \geq 1$ satisfy: $t_\circ + (N-1)\Delta \leq t \leq t_\circ + N\Delta$, then using (10), (12) we obtain that

$$\begin{aligned} \|x(t,\lambda,t_\circ,x_\circ)\| & \leq \mathrm{e}^{-1} \|x(t-\Delta,\lambda,t_\circ,x_\circ)\| \\ & \leq \mathrm{e}^{-N} \|x(t-N\Delta,\lambda,t_\circ,x_\circ)\| \\ & \leq c_1 \mathrm{e}^{-N} \|x_\circ\| \\ & \leq c_1 \mathrm{e}^{-(t-t_\circ)/\Delta} \|x_\circ\| \\ & = c_1 \|x_\circ\| \mathrm{e}^{-\gamma(t-t_\circ)}, \end{aligned}$$

Notice that c_1 and γ are independent of λ. The result follows. ■

We finish this section with the following result for a first order LTV parameterized system.

Lemma 3 *Consider the system*

$$\dot{x} = -B(t,\lambda)B(t,\lambda)^\top x \tag{14}$$

where $\lambda \in \mathcal{D}$ and $B(t,\lambda)$ satisfies the following assumptions.

A1. There exists a constant $\phi_M > 0$ such that, for almost all $t \geq 0$ and all $\lambda \in \mathcal{D}$ the following bound is satisfied

$$\max\left\{ \|B(t,\lambda)\|, \left\| \frac{\partial B(t,\lambda)}{\partial t} \right\| \right\} \leq \phi_M \,. \tag{15}$$

A2. There exist constants T, $\mu > 0$ such that

$$\mu I \leq \int_t^{t+T} B(\tau,\lambda)B(\tau,\lambda)^\top d\tau \tag{16}$$

for all $t \geq 0$ and all $\lambda \in \mathcal{D}$.

Then the system (14) is λ-UGES. □

Remark 1 Assumption **A2** is similar to the well known concept of persistency of excitation (PE). Notice that the inequality (15) is assumed to hold uniformly in λ. New notions of *uniform* persistency of excitation for regressors depending on the state have been introduced in [15, 13]. Also, in [11] the authors use the notion of persistency of excitation for matrix functions of time only. □

Proof. Let $V(t,x) := \frac{1}{2}\|x\|^2$, then

$$\dot{V}_{(14)}(\tau, x(\tau,\lambda)) \leq -x(t,\lambda)^\top B(\tau,\lambda)B(\tau,\lambda)^\top x(\tau,\lambda) \,, \tag{17}$$

therefore defining $v(t,\lambda) := V(t, x(t,\lambda))$,

$$v(t+T,\lambda) - v(t,\lambda) = -\int_t^{t+T} \left\|B(\tau,\lambda)^\top x(\tau,\lambda)\right\|^2 d\tau \,, \tag{18}$$

where the solution

$$x(\tau,\lambda) = x(t,\lambda) - \int_t^\tau B(s,\lambda)B(s,\lambda)^\top x(s,\lambda) ds \,. \tag{19}$$

Substituting (19) in (18) we obtain, using Schwartz inequality, $(a-b)^2 \geq 0.5a^2 - b^2$, and Assumptions **A1**, **A2**

$$\begin{aligned} v(t+T,\lambda) - v(t,\lambda) \quad &\leq \quad -\frac{1}{2}\int_t^{t+T} \left\|B(\tau,\lambda)^\top x(t,\lambda)\right\|^2 d\tau \\ &+ \int_t^{t+T} \|B(\tau,\lambda)\|^2 d\tau \ \times \\ &\int_t^{\tau} \|B(s,\lambda)\|^2 \left\|B(s,\lambda)^\top x(s,\lambda)\right\|^2 \, ds\, d\tau \\ &\leq -\frac{\mu}{2}\|x(t,\lambda)\|^2 + \\ &\phi_M^4 \int_t^{t+T}\int_t^{\tau} \left\|B(s,\lambda)^\top x(s,\lambda)\right\|^2 \, ds\, d\tau \,. \end{aligned} \tag{20}$$

