Tracking and Regulation Control of an Underactuated Surface Vessel with Nonintegrable Dynamics*

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Abstract

In this paper, a continuous, time-varying tracking controller is designed that globally exponentially forces the position/orientation tracking error of an underactuated surface vessel to a neighborhood about zero that can be made arbitrarily small (*i.e.*, global uniformly ultimately bounded). The result is facilitated by fusing a filtered tracking error transformation with the dynamic oscillator design presented in [4]. We also illustrate that the proposed tracking controller yields a GUUB result for the regulation problem. In addition, an extension is provided that illustrates that the proposed unified tracking/regulation controller can be applied to a twin rotor helicopter model.

1. Introduction

Over the past decade, many researchers have studied the control problem for underactuated systems with nonintegrable constraints. The majority of this research has targeted nonholonomic systems (*i.e.*, systems with nonintegrable velocity constraints), such as wheeled mobile robots and the general chained-form system (for a survey of research that has targeted tracking and regulation control of nonholonomic systems see [2], [5], [4], [9], [10], [11], [19], [20], and the references within). However, motivated by the challenging theoretical aspects and numerous practical applications, researchers have also attacked underactuated systems with nonintegrable dynamics (*e.g.*, surface vessels, twin rotor helicopters, underwater vehicles, V/CTOL aircrafts, *etc.*). For example, in [17], Reyhanoglu *et al.* provides a detailed discussion on the controllability and the stabilizability of underactuated mechanical systems with nonintegrable dynamics. The conclusion from this discussion is a result similar to Brockett's condition [1] for nonholonomic systems. That is, Reyhanoglu *et al.* illustrated that underactuated systems with nonintegrable dynamics cannot be asymptotically stabilized by a continuous, time-invariant feedback law. In [12], Pettersen *et al.* showed that underactuated surface vessels cannot be asymptotically stabilized by either continuous or discontinuous time-invariant feedback laws. In addition, Pettersen *et al.* [12] proposed a time-varying feedback controller for an underactuated surface vessel that contained explicit time-periodic sinusoidal terms (similar in structure to [19]) to obtain local exponential regulation. In [13], Pettersen *et al.* modified the continuous time-varying feedback law of [12] to design a controller that also locally exponentially regulates the position and orientation of an underactuated surface vessel.

In addition to the regulation problem, several controllers have also been proposed for the tracking control problem. Specifically, in [8], Godhavn utilized a continuous time-invariant state feedback controller to achieve global exponential position tracking provided the desired surge velocity is always positive; however, due to the control structure, the orientation of the surface vessel is not controlled. In [14], Pettersen *et al.* proposed a tracking controller that achieved global exponential *practical* stability (*i.e.*, global exponential stability of an arbitrarily small neighborhood of the desired trajectory) of an underactuated surface vessel. In [15], Pettersen *et al.*, proposed a continuous time-invariant control law that obtained semi-global exponential position and orientation tracking, provided the desired angular trajectory remains positive. That is, Pettersen *et al.* proved semi-global exponential position and orientation tracking for a class of desired trajectories (*i.e.*, a straight line or a sinusoidal trajectory cannot be tracked).

In this paper, we design a continuous time-varying tracking controller that yields global uniformly ultimately bounded (GUUB) position/orientation tracking. Specifically, we first manipulate a reference model generator and the dynamic model of an underactuated surface vessel into a form that allows a Lyapunov-based control structure to be developed. That is, motivated by the dynamic oscillator designed in [4] by Dixon *et al.* for wheeled mobile robots, a time-varying dynamic oscillator is constructed that globally exponentially forces the position/orientation tracking error to a neighborhood about zero that can be made arbitrarily small. The new result is facilitated by fusing a filtered tracking error transformation with the dynamic oscillator design. In addition, since the only restriction we place on the desired trajectory is that the reference generator remain bounded, it is straightforward to illustrate that the proposed controller also yields a GUUB result for the regulation problem.

The paper is organized as follows. In Section 2, we present the kinematic and dynamic model for an underactuated surface vessel and then transform the open-loop tracking dynamics into a more convenient form for the subsequent controller development and the stability analysis. In Section 3, we present the proposed GUUB tracking control design. The corresponding closed-loop error system is given in Section 4 while the stability analysis is given in Section 5. An extension that illustrates that the proposed tracking controller also solves the regulation problem is given in Section 6. In Section 7, we illustrate that the proposed tracking/regulating controller for underactuated surface vessels can also be applied to other underactuated systems with nonintegrable dynamics such as twin rotor helicopters. Concluding remarks are presented in Section 8.

