Stability Tests for a Class of Differential Linear Repetitive Processes with Dynamic Boundary Conditions

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Abstract: Repetitive processes are a distinct class of 2D systems of both practical and theoretical interest. Their essential characteristic is repeated sweeps, termed passes, through a set of dynamics defined over a finite duration with explicit interaction between the outputs, or pass profiles, produced as the process dynamics evolve. Experience has shown that these processes cannot be studied/controlled by direct application of existing theory (in all but a few very restrictive special cases). This fact, and the growing list of applications areas, has prompted an ongoing research programme into the development of a 'mature' systems theory for these processes for onward translation into reliable generally applicable controller design algorithms. This paper develops stability tests for a sub-class of so-called differential linear repetitive processes in the presence of a general set of initial conditions, where it is known that the structure of these conditions is critical to their stability properties.

1 Introduction

The essential unique characteristic of a repetitive, or multipass, process can be illustrated by considering machining operations where the material or workpiece involved is processed by a sequence of passes of the processing tool. Assuming that the pass length α (i.e. the duration of a pass of the processing tool) is finite and constant, the output, or pass profile, $y_k(t)$, $0 \leq t \leq \alpha$, (t being the independent spatial or temporal variable) produced on the $k\,th$ pass acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile $y_{k+1}(t)$, $0 \leq t \leq \alpha$, $k \geq 0$.

Repetitive processes clearly have a two-dimensional, or 2D, structure, i.e. information propagation occurs along a given pass (t direction) and from pass to pass (k direction). They are distinct from all other classes of 2D linear systems due to the fact that information propagation along the pass only occurs over a finite and fixed interval - the pass length α.

The basic unique control problem for repetitive processes is that the output sequence of pass profiles can contain oscillations that increase in amplitude in the pass to pass direction (i.e. in the k - direction in the notation for variables used here). Early approaches to stability analysis and controller design for (linear single-input single-output (SISO)) repetitive processes and, in particular, long-wall coal cutting was based on first converting the dynamics into those of an equivalent infinite-length single-pass process. This resulted, for example, in a scalar algebraic/delay system to which standard scalar inverse-Nyquist stability criteria could then be applied.

In general, however, it was soon established that this approach to stability analysis and controller design would, except in a few very restrictive special cases, lead to incorrect conclusions. The basic reason for this is that such an approach effectively neglects their finite pass length repeatable nature and the effects of resetting the initial conditions before the start of each pass. To remove these difficulties, a rigorous stability theory has been developed [1] based on an abstract model in a Banach space setting which includes all linear dynamics constant pass length processes as special cases.

Use of this theory confirms that the initial conditions at the start of each pass, termed boundary conditions, have a critical role in determining the stability properties of a given example. In [2], a stability analysis for so-called differential linear repetitive processes, a sub-class of major interest in terms of both physical and algorithmic applications, was given in the presence of a general set of boundary conditions. This paper develops stability tests for this case using a so-called 1D Lyapunov equation and, as an alternative, the so-called strictly bounded real lemma [3].

2 Background

The state space model of the sub-class of differential linear repetitive processes considered here has the following form over $0 \leq t \leq \alpha$, $k \geq 0$,

$$\dot{y}_{k+1}(t) = Ay_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t). \quad (1)$$

Here on pass k, $y_k(t)$ is the $n \times 1$ pass profile vector, and $u_k(t)$ is the $l \times 1$ vector of control inputs.

To complete the process description, it is necessary to specify the initial conditions, termed the boundary conditions. The most general form of boundary conditions for (1) arise when $y_{k+1}(0)$ is an explicit function of a finite number of points along the previous pass profile. In this paper we will focus on the case when these are of the form

$$y_{k+1}(0) = d_{k+1} + y_k(\alpha),\ k \geq 0, \quad (2)$$

which is of particular interest in terms of links with delay differential systems and also repetitive control schemes. The $n \times 1$ vector d_{k+1} has constant entries.

The stability theory [1] for linear repetitive processes consists of two distinct concepts termed asymptotic stability and stability along the pass respectively. In effect, asymptotic stability demands that bounded (in a well defined sense) sequences of inputs produce bounded sequences of pass profiles over the (finite and constant) pass length and stability along the pass demands this property uniformly, i.e. independent of the pass length α. Here we focus on stability along the pass which is what is required in the vast majority of cases. The following result gives necessary and sufficient conditions for asymptotic stability of processes described by (1) and (2).