Changing the order of integration for the second term on the right hand side of (20) we obtain that

$$\begin{aligned} \int_t^{t+T}\int_t^{\tau} \left\|B(s,\lambda)^\top x(s,\lambda)\right\|^2 ds d\tau \quad &= \\ \int_t^{t+T} \left\|B(s,\lambda)^\top x(s,\lambda)\right\|^2 \int_s^{t+T} d\tau ds& \\ = \int_t^{t+T} \left\|B(s,\lambda)^\top x(s,\lambda)\right\|^2 \, [t+T-s] ds& \\ \leq T \int_t^{t+T} \left\|B(s,\lambda)^\top x(s,\lambda)\right\|^2 ds& \\ \leq T[v(t+T,\lambda) - v(t,\lambda)] \,.& \end{aligned} \tag{21}$$

Hence, using (21) in (20) we obtain that

$$\frac{\mu}{2}\|x(t,\lambda)\|^2 \leq (1+\phi_M^4 T)[v(t,\lambda) - v(t+T,\lambda)] \tag{22}$$

which, observing that

$$\int_{t_\circ}^{\infty} v(t+T)dt = \int_{t_\circ}^{\infty} v(t+T)d(t+T) = \int_{t_\circ + T}^{\infty} v(s)ds$$

implies that

$$\begin{aligned} \int_{t_\circ}^{\infty} \|x(t,\lambda)\|^2 \quad &\leq \quad \frac{2}{\mu}(1+\phi_M^4 T)\int_{t_\circ}^{t_\circ + T} v(\tau,\lambda)d\tau \\ &\leq \quad \frac{2T}{\mu}(1+\phi_M^4 T)v(t_\circ,\lambda) \\ &\leq \quad \frac{T}{\mu}(1+\phi_M^4 T)\|x_\circ\|^2 \,. \end{aligned}$$

The proof finishes invoking Lemma 2 observing that $c := \frac{T}{\mu}(1+\phi_M^4 T)$ is independent of λ. ■

3 Main result

We consider families of LTV multivariable systems of the form

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A(t,\lambda) & B(t,\lambda)^\top \\ -C(t,\lambda) & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \; z := \begin{bmatrix} e \\ \theta \end{bmatrix} \tag{23}$$

where $e \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$, $A(t,\lambda) \in \mathbb{R}^{n\times n}$, $B(t,\lambda) \in \mathbb{R}^{n\times p}$, $C(t,\lambda) \in \mathbb{R}^{n\times p}$ and $\lambda \in \mathcal{D} \subset R^l$. We assume that

A3. *A3.1.* The system $\dot{x} = A(t,\lambda)x$ is λ-UGES.

A3.2. There exist symmetric matrices $P(t,\lambda)$ and $Q(t,\lambda)$ such that $P(t,\lambda)B(t,\lambda) = C(t,\lambda)$ and $-Q(t,\lambda) := A(t,\lambda)^\top P(t,\lambda) + P(t,\lambda)A(t,\lambda) + \dot{P}(t,\lambda)$. Furthermore, $\exists\, p_m$, q_m, p_M, and $q_M > 0$ such that, for all $\lambda \in \mathcal{D}$, $p_m I \leq P(t,\lambda) \leq p_M I$ and $q_m I \leq Q(t,\lambda) \leq q_M I$.

Theorem 1 (UGES of LTV) *The system (23) under Assumptions* **A1** *and* **A3** *is λ-UGES if and only if* **A2** *holds.* □

Proof. Sufficiency: Using the Lyapunov function $V_1(t,\lambda,z) := 0.5z^\top R(t,\lambda)z$, with $R :=$ block-diag$\{P(t,\lambda), I\}$, we obtain that $\dot{V}_1(t,\lambda,z) \leq -q_m\|e\|^2$. Hence, (23) is uniformly globally stable (UGS).