2. Kinematic and Dynamic Model Development

2.1. Model Formulation

As described in [7], the kinematic equations of motion of the center of mass (COM) for a surface vessel (SV) can be written as follows

$$\dot{q} = S(q)v \tag{1}$$

where $\dot{q}(t) = \begin{bmatrix} \dot{x}_c(t) & \dot{y}_c(t) & \dot{\theta}(t) \end{bmatrix}^T \in \Re^3$ represents the time derivative of $q(t) = [x_c(t) \quad y_c(t) \quad \theta(t)]^T \in \Re^3$, $x_c(t)$, $y_c(t)$, and $\theta(t) \in \Re^1$ denote the Cartesian position and orientation, respectively, of the COM of the SV, the transformation matrix $S(q) \in \Re^{3\times 3}$ is defined as follows

$$S(q) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

and the velocity vector $v(t) \in \Re^3$ is defined as

$$v = [v_1 \quad v_2 \quad v_3]^T \tag{3}$$

where $v_1(t), v_2(t)$, and $v_3(t) \in \Re^1$ denote the surge, sway, and yaw velocities of the SV, respectively. Under the assumptions that: *i*)

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the body-fixed coordinate axis coincides with the center of gravity (CG), *ii*) the mass distribution is homogeneous, and *iii*) the heave, pitch, and roll modes can be neglected, the dynamic model for the SV can be expressed in the following form [7]

$$M\dot{v} + D(v)v = \tau_0 \tag{4}$$

where $\dot{v}(t) \in \Re^3$ denotes the time derivative of $v(t)$ defined in (3), $M \in \Re^{3\times 3}$ represents the constant, diagonal, positive definite inertia matrix, which is explicitly defined as

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_o \end{bmatrix} \tag{5}$$

m, $I_o \in \Re^1$ represent the mass and inertia of the SV, respectively, $D(v) \in \Re^{3\times 3}$ represents the Centripetal-Coriolis and hydrodynamic damping effects, and is explicitly defined as follows

$$D(\nu) = \begin{bmatrix} -X_{v1} & 0 & -mv_2 \\ 0 & -Y_{v2} & mv_1 - Y_{v3} \\ 0 & -N_{v2} & -N_{v3} \end{bmatrix}, \tag{6}$$

X_{v1}, Y_{v2}, Y_{v3}, N_{v2}, and $N_{v3} \in \mathbb{R}^{\not{k}}$ denote constant damping coefficients, and the force-torque control input vector denoted by $\tau_0(t) \in \Re^3$ is explicitly defined as

$$\tau_0(t) = [F \quad 0 \quad \tau]^T \tag{7}$$

where $F(t) \in \Re^1$ denotes a control force that is applied to produce a forward thrust, and $\tau(t) \in \Re^1$ denotes a torque that is applied about the CG (see Figure 1).

In order to simplify the subsequent control development and stability analysis, we first design an outer-loop controller for $F(t)$ and $\tau(t)$ as follows

$$F = -X_{v1}v_1 + mF_1 \tag{8}$$

and

$$\tau = -N_{v2}v_2 - N_{v3}v_3 + I_o\tau_1 \tag{9}$$

where $F_1(t), \tau_1(t) \in \Re^1$ denote subsequently designed auxiliary control inputs. Based on (3), (4), (5), (6), (7), (8), and (9), we can rewrite the expression for the dynamic model given in (4) as follows

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} F_1 + v_2 v_3 \\ \frac{1}{m}\left(Y_{v2}v_2 + Y_{v3}v_3\right) - v_1 v_3 \\ \tau_1 \end{bmatrix}. \tag{10}$$

2.2. Reference Model Development

Motivated by the desire to generate a reference model that satisfies the same dynamics as that given in (4), we take the time derivative of $\dot{x}_c(t), \dot{y}_c(t)$ given in (1) and then use (2), (3), and (10) to obtain the following expression

$$\begin{bmatrix} \ddot{x}_c \\ \ddot{y}_c \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} F_1\cos\theta + \chi_1 \\ F_1\sin\theta + \chi_2 \\ v_3 \end{bmatrix}. \tag{11}$$

where χ_1, $\chi_2 \in \Re^1$ are explicitly defined as

$$\begin{aligned} \chi_1 &= -\frac{Y_{v2}}{m}\left(\dot{y}_c\cos\theta - \dot{x}_c\sin\theta\right)\sin\theta - \frac{Y_{v3}}{m}v_3\sin\theta \\ \chi_2 &= \frac{Y_{v2}}{m}\left(\dot{y}_c\cos\theta - \dot{x}_c\sin\theta\right)\cos\theta + \frac{Y_{v3}}{m}v_3\cos\theta \end{aligned} \tag{12}$$