Theorem 1 *Suppose that the pair $\{A, B_0\}$ is controllable and that all eigenvalues of the matrix A have strictly negative real parts. Then linear repetitive processes described by (1) and (2) are stable along the pass if, and only if,*
(a) all solutions of

$$|zI_n - e^{(A+z^{-1}B_0)\alpha}| = 0, \quad (3)$$

have modulus strictly less than unity $\forall\, \alpha \geq 0$, and
(b)

$$\sup_{\omega \geq 0} r(G(\imath\omega)) < 1, \quad (4)$$

where

$$G(s) = (sI_n - A)^{-1}B_0. \quad (5)$$

3 1D Lyapunov Equation Stability Tests

In terms of testing a given example for stability along the pass using Theorem 1, first note that it is only condition (a) which cannot be directly tested by direct application of 1D linear systems tests. In this section, we develop a so-called 1D Lyapunov equation based interpretation of this condition. The designation of the Lyapunov equation used as 1D is to highlight the fact that the structure of this equation is identical to its well known counterpart for 1D differential linear systems but here the defining matrices are functions of a complex variable.

Write $z = e^{s\alpha}$, and hence (3) reduces to the requirement that all solutions of

$$\det(sI_n - F(s)) = 0, \quad (6)$$

have strictly negative real parts where

$$F(s) = A + B_0 e^{-s\alpha}. \quad (7)$$

Also it can be shown (using results in [4]) that (6) reduces to the requirement that

$$\det(sI_n - F(e^{-\imath\omega\alpha})) \neq 0, \forall \mathrm{Re}(s) \geq 0, \forall\, \omega \in [0, 2\pi]. \quad (8)$$

The following result now expresses the condition of (8) in terms of a so-called 1D Lyapunov equation.

Theorem 2 *The condition of (8) holds if, and only if, for a given positive definite Hermitian (P. D. H.) matrix $Q(e^{\imath\omega})$, $\omega \in [0, 2\pi]$, the solution, $P(e^{\imath\omega})$, of the matrix Lyapunov equation*

$$F^*(e^{\imath\omega})P(e^{\imath\omega}) + P(e^{\imath\omega})F(e^{\imath\omega}) = -Q(e^{\imath\omega}), \quad (9)$$

is P. D. H. $\forall\, \omega \in [0, 2\pi]$, where $*$ denotes the complex conjugate transpose operation.

Proof: To show sufficiency, first note that for any fixed $\omega_o \in [0, 2\pi]$, the matrix $F(e^{\imath\omega_o})$ is an $n \times n$ matrix with complex elements. Also let λ_o be an eigenvalue of this matrix and w_o the corresponding eigenvector. Then

$$\begin{aligned} F(e^{\imath\omega_o})w_o &= \lambda_o w_o \\ w_o^* F^*(e^{\imath\omega_o}) &= \overline{\lambda}_o w_o^*, \end{aligned} \quad (10)$$

where the bar denotes the complex conjugate operation. Now pre-multiply (9) by w_o^* and then post-multiply this same equation by w_o to yield

$$\begin{aligned} w_o^* Q(e^{\imath\omega_o})w_o &= -w_o^*(F^*(e^{\imath\omega_o})P(e^{\imath\omega_o}) \\ &+ P(e^{\imath\omega_o})F(e^{\imath\omega_o}))w_o. \end{aligned} \quad (11)$$

Then, since both $P(e^{\imath\omega})$ and $Q(e^{\imath\omega})$ are P. D. H. matrices $\forall\, \omega \in [0, 2\pi]$, it follows that

$$w_o^* Q(e^{\imath\omega_o})w_o = -(\overline{\lambda}_o + \lambda_o)w_o^* P(e^{\imath\omega_o})w_o, \quad (12)$$

and that

$$\mathrm{Re}(\lambda) = \frac{1}{2}(\overline{\lambda} + \lambda) = -\frac{1}{2}\left(\frac{w^* Q(e^{\imath\omega})w}{w^* P(e^{\imath\omega})w}\right) < 0, \quad (13)$$

where now λ is any eigenvalue of $F(e^{\imath\omega})$ and w is the corresponding eigenvector. Hence condition (a) of Theorem 1 holds.