Similarly to [18, 12], let $a > 0$ and partition the right hand side of (23) to rewrite this system as

$$\dot{z} = \mathcal{A}(t,\lambda)z + \mathcal{G}(t,\lambda,e) \tag{24}$$

where

$$\mathcal{A} := \begin{bmatrix} -aI & B(t,\lambda)^\top \\ -B(t,\lambda) & 0 \end{bmatrix} \tag{25}$$

$$\mathcal{G} := \begin{bmatrix} [A(t,\lambda) + aI]e \\ B(t,\lambda)[I - P(t,\lambda)]e \end{bmatrix} . \tag{26}$$

Notice that under Assumptions **A1** and *A3.2* there exists k_g such that $\|\mathcal{G}(t,\lambda,e)\| \leq k_g\|e\|$ for all $\lambda \in \mathcal{D}$. We claim that (23) is λ-UGES if $\dot{z} = \mathcal{A}(t,\lambda)z$ is λ-UGES. Indeed, if the latter is true, it follows from **A1** and Lemma 1, that there exists $V_2(t,\lambda,z)$ satisfying (7) – (9). Now let π be a positive constant satisfying $\pi \geq \frac{k_g^2 c_2^2}{2q_m c_3}$ and define $V_3(t,z,\lambda) := \pi V_1(t,\lambda,z) + V_2(t,\lambda,z)$. Taking the time derivative along the trajectories of (24), we obtain that $\dot{V}_3(t,z,\lambda) \leq -\pi q_m\|e\|^2 - c_3\|z\|^2 + k_g c_2\|z\|\|e\|$, therefore $\dot{V}_3(t,\lambda,z) \leq -\frac{c_3}{2}\|z\|^2$. The result follows.

It is only left to prove that $\dot{z} = \mathcal{A}(t,\lambda)z$ is λ-UGES. To that end we find it convenient to introduce the following change of coordinates:

$$\zeta := ke - B(t,\lambda)^\top\theta\,, \qquad k > 0\,, \tag{27}$$

then we have that

$$\dot{\zeta} \quad = \quad -k\zeta + [k^2 + B(t,\lambda)^\top B(t,\lambda) - ka]e - \dot{B}(t,\lambda)^\top\theta\,.$$

Defining $v_1(t,\lambda,e) := [k^2 + B(t,\lambda)^\top B(t,\lambda) - ka]e$ and $c_4 := k^2 + ka + \phi_M^2$, we have from **A1** that

$$\|v_1(t,\lambda,e)\| \leq c_4\|e\|\,, \qquad \text{a.e.} \tag{28}$$

$$\left\|\dot{B}(t,\lambda)^\top\theta\right\| \leq \phi_M\|\theta\|\,, \qquad \text{a.e.} \tag{29}$$

On the other hand, defining $v_2(t,\lambda,e) := [k-1]B(t,\lambda)e$ we obtain that

$$\dot{\theta} \quad = \quad -B(t,\lambda)B(t,\lambda)^\top\theta - B(t,\lambda)\zeta + v_2(t,\lambda,e)\,.$$

Thus, in compact form, $\dot{z} = \mathcal{A}(t,\lambda)z$ is equivalent to

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -kI & -\dot{B}(t,\lambda) \\ -B(t,\lambda) & -B(t,\lambda)B(t,\lambda)^\top \end{bmatrix} \begin{bmatrix} \zeta \\ \theta \end{bmatrix} + \begin{bmatrix} v_1(t,\lambda,e) \\ v_2(t,\lambda,e) \end{bmatrix} . \tag{30}$$