Based on (11), we construct a reference trajectory generator as follows

$$\begin{bmatrix} \ddot{x}_{rc} \\ \ddot{y}_{rc} \\ \dot{\theta}_r \end{bmatrix} = \begin{bmatrix} F_{1r}\cos\theta_r + \chi_{r1} \\ F_{1r}\sin\theta_r + \chi_{r2} \\ v_{3r} \end{bmatrix} \tag{13}$$

where χ_{r1}, $\chi_{r2} \in \Re^1$ are explicitly defined as

$$\begin{aligned} \chi_{r1} &= -\frac{\sin\theta_r}{m}\left(Y_{v2}\left(\dot{y}_{rc}\cos\theta_r - \dot{x}_{rc}\sin\theta_r\right) + Y_{v3}v_{3r}\right) \\ \chi_{r2} &= \frac{\cos\theta_r}{m}\left(Y_{v2}\left(\dot{y}_{rc}\cos\theta_r - \dot{x}_{rc}\sin\theta_r\right) + Y_{v3}v_{3r}\right) \end{aligned} \tag{14}$$

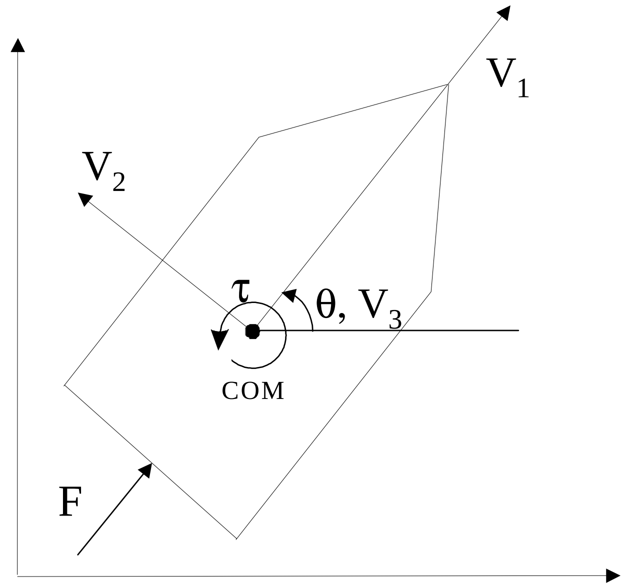

Figure 1 – Actuator Diagram for an Underactuated Surface Vessel

where $x_{cr}(t), y_{cr}(t), \theta_r(t) \in \Re^1$ represent the Cartesian position and orientation of the reference SV, respectively, and $F_{1r}(t), v_{3r}(t) \in \Re^1$ denote reference input signals. It is assumed that the reference model is constructed such that $x_{cr}(t)$, $y_{cr}(t)$, $\theta_r(t)$, $\dot{x}_{cr}(t)$, $\dot{y}_{cr}(t)$, $\dot{\theta}_r(t)$, $\ddot{x}_{cr}(t)$, $\ddot{y}_{cr}(t)$, $\dot{v}_{3r}(t)$, $F_{1r}(t) \in \mathcal{L}_\infty$ where $\dot{v}_{3r}(t)$ denotes the time derivative of $v_{3r}(t)$ defined in (13). Note that the reference orientation is generated by a reference velocity input rather than a reference force or torque input to facilitate the subsequent stability analysis.

2.3. Open-Loop Error System Formulation

To rewrite the open-loop tracking error system in a more convenient form, we define the following global invertible transformation

$$\begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\tilde{\theta}\cos\theta + 2\sin\theta & -\tilde{\theta}\sin\theta - 2\cos\theta & 2\frac{Y_{v3}}{m} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ \tilde{\theta} \end{bmatrix} \tag{15}$$

where $w(t) \in \Re^1$, $z(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T \in \Re^2$ are auxiliary tracking error variables, $r_x(t)$, $r_y(t) \in \Re^1$ are filtered tracking error variables defined as

$$r_x = \dot{\tilde{x}} + \mu\tilde{x} \qquad r_y = \dot{\tilde{y}} + \mu\tilde{y} \tag{16}$$

$\mu \in \Re^1$ is a positive constant control gain, and $\dot{\tilde{x}}(t)$, $\dot{\tilde{y}}(t) \in \Re^1$ represents the time derivative of $\tilde{x}(t)$, $\tilde{y}(t)$ where $\tilde{x}(t)$, $\tilde{y}(t)$, $\tilde{\theta}(t) \in \Re^1$ denote the difference between the actual position/orientation and the reference position/orientation of the SV as follows

$$\tilde{x} = x_c - x_{cr} \qquad \tilde{y} = y_c - y_{cr} \qquad \tilde{\theta} = \theta - \theta_r \tag{17}$$

and $x_{cr}(t)$, $y_{cr}(t)$, $\theta_r(t)$ are generated by the reference generator defined in (13). By taking the time derivative of (15), and using (1), (2), (3), (11), (13), (16), and (17), we can rewrite the open-loop tracking error dynamics in the following advantageous form

$$\begin{aligned} \dot{w} &= u^T J^T z + f \\ \dot{z} &= u \\ \dot{u}_2 &= -\dot{v}_{3r} + \tau_1 \end{aligned} \tag{18}$$