To show necessity, consider (9) with an arbitrary P. D. H. matrix $Q(e^{\imath\omega})$ on $[0, 2\pi]$. Then if (8) holds, it can be shown [4] that all eigenvalues of the matrix $F(e^{\imath\omega})$ have strictly negative real parts $\forall\, \omega \in [0, 2\pi]$. Now define

$$P(e^{\imath\omega}) = \int_0^{\infty} e^{F^*(e^{\imath\omega})t} Q(e^{\imath\omega}) e^{F(e^{\imath\omega})t}\, dt, \quad (14)$$

which is well defined since the eigenvalues of $F(e^{\imath\omega})$ (and $F^*(e^{\imath\omega})$) are in the left-half of the complex plane. Also $P^*(e^{\imath\omega}) = P(e^{\imath\omega})$, $\forall\, \omega \in [0, 2\pi]$ and it follows from routine analysis that

$$F^*(e^{\imath\omega})P(e^{\imath\omega}) + P(e^{\imath\omega})F(e^{\imath\omega}) = -Q(e^{\imath\omega}). \quad (15)$$

Hence we have that $\forall\, \omega \in [0, 2\pi]$,

$$F^*(e^{\imath\omega})P(e^{\imath\omega}) + P(e^{\imath\omega})F(e^{\imath\omega}) = -Q(e^{\imath\omega}), \quad (16)$$

and $P^*(e^{\imath\omega}) = P(e^{\imath\omega})$, $\forall\, \omega \in [0, 2\pi]$.

Now define

$$F(z)|_{z=e^{\imath\omega}} = F_1(\omega) + \imath F_2(\omega), \quad (17)$$

where $F_1(\omega)$ and $F_2(\omega)$ are real $n \times n$ matrices. Also for a fixed $\omega_o \in [0, 2\pi]$, $F(e^{\imath\omega_o})$ is an $n \times n$ matrix with complex entries which can be written as

$$F(e^{\imath\omega_o}) = F_1(\omega_o) + \imath F_2(\omega_o). \quad (18)$$

The system $\dot{y} = F(e^{\imath\omega_o})y$ can be rewritten as

$$\dot{y}_r + \imath\dot{y}_i = (F_1(\omega_o) + \imath F_2(\omega_o))(y_r + \imath y_i), \quad (19)$$

where y_r and y_i denote the real and imaginary parts of y respectively.

Separating (19) into real and imaginary parts now yields the equivalent expression

$$\begin{bmatrix} \dot{y}_r \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} F_1(\omega_o) & -F_2(\omega_o) \\ F_2(\omega_o) & F_1(\omega_o) \end{bmatrix} \begin{bmatrix} y_r \\ y_i \end{bmatrix}. \quad (20)$$

Now introduce

$$\hat{F}(\omega) = \begin{bmatrix} F_1(\omega) & -F_2(\omega) \\ F_2(\omega) & F_1(\omega) \end{bmatrix}. \quad (21)$$

Then in the SISO case, a necessary and sufficient condition for condition (a) of Theorem 1 to hold is that $F_1(\omega) < 0$, $\forall\, \omega \in [0, 2\pi]$, i.e. 1D stability of the real part of $F(e^{\imath\omega})$ (defined by (17)). Also in this case

$$\det(sI - \hat{F}(\omega)) = s^2 - 2F_1(\omega)s + F_1^2(\omega) + F_2^2(\omega). \quad (22)$$

Hence in the SISO case, a necessary and sufficient condition for (a) of Theorem 1 is that $F_1(\omega) < 0$, $\forall\, \omega \in [0, 2\pi]$.

4 Strict Positive Realness Based Tests

First rewrite (8) is obvious notation as

$$\Delta(s, z) \equiv \det(sI_n - F(z)) \neq 0, \mathrm{Re}(s) \geq 0,\ |z| = 1, \quad (23)$$

or

$$\Delta(s, e^{\imath\omega}) \neq 0,\ \mathrm{Re}(s) \geq 0,\ \omega \in [0, 2\pi] \quad (24)$$

This is an equation with complex coefficients which are polynomials in $e^{\imath\omega}$ and it is required that all its roots should lie in the open-left half of the s - plane. Using 'classical' root clustering theory, the condition for this (see, for example, [5]) is that the Hermite matrix obtained from the coefficients in $\Delta(s, e^{\imath\omega})$ is positive definite or, alternatively, the inner-wise matrix obtained from the coefficients must be positive inner-wise.