The interest of this partition is the following. From Lemmas 3 and 1 we have that there exist c_1, c_2, and $c_3 > 0$ such that, for each $\lambda \in \mathcal{D}$, there exists a positive definite matrix $P_\circ(t,\lambda)$ such that, defining $V_\circ(t,\lambda,\theta) := \theta^\top P_\circ(t,\lambda,\theta)\theta$, we have

$$c_1 \|\theta\|^2 \le V_\circ(t,\lambda,\theta) \le c_2 \|\theta\|^2 \tag{31}$$

and the derivative of $V_\circ$ along the trajectories of

$$\dot{\theta} = -B(t,\lambda)B(t,\lambda)^\top \theta \tag{32}$$

satisfies

$$\dot{V}_\circ(t,\lambda,\theta) \le -c_3 \|\theta\|^2 . \tag{33}$$

Finally, let $\alpha > 0$ to be defined later and consider the Lyapunov function candidate

$$W(t,\lambda,\zeta) := \frac{1}{2}\left(\|\zeta\|^2 + \alpha\|z\|^2\right) + V_\circ(t,\lambda,\theta) , \tag{34}$$

and evaluate its time derivative along the trajectories of $\dot{z} = \mathcal{A}(t,\lambda)$ and (30), then using (33), (28) and (29), we obtain that

$$\begin{aligned} \dot{W}_{(30)}(t,\lambda,\zeta) \quad \le \quad & -k\|\zeta\|^2 + c_4\|e\|\,\|\zeta\| + \phi_M\|\theta\|\,\|\zeta\| \\ & -c_3\|\theta\|^2 - a\alpha\|e\|^2 + \theta^\top(P_\circ(t,\lambda) \\ & +P_\circ(t,\lambda)^\top)[-B(t,\lambda)\zeta + v_2(t,\lambda,e)] \\ \le -\frac{1}{2}&\left(k\|\zeta\|^2 + c_3\|\theta\|^2 + a\alpha\|e\|^2\right) \\ -\frac{1}{4}&\begin{bmatrix} \|\zeta\| \\ \|e\| \end{bmatrix}^\top \begin{bmatrix} k & -2c_4 \\ -2c_4 & a\alpha \end{bmatrix} \begin{bmatrix} \|\zeta\| \\ \|e\| \end{bmatrix} \\ -\frac{1}{4}&\begin{bmatrix} \|\theta\| \\ \|e\| \end{bmatrix}^\top \begin{bmatrix} c_3 & -4c_5p_M \\ -4c_5p_M & a\alpha \end{bmatrix} \begin{bmatrix} \|\theta\| \\ \|e\| \end{bmatrix} \\ -\frac{1}{4}&\begin{bmatrix} \|\zeta\| \\ \|\theta\| \end{bmatrix}^\top \begin{bmatrix} k & -c_6 \\ -c_6 & c_3 \end{bmatrix} \begin{bmatrix} \|\zeta\| \\ \|\theta\| \end{bmatrix} , \end{aligned} \tag{35}$$

where we defined $c_5 := (k-1)\phi_M$ so that $\|v_2(t,\lambda,e)\| \le c_5\|e\|$ and $c_6 := 2\phi_M(2p_M+1)$. Thus, $\dot{z} = \mathcal{A}(t,\lambda)z$ is λ-UGES if[2]

$$\alpha \ge \max\left\{\frac{4c_4^2}{ak}, \frac{16c_5^2(2p_M+1)^2}{ac_3}\right\} \quad \text{and}$$

$$k \ge \max\left\{\frac{16\phi_M^2 p_M^2}{c_3}, 1\right\}$$

Necessity[3]: We prove now that **A1**, **A3** and λ-UGES imply **A2**. To compact the notation, let us define the matrix in (23) as $\mathbf{A}(t,\lambda)$, i.e., let (23) be written as $\dot{z} = \mathbf{A}(t,\lambda)z$. From Lemma 1 it follows that for each $\lambda \in \mathcal{D}$ there exists $V_\circ(t,\lambda,z)$ satisfying (7)–(9) hence, for all $t \ge t_\circ$ and each $z \in \mathbb{R}^{n+m}$,

$$z^\top[P_\circ(t,\lambda)\mathbf{A}(t,\lambda) + \mathbf{A}(t,\lambda)^\top P_\circ(t,\lambda)]z + z^\top \dot{P}_\circ(t,\lambda)z \le -c_3\|z\|^2 .$$