where the auxiliary control signal $u(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T \in \Re^2$ is related to the open-loop tracking error variables defined in (18) according to the following globally invertible transformation

$$u = T^{-1}\begin{bmatrix} F_1 \\ v_3 \end{bmatrix} - \Pi \quad \begin{bmatrix} F_1 \\ v_3 \end{bmatrix} = T\left(u + \Pi\right), \tag{19}$$

the matrix $T(\cdot) \in \Re^{2\times 2}$ and the auxiliary vector $\Pi(\cdot) \in \Re^2$ are defined as

$$T = \begin{bmatrix} r_x \sin\theta - r_y \cos\theta & 1 \\ 1 & 0 \end{bmatrix} \tag{20}$$

$$\Pi = \begin{bmatrix} v_{3r} \\ F_{1r}\cos z_1 + \frac{1}{m} Y_{v2} \left(\dot{y}_{rc}\cos\theta_r - \dot{x}_{rc}\sin\theta_r\right)\sin z_1 \\ -\mu\left(v_1 - \dot{x}_{rc}\cos\theta - \dot{y}_{rc}\sin\theta\right) + \frac{Y_{v3}}{m} v_{3r}\sin z_1 \end{bmatrix}, \tag{21}$$

$J \in \Re^{2\times 2}$ is a skew-symmetric matrix defined as

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{22}$$

and the auxiliary signal $f(\cdot) \in \Re^1$ is defined as

$$\begin{aligned} f = 2\Big(&v_{r3}z_2 - F_{1r}\sin z_1 + \mu\left(\sin\theta\,\dot{\tilde{x}} - \cos\theta\,\dot{\tilde{y}}\right) \\ &+ \frac{1}{m} Y_{v2}\left(\left(\dot{y}_{rc}\cos\theta_r - \dot{x}_{rc}\sin\theta_r\right)\cos z_1 - v_2\right) \\ &+ \frac{Y_{v3}v_{3r}}{m}\left(\cos z_1 - 1\right). \end{aligned} \tag{23}$$

Remark 1 *The open-loop tracking error system given in (18) is in terms of the subsequently designed control inputs* $u_1(t)$ *and* $\tau_1(t)$*. From* $u_1(t)$ *and* $\tau_1(t)$*, we can utilize (8), (9), and (19) to recover the original control inputs* $F(t)$ *and* $\tau(t)$*.*

Remark 2 *Based on the definition of* $r_x(t)$ *and* $r_y(t)$ *given in (16), standard arguments [3] can be utilized to prove that: i) if* $r_x(t)$, $r_y(t) \in \mathcal{L}_\infty$ *then* $x_c(t)$, $\tilde{x}(t)$, $y_c(t)$, $\tilde{y}(t) \in \mathcal{L}_\infty$*, and ii) if* $r_x(t)$, $r_y(t)$ *are GUUB, then* $x_c(t)$, $\tilde{x}(t)$, $y_c(t)$, $\tilde{y}(t)$ *are GUUB.*

3. Control Development

Our control objective is to design a controller that exponentially forces the tracking error to a neighborhood about zero that can be made arbitrarily small (*i.e.*, GUUB). To this end, we define an auxiliary error signal $\tilde{z}(t) \in \Re^2$ as the difference between the subsequently designed auxiliary signal $z_d(t) \in \Re^2$ and the transformed variable $z(t)$ defined in (15) as follows

$$\tilde{z} = \begin{bmatrix} \tilde{z}_1 & \tilde{z}_2 \end{bmatrix}^T = z_d - z. \tag{24}$$

In addition, we define an auxiliary error signal $\eta(t) \in \Re^1$ as the difference between the subsequently designed auxiliary signal $u_{d1}(t) \in \Re^1$ and the auxiliary signal $u_1(t)$ defined in (19) as shown below

$$\eta = u_{d1} - u_1. \tag{25}$$

3.1. Control Formulation

Based on the structure of the open-loop error system given by (18) and the subsequent stability analysis, we design the auxiliary signals $u_{d1}(t)$ and $u_2(t)$ as follows

$$\begin{bmatrix} u_{d1} & u_2 \end{bmatrix}^T = u_a - k_2 z \tag{26}$$

where the auxiliary control signal $u_a(t) \in \Re^2$ is defined as

$$u_a = \left(\frac{k_1 w + f}{\delta_d^2}\right) J z_d + \Omega_1 z_d, \tag{27}$$

the auxiliary signal $z_d(t) \in \Re^2$ is defined by the following oscillator-like relationship

$$\dot{z}_d = \frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{k_1 w + f}{\delta_d^2} + w\Omega_1\right) J z_d \qquad z_d^T(0) z_d(0) = \delta_d^2(0), \tag{28}$$

the auxiliary terms $\Omega_1(t)$, $\delta_d(t) \in \Re^1$ are defined as follows