Consider the complex polynomial

$$B(s) = \sum_{i=0}^{n} b_i s^i. \quad (25)$$

Then the associated Hermite matrix, H, is obtained as follows

$$H = \{h_{p,q}\},\ \ h_{p,q} = h_{q,p} \quad (26)$$

where the elements $h_{p,q}$ are constructed from the b_i by a well documented formula which is omitted here for brevity. Also it can be shown that [4, 5] H positive definite $\forall \omega \in [0, 2\pi]$ (or $|e^{\imath\omega}| \in [-1, 1]$) is equivalent to the following conditions

$$\begin{aligned} H(e^{\imath 0}) &= H(1) > 0 & (27) \\ \det(H(e^{\imath\omega})) &> 0,\ \forall\, \omega \in [0, 2\pi]. & (28) \end{aligned}$$

The checking of (27) is straightforward and to check the more difficult condition of (28) it is possible to use a positivity test. This is based on the fact that $\det(H(e^{\imath\omega}))$ is a function of $\cos\omega, \cos 2\omega, \cdots$ and on setting $x = \cos\omega$, $\det(H(e^{\imath\omega}))$ becomes a function of x and its powers. Hence (28) becomes

$$\det(H(e^{\imath\omega})) = F(x) > 0,\ x \in [-1, 1]. \quad (29)$$

This last condition holds provided $F(x)$ has no real roots in the interval $[-1, 1]$. Also introduce the change of variable (a bilinear transform)

$$x = \frac{u-1}{u+1}, \quad (30)$$

into (29) to yield (in obvious notation) the equivalent condition in the form

$$F_1(u) > 0,\ u \in [0, \infty]. \quad (31)$$

This condition can be checked computationally using any of the computational positivity tests [5].

In the remainder of this section we develop a computationally more feasible alternative to the approach just presented. The starting point is to note that the condition to be tested here can be expressed as the requirement that a two variable polynomial of the general form

$$a(s,z) = s^p + \sum_{j=0}^{p-1}\sum_{i=0}^{q} a_{ij}s^j z^i, \qquad (32)$$

should satisfy

$$a(s,z) \neq 0,\ \mathrm{Re}(s) \geq 0,\ |z| \leq 1. \qquad (33)$$

Next we describe how to reduce (33) to a 1D problem by showing that this condition is equivalent to positive realness of a certain 1D rational transfer function matrix. This leads to a numerically efficient testing algorithm and requires, as background, the results summarized next relating to the so-called strictly bounded real lemma [3].

Definition 1 *A real rational transfer function matrix $G(s) = C_1(sI - A_1)^{-1}B_1$ is termed strictly bounded real if, and only if, the matrix A_1 is Hurwitz (i.e. all its eigenvalues have negative real parts) and*

$$I - G^T(-\imath\omega)G(\imath\omega) > 0,\ \forall\, \omega \in \mathbb{R}. \qquad (34)$$

The well known strictly bounded real lemma [3] takes the following form here.

Definition 2 *Suppose that $G(s)$ is a proper rational transfer function matrix and let $\{A_1, B_1, C_1, D_1\}$ be an associated minimal realization. Then this transfer function matrix is strictly bounded real if, and only if, $\exists$ a real symmetric positive definite matrix P such that*

$$\begin{aligned} M &= \begin{bmatrix} H_1 & H_2 \\ H_2^T & D_1^T D_1 - I \end{bmatrix} \\ &< 0, \end{aligned} \qquad (35)$$

where

$$\begin{aligned} H_1 &= A_1^T P + PA_1 + C_1^T C_1 \\ H_2 &= PB_1 + C_1^T D_1 \end{aligned} \qquad (36)$$

One characterization of this strictly bounded real property (for the proof see, for example, [6]) is that $G(s)$ has this property if, and only if, for any given real symmetric matrix $Q > 0$, $\exists\, \epsilon > 0$ such that:

(a) $I - D_1^T D_1 > 0$, and
(b) the algebraic Riccati equation

$$\begin{aligned} A_1^T P &+ PA_1 + (PB_1 \\ &+ C_1^T D_1)(I - D_1^T D_1)^{-1}(B_1^T P + D_1^T C_1) \\ &+ C_1^T C_1 + \epsilon Q = 0, \end{aligned} \qquad (37)$$

has a positive definite solution matrix P. Also the requirement for a minimal realization can be relaxed by the following result (also proved in [6]).