This and (8) imply that for all $t_1 \ge t_\circ \ge 0$,

$$\begin{aligned} -\int_{t_\circ}^{t_1} & z^\top[P_\circ(t,\lambda)\mathbf{A}(t,\lambda) + \mathbf{A}(t,\lambda)^\top P_\circ(t,\lambda)]z \\ & \ge c_3\|z\|^2(t_1 - t_\circ) - z^\top \int_{t_\circ}^{t_1} \dot{P}_\circ(t,\lambda)dt\, z \\ & \ge c_3\|z\|^2(t_1 - t_\circ) - z^\top[P_\circ(t_1,\lambda) - P_\circ(t_\circ,\lambda)]z \\ & \ge c_3\|z\|^2(t_1 - t_\circ) - c_2\|z\|^2 . \end{aligned} \tag{36}$$

On the other hand, defining the unitary vector $\xi \in \mathbb{R}^{n+m}$, such that $z = \|z\|\,\xi$, we obtain

$$\begin{aligned} -z^\top[P_\circ(t,\lambda)\mathbf{A}(t,\lambda) + \mathbf{A}(t,\lambda)^\top P_\circ(t,\lambda)]z & \\ \le 2c_2\|z\|\,\|\mathbf{A}(t,\lambda)z\| & \qquad (37) \\ \le 2c_2\|z\|^2\,\|\mathbf{A}(t,\lambda)\xi\| . & \qquad (38) \end{aligned}$$

From (36) and (38) it follows that, for all unitary vectors $\xi \in \mathbb{R}^{n+m}$,

$$\int_{t_\circ}^{t_1} \|\mathbf{A}(t,\lambda)\xi\|\,dt \ge \frac{c_3}{2c_2}(t_1 - t_\circ) - 1 \tag{39}$$

which, defining $T > 0$ such that $\beta := \frac{c_3 T}{2c_2} - 1 > 0$, implies that

$$\int_{t_\circ}^{t_\circ+T} \|\mathbf{A}(t,\lambda)\xi\|\,dt \ge \beta \qquad \forall\, t \ge 0 . \tag{40}$$

Since (40) holds for all $\xi \in \mathbb{R}^{n+m} : \|\xi\| = 1$, it also holds for $\bar{\xi} := [0, \xi_1]$ where $\xi_1 \in \mathbb{R}^m$ is also a unitary vector. To that end, noticing that $\mathbf{A}(t,\lambda)\bar{\xi} = B(t,\lambda)^\top \xi_1$, (40) implies that

$$\int_t^{t+T} \left\|B(\tau,\lambda)^\top \xi_1\right\| d\tau \ge \beta \qquad \forall\, t \ge 0 . \tag{41}$$

Finally, using the Cauchy-Schwartz inequality, we obtain that

$$\xi_1^\top \int_t^{t+T} B(\tau,\lambda)B(\tau,\lambda)^\top d\tau\, \xi_1 \ge \frac{\beta^2}{T} =: \mu \qquad \forall\, t \ge 0 .$$

which is equivalent to (16). ■

4 Conclusions

We have presented here sufficient and necessary conditions for uniform global exponential stability of families of multivariable linear time-varying systems. Our results are formulated in terms of the well known concepts of persistency of excitation and uniform complete observability.

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[2] Notice that the numbers α and k are introduced in the proof, they are not parameters of the system. See (34) and (27) Also, we recall that $c_4 = c_4(a)$.

[3] This part of the proof is inspired on [16, 2].

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