$$\Omega_1 = k_2 + \frac{\dot{\delta}_d}{\delta_d} + \frac{k_1 w^2 + wf}{\delta_d^2} \tag{29}$$

$$\delta_d = \gamma_0 \exp(-\gamma_1 t) + \varepsilon_1, \tag{30}$$

k_1, k_2, γ_0, γ_1, $\varepsilon_1 \in \Re^1$ are positive, constant design parameters, and $f(\cdot)$ was defined in (23). Based on (18) and the subsequent stability analysis, we design the control torque input $\tau_1(t)$ given in (9) as follows

$$\tau_1 = \dot{u}_{d1} + \dot{v}_{3r} + k_3\eta - w z_2 + \tilde{z}_1 \tag{31}$$

where $\dot{u}_{d2}(t) \in \Re^1$ denotes the time derivative of $u_{d2}(t)$ defined in (26) (see Appendix A for an explicit expression for $\dot{u}_{d2}(t)$).

Remark 3 *Motivation for the structure of (28) is obtained by taking the time derivative of* $z_d^T z_d$ *as follows*

$$\frac{d}{dt}\left(z_d^T z_d\right) = 2 z_d^T \dot{z}_d = 2 z_d^T \left(\frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{k_1 w + f}{\delta_d^2} + w\Omega_1\right) J z_d\right) \tag{32}$$

where (28) has been utilized. After noting that the matrix J *of (22) is skew symmetric, we can rewrite (32) as follows*

$$\frac{d}{dt}\left(z_d^T z_d\right) = 2\frac{\dot{\delta}_d}{\delta_d} z_d^T z_d. \tag{33}$$

As result of the selection of the initial conditions given in (28), it is easy to verify that

$$z_d^T z_d = \|z_d\|^2 = \delta_d^2 \tag{34}$$

is a unique solution to the differential equation given in (33). The relationship given by (34) will be used during the subsequent error system development and stability analysis.

4. Closed-Loop Error System Development

To facilitate the closed-loop error system development for $w(t)$, we inject the auxiliary control input $u_{d2}(t)$ by adding and subtracting the term $u_{d1}z_2$ to the right-side of the open-loop dynamic expression for $w(t)$ given in (18) and then utilizing (25) to obtain the following expression

$$\dot{w} = \begin{bmatrix} u_{d1} & u_2 \end{bmatrix} J^T z - \eta z_2 + f. \tag{35}$$

After substituting (26) for $\begin{bmatrix} u_{d1} & u_2 \end{bmatrix}$, adding and subtracting $u_a^T J z_d$ to the resulting expression, utilizing (24), and exploiting the skew symmetry of J defined in (22), we can rewrite the dynamics for $w(t)$ as follows

$$\dot{w} = -u_a^T J z_d + u_a^T J \tilde{z} - \eta z_2 + f \tag{36}$$

where we have utilized the fact that $J^T = -J$. Finally, by substituting (27) for only the first occurrence of $u_a(t)$ in (36) and then utilizing the equality given by (34), the skew symmetry of J defined in (22), and the fact that $J^T J = I_2$ (Note that I_2 denotes the standard 2×2 identity matrix), we obtain the final expression for the closed-loop error system for $w(t)$ as follows

$$\dot{w} = -k_1 w + u_a^T J \tilde{z} - \eta z_2. \tag{37}$$

To determine the closed-loop error system for $\tilde{z}(t)$, we take the time derivative of (24), substitute (28) for $\dot{z}_d(t)$, and then substitute (18) for $\dot{z}(t)$ to obtain

$$\dot{\tilde{z}} = \frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{k_1 w + f}{\delta_d^2} + w\Omega_1\right) J z_d - \begin{bmatrix} u_{d1} & u_2 \end{bmatrix}^T + \begin{bmatrix} \eta & 0 \end{bmatrix}^T \tag{38}$$

where the auxiliary control input $u_{d1}(t)$ was injected by adding and subtracting $\begin{bmatrix} u_{d1} & 0 \end{bmatrix}^T$ to the right-side of (38) and then (25) was utilized. After substituting (26) for $\begin{bmatrix} u_{d1} & u_2 \end{bmatrix}^T$, and then substituting (27) for $u_a(t)$ in the resulting expression, we can rewrite the expression given by (38) as follows

$$\dot{\tilde{z}} = \frac{\dot{\delta}_d}{\delta_d} z_d + w\Omega_1 J z_d - \Omega_1 z_d + k_2 z + \begin{bmatrix} \eta & 0 \end{bmatrix}^T. \tag{39}$$