Lemma 1 *Suppose that $G(s)$ is strictly proper and let $\{A_1, B_1, C_1\}$ be a state space realization with the pair $\{A_1, B_1\}$ controllable. Then $G(s)$ is strictly bounded real if, and only if, for any given real symmetric matrix $Q > 0$, $\exists$ a scalar $\epsilon > 0$ such that the algebraic Riccati equation*

$$A_1^T P + PA_1 + PB_1B_1^T P + C_1^T C_1 + \epsilon Q = 0, \quad (38)$$

has a positive definite solution matrix P.

If (38) has a solution $P > 0$ for $\epsilon^* > 0$ then for any $\epsilon \in [0, \epsilon^*]$, this equation admits at least one positive definite solution. Also if $G(s)$ is not strictly proper, an analogous result (again from [6]) can be used.

To proceed, first note the following result proved in [7].

Lemma 2 *Consider the two variable polynomial $a(s,z)$ and suppose that $a(0,z) \neq 0$, $\forall\, |z| = 1$. Then (33) holds if, and only if,*
(a) $a(s,0)$ is Hurwitz, and
(b)

$$a(s,z) \neq 0,\ \mathit{Re}(s) = 0,\ |z| \leq 1. \qquad (39)$$

Clearly it is the second of these conditions which is the most difficult to test. In what follows we develop a numerically efficient test based on treating $a(s,z)$ as a polynomial, denoted $a_s(z)$, in z with coefficients which are polynomials in s with s taking values on the extended imaginary axis of the complex plane.

The key point to note now is that (39) is true if, and only if, $a_s(z)$ has all its roots outside the unit circle for all s on the extended imaginary axis. Hence we can apply a 1D stability test to this condition using a point-wise approach, and here we use the Schur-Cohn test expressed in the following form (from [8]).

Lemma 3 *Let $a(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_0 \neq 0$, $a_n \neq 0$, be a polynomial with complex coefficients a_i, $i = 0, 1, \cdots, n$. Define also the triangu-*

lar Toeplitz matrices

$$D = \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_{n-2} & \overline{a}_{n-1} \\ 0 & \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \overline{a}_0 & \overline{a}_1 \\ 0 & 0 & \cdots & \cdots & \overline{a}_0 \end{bmatrix}, \qquad (40)$$

and

$$N = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & a_n & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_n & a_{n-1} \\ 0 & 0 & \cdots & \cdots & a_n \end{bmatrix}. \qquad (41)$$

Then $a(z) \neq 0, \ \forall\, |z| \leq 1$, if, and only if, the Hermitian matrix

$$\Phi = D^* D - N^* N, \qquad (42)$$

is strictly positive (where again $*$ denotes the complex conjugate transpose operation).

Note also that if $a_0 \neq 0$ then if Φ is P. D. H. $\Leftrightarrow$ the matrix $G = ND^{-1}$ is a strict contraction.

In the case under consideration here, the coefficients in the polynomial $a_s(z)$ are polynomials in s, $s = \imath\omega$, $\omega \in \mathbb{R}$. Hence, denoting these by $a_i(s)$, we have that $\overline{a}_i(s) = a_i(-s)$, $i = 0, 1, \cdots, q$. Also the triangular Toeplitz matrices D and N of (40) and (41) respectively can be constructed for this case. Similarly, define

$$\Phi(s) = D^T(-s)D(s) - N^T(-s)N(s), \qquad (43)$$

and

$$G(s) = N(s)D^{-1}(s). \qquad (44)$$

Then a simple controllable realization for $G(-s)$ is defined as follows

$$\begin{aligned} \hat{A} &= \begin{bmatrix} -\hat{A}_1 & -\hat{A}_2 & -\hat{A}_3 & \cdots & -\hat{A}_p \\ I_q & 0 & 0 & \cdots & 0 \\ 0 & I_q & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & I_q & 0 \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ \hat{C} &= \begin{bmatrix} \hat{C}_1 & \cdots & \hat{C}_p \end{bmatrix}, \end{aligned} \qquad (45)$$

where

$$\hat{A}_{p-j} = \begin{bmatrix} a_{0j} & a_{ij} & a_{2j} & \cdots & a_{q-1j} \\ 0 & a_{0j} & a_{1j} & \cdots & a_{q-2j} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{0j} & a_{1j} \\ 0 & 0 & \cdots & 0 & a_{0j} \end{bmatrix}, \qquad (46)$$

and

$$\hat{C}_{p-j} = (-1)^j \tilde{C}_{p-j}, \qquad (47)$$

where

$$\tilde{C}_{p-j} = \begin{bmatrix} a_{qj} & a_{q-1j} & a_{q-2j} & \cdots & a_{1j} \\ 0 & a_{qj} & a_{q-1j} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{qj} & a_{q-1j} \\ 0 & 0 & \cdots & 0 & a_{qj} \end{bmatrix} \qquad (48)$$

are upper triangular Toeplitz matrices with a_{ij} real as defined in (32).