After substituting (29) for only the second occurrence of $\Omega_1(t)$ in (39) and using the fact that $JJ = -I_2$, we can cancel common terms and then rearrange the resulting expression to obtain

$$\dot{\tilde{z}} = -k_2\tilde{z} + wJ\left[\left(\frac{k_1 w + f}{\delta_d^2}\right) J z_d + \Omega_1 z_d\right] + \begin{bmatrix} \eta & 0 \end{bmatrix}^T \tag{40}$$

where (24) was utilized. Finally, since the bracketed term in (40) is equal to $u_a(t)$ defined in (27), we can obtain the final expression for the closed-loop error system for $\tilde{z}(t)$ as follows

$$\dot{\tilde{z}} = -k_2\tilde{z} + wJu_a + \begin{bmatrix} \eta & 0 \end{bmatrix}^T. \quad (41)$$

To develop the closed-loop error system for $\eta(t)$, we take the time derivative of (25), substitute (18) for $\dot{u}_1(t)$, and then rearrange the resulting expression to obtain

$$\dot{\eta} = \dot{u}_{d1} + \dot{v}_{3r} - \tau_1. \quad (42)$$

After substituting for the auxiliary control torque input $\tau_1(t)$ given in (31), we obtain the closed-loop error system for $\eta(t)$ as follows

$$\dot{\eta} = -k_3\eta + wz_2 - \tilde{z}_1. \quad (43)$$

5. Stability Analysis

Theorem 1 *Given the closed-loop system of (37), (41), and (43), the position/orientation tracking error defined in (16) and (17) is GUUB in the sense that*

$$|\tilde{x}(t)|, |\tilde{y}(t)|, \left|\tilde{\theta}(t)\right| \leq \beta_0 \exp(-\lambda_0 t) + \beta_1\varepsilon_1 \quad (44)$$

where β_0, β_1, and $\lambda_0 \in \Re^1$ are positive constants, and ε_1 was originally defined in (30).

Proof: To prove *Theorem 1*, we define a non-negative, scalar function, denoted by $V(t) \in \Re^1$, as follows

$$V = \frac{1}{2}w^2 + \frac{1}{2}\eta^2 + \frac{1}{2}\tilde{z}^T\tilde{z}. \quad (45)$$

After taking the time derivative of (45) and making the appropriate substitutions from (37), (41), and (43), we obtain the following expression

$$\begin{aligned} \dot{V} = \; & w\left[-k_1 w + u_a^T J\tilde{z} - \eta z_2\right] \\ & + \tilde{z}^T\left[-k_2\tilde{z} + wJu_a + \begin{bmatrix} \eta & 0 \end{bmatrix}^T\right] \\ & + \eta\left[-k_3\eta + wz_2 - \tilde{z}_2\right]. \end{aligned} \quad (46)$$

After utilizing the fact that $J^T = -J$, cancelling common terms, and utilizing (45), we can upper bound $\dot{V}(t)$ as follows

$$\dot{V} \leq -2\min\{k_1,\, k_2,\, k_3\} V. \quad (47)$$

Standard arguments can now be employed to solve the differential inequality given in (47) as follows

$$V(t) \leq \exp(-2\min\{k_1,\, k_2, k_3\}\, t)V(0). \quad (48)$$

Finally, based on (45) the expression given in (48) can be rewritten as

$$\|\Psi(t)\| \leq \exp(-\min\{k_1, k_2, k_3\}\, t)\, \|\Psi(0)\| \quad (49)$$

where $\Psi(t) \in \Re^4$ is defined as

$$\Psi = \begin{bmatrix} w & \eta & \tilde{z}^T \end{bmatrix}^T. \quad (50)$$

From (49) and (50), it is straightforward to see that $w(t)$, $\eta(t)$, $\tilde{z}(t) \in \mathcal{L}_\infty$. After utilizing (24), (34), and the fact that $\tilde{z}(t), \delta_d(t) \in \mathcal{L}_\infty$, we can conclude that $z(t)$, $z_d(t) \in \mathcal{L}_\infty$. From the fact that $z(t)$, $w(t) \in \mathcal{L}_\infty$ we can use the inverse transformation of (15), given below

$$\begin{bmatrix} r_x \\ r_y \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sin\theta & -\frac{Y_{v3}}{m}\sin\theta & \frac{1}{2}\tilde{\theta}\sin\theta + 2\cos\theta \\ -\frac{1}{2}\cos\theta & \frac{Y_{v3}}{m}\cos\theta & -\frac{1}{2}\tilde{\theta}\cos\theta - 2\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} \quad (51)$$

to conclude that $r_x(t), r_y(t), \tilde{\theta}(t) \in \mathcal{L}_\infty$. Based on the fact that $r_x(t), r_y(t), \tilde{\theta}(t) \in \mathcal{L}_\infty$ and the fact that the reference trajectory is selected so that $x_{cr}(t)$, $y_{cr}(t)$, $\theta_r(t)$, $\dot{x}_{cr}(t)$, $\dot{y}_{cr}(t)$, $\dot{\theta}_r(t) \in \mathcal{L}_\infty$, we can utilize (16), (17), and Remark 2 to conclude that $\dot{x}_c(t)$, $\dot{y}_c(t)$, $x_c(t)$, $y_c(t)$, $\theta(t) \in \mathcal{L}_\infty$. From (1) and the fact that $\dot{x}_c(t)$, $\dot{y}_c(t) \in \mathcal{L}_\infty$, we can conclude that $v_1(t)$, $v_2(t) \in \mathcal{L}_\infty$. Using the fact that $z(t)$, $\dot{x}_c(t)$, $\dot{y}_c(t)$, $v_1(t)$, $v_2(t) \in \mathcal{L}_\infty$, we can conclude that $f(\cdot), T(\cdot), \Pi(\cdot) \in \mathcal{L}_\infty$ from (23), (20), and (21). Based on these facts, we can now utilize (25), (26), (27), (28), (29), and (30), to show that $u_{d1}(t)$, $u_a(t)$, $\dot{z}_d(t)$, $\Omega_1(t)$, $u_1(t)$, $u_2(t) \in \mathcal{L}_\infty$. From (11), (19), and (20), we can now conclude that $F_1(t)$, $\dot{\theta}(t)$, $v_3(t) \in \mathcal{L}_\infty$. Based on the previous facts, it is easy to show that $\dot{u}_{d1}(t) \in \mathcal{L}_\infty$ (see Appendix A), and hence, from (31) we can conclude that $\tau_1(t) \in \mathcal{L}_\infty$. Since $v_1(t)$, $v_3(t)$, $F_1(t)$, $\tau_1(t) \in \mathcal{L}_\infty$, we can conclude from (8) and (9) that $\tau(t), F(t) \in \mathcal{L}_\infty$. We can now employ standard signal chasing arguments to conclude that all of the remaining signals in the control and the system remain bounded during closed-loop operation.

To prove (44), we first show that $z(t)$ defined in (15) is GUUB by applying the triangle inequality to (24), and hence, obtain the following bound for $z(t)$

$$\begin{aligned} \|z\| \leq \|\tilde{z}\| + \|z_d\| \leq \; & \exp(-\min\{k_1, k_2, k_3\}\, t)\, \|\Psi(0)\| \\ & + \gamma_0 \exp(-\gamma_1 t) + \varepsilon_1 \end{aligned} \quad (52)$$

where (30), (34), (49), and (50) have been utilized. The main result given by (44) can now be directly obtained from Remark 2, (16), (49), (50), (51), and (52). □

6. Setpoint Extension

Since the only restrictions placed on the desired trajectory are that the reference generator remain bounded, the position/orientation tracking problem reduces to the position and orientation regulation problem. That is, if the control objective is targeted at the regulation problem, the desired position and orientation vector, denoted by $q_r = \begin{bmatrix} x_{cr} & y_{cr} & \theta_r \end{bmatrix}^T \in \Re^3$, becomes an arbitrary desired constant vector. Based on the fact that q_r is now defined as a constant vector, it is straightforward to see that $F_{1r}(t)$, and $v_{3r}(t)$ equal zero. Moreover, $f(\cdot)$ defined in (23) reduces to the following expression

$$f = -2\left(\frac{1}{m}Y_{v2}v_2 + \mu v_2\right), \quad (53)$$

and $\Pi(\cdot)$ defined in (21) reduces to

$$\Pi = \begin{bmatrix} 0 \\ -\mu v_1 \end{bmatrix}. \quad (54)$$

Based on the above simplifications, it is straightforward to illustrate that the GUUB result given in *Theorem 1* is also valid for the regulation problem.

7. Twin Rotor Helicopter Extension

In this section, we illustrate how the proposed SV controller can be applied to other systems with nonintegrable dynamics. Specifically, we illustrate how the open-loop tracking error dynamics for a twin rotor helicopter (TRH) can be cast into the same form as the SV given in (18). Based on the fact that the open-loop tracking error dynamics can be represented in the same form, it is straightforward to show that the same design procedure can be applied to develop a TRH tracking controller.

7.1. Model Formulation

Based on the assumptions that: *i*) the friction of the rotary parts and the centrifugal forces are small enough to be neglected, and *ii*) the inertia coupling is neglected, the state space equations for the TRH can be written in a manner similar to [18] as follows

$$\begin{aligned} \ddot{\alpha}_1 = \; & L_1 f_a \sin\theta_H \\ \ddot{\alpha}_2 = \; & L_2 f_a \cos\theta_H - Mg\cos\alpha_2 \\ \ddot{\theta}_H = \; & L_3 f_b \end{aligned} \quad (55)$$

where $f_a(t), f_b(t) \in \Re^1$ are auxiliary control inputs that are related to the actual control input forces $f_1(t), f_2(t) \in R^1$ (see Figure 2) as shown below

$$f_1 = \frac{1}{2}\left(f_a - \frac{f_b}{r}\right) \qquad f_2 = \frac{1}{2}\left(f_a + \frac{f_b}{r}\right) \quad (56)$$