The next stage is to show that the conditions of Lemma 2 are equivalent to $G(-s)$ being bounded real. To do this, first take $G(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ as defined by (45). Then (a) of Lemma 2 implies that $\det(sI - \hat{A})) = \det(D(-s)) = (a(s,0))^q$ is Hurwitz and hence $G(-s)$ is stable. Using (b) of Lemma 2 we now have that $\Phi(\imath\omega)$ (or $\Phi(-\imath\omega)$) is P. D. H. $\forall \omega \in \mathbb{R}$ and this, in turn, is equivalent to $G(-\imath\omega)$ being a strict contraction for each $\omega \in \mathbb{R}$. Hence $G(-s)$ is strictly bounded real.

Suppose now that $G(-s)$ is strictly bounded real. Then $\det(sI - \hat{A})$ is Hurwitz and hence (a) of Lemma 2 holds. Also since $G(-\imath\omega)$ is a strict contraction for each ω implies, by the Schur-Cohn test, that (b) of Lemma 2 holds.

The arguments just given establish the following result.

Theorem 3 *Consider the two-variable polynomial $a(s, z)$ defined by (32) and $G(-s)$ defined by the state space matrices of (45). Suppose also that $a(0, z) \neq 0, \forall\, |z| = 1$. Then this polynomial satisfies (33) if, and only if, $G(-s)$ is strictly bounded real.*

This leads immediately to the following algorithm for testing (33).

1. Input p, q, and a_{ij} as defined in (32).
2. Test if $a(s, 0)$ is Hurwitz and if not then stop since (33) does not hold (and hence the example under consideration is not stable along the pass).
3. Construct the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and choose a positive-definite matrix Q and a positive real scalar ϵ to solve the algebraic Riccati equation (38). If this equation has a solution then (32) holds. In which case proceed to test the other conditions for stability along the pass.

Note that the realization defined by (45) may not be minimal and hence there could be numerical problems in solving the algebraic Riccati equation if the product pq is large. Hence an input normal realization [9] should be used to obtain a minimal realization prior to testing $G(-s)$ for the strictly bounded realness property.

It is possible to avoid computing the solution of the algebraic Riccati equation here. This is based on the fact that since $G(-s)$ is strictly proper, it is guaranteed to be strictly bounded real if $\det(\Phi(s)) \neq 0, \forall \operatorname{Re}(s) = 0$. Also since we are using a minimal realization of $G(-s)$ it can be shown that this transfer function matrix is strictly bounded real if, and only if, the Hamiltonian matrix

$$H_a := \begin{bmatrix} \hat{A} & \hat{B}\hat{B}^T \\ -\hat{C}^T\hat{C} & -\hat{A}^T \end{bmatrix} \tag{49}$$

has no purely imaginary eigenvalues. Note that the dimensions of this matrix are $2pq \times 2pq$ and hence if pq is 'large' then the eigenvalue computation cannot be expected to produce 'high accuracy' results.

5 Conclusions

Previous work had shown that the stability of differential linear repetitive processes is critically dependent on the structure of the boundary conditions and, in particular, on the structure of the pass state initial vector sequence. This was established by defining a general set of such boundary conditions and then applying the stability theory based on the abstract model in a Banach space setting. In this paper the problem of testing these conditions has been addressed in one case of particular interest.

The results given here have resulted in tests based on a so-called 1D Lyapunov equation and the well known concept of a rational strictly bounded real transfer function matrix. Of these, the tests based on the latter approach are more suited to numerical computation whereas the 1D Lyapunov equation based tests offer the potential of developing performance bounds. Note that such performance bounds have already been reported for the simplest possible set of pass state initial vectors [10]. Work is proceeding in this and related areas and will be reported on in due course.

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