$\alpha_1(t)$, $\alpha_2(t)$, $\theta_H(t) \in R^1$ represent the yaw, pitch, and roll angles of the TRH, respectively, r, L_1, L_2, L_3, and $M \in R^1$ are known, constant mechanical parameters, and $g \in R^1$ is the acceleration due to gravity. Based on (55), we construct the following reference generator

$$\begin{aligned} \ddot{\alpha}_{1r} = \; & L_1 f_{ar} \sin\theta_{Hr} \\ \ddot{\alpha}_{2r} = \; & L_2 f_{ar} \cos\theta_{Hr} - Mg\cos\alpha_{2r} \\ \dot{\theta}_{Hr} = \; & \omega_r \end{aligned} \quad (57)$$

to generate the reference yaw, pitch, and roll angles, denoted by $\alpha_{1r}(t),\ \alpha_{2r}(t),\ \theta_{Hr}(t) \in \mathbb{R}^{\nvdash}$, respectively, where L_1, L_2, L_3, M, and g were defined in (55) and $f_{ar}(t), \omega_r(t) \in \Re^1$ denote reference input variables. It is assumed that the reference model is constructed such that $\alpha_{1r}(t),\ \alpha_{2r}(t),\ \theta_{Hr}(t),\ \dot{\alpha}_{1r}(t),\ \dot{\alpha}_{2r}(t),\ \dot{\theta}_{Hr}(t),\ \ddot{\alpha}_{1r}(t),\ \ddot{\alpha}_{2r}(t),\ \ddot{\theta}_{Hr}(t), \in \mathcal{L}_\infty$.

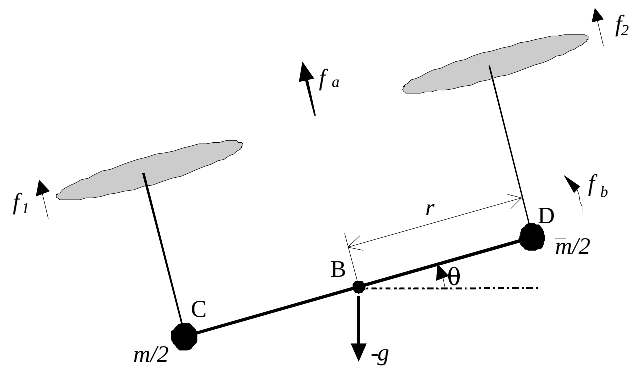

Figure 2 – Actuator Diagram for a Twin Rotor Helicopter

7.2. *Open-Loop Error System Development*

To facilitate the subsequent control synthesis and the corresponding stability proof, we define the following global invertible transformation

$$\begin{bmatrix} \bar{w} \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = \Omega \begin{bmatrix} r_{\alpha 1} \\ r_{\alpha 2} \\ \tilde{\theta}_H \end{bmatrix}, \qquad (58)$$

where $\Omega(\cdot) \in \Re^{3\times 3}$ is a globally invertible transformation matrix constructed as

$$\Omega = \begin{bmatrix} \sigma_1 & \sigma_2 & 0 \\ 0 & 0 & 1 \\ \dfrac{1}{L_1}\sin\theta_H & \dfrac{1}{L_2}\cos\theta_H & 0 \end{bmatrix}, \qquad (59)$$

$\sigma_1(\cdot),\ \sigma_2(.) \in \Re^1$ are defined explicitly as follows

$$\begin{aligned} \sigma_1 &= -\frac{1}{L_1}\left(\tilde{\theta}_H \sin\theta_H + 2\cos\theta_H\right) \\ \sigma_2 &= -\frac{1}{L_2}\left(\tilde{\theta}_H \cos\theta_H - 2\sin\theta_H\right) \end{aligned}, \qquad (60)$$

$\bar{w}(t) \in \Re^1$ and $\bar{z}(t) = \begin{bmatrix} \bar{z}_1(t) & \bar{z}_2(t) \end{bmatrix}^T \in \Re^2$ are auxiliary tracking error variables, $r_{\alpha 1}(t),\ r_{\alpha 2}(t) \in \Re^1$ are filter tracking error variables defined as follows

$$r_{\alpha 1} = \dot{\tilde{\alpha}}_1 + \mu_1 \tilde{\alpha}_1 \qquad r_{\alpha 2} = \dot{\tilde{\alpha}}_2 + \mu_1 \tilde{\alpha}_2 \qquad (61)$$

$\mu_1 \in \Re^1$ is a positive constant control gain, and $\tilde{\alpha}_1(t), \tilde{\alpha}_2(t), \tilde{\theta}_H(t) \in \Re^1$ denote the difference between the actual yaw, pitch, and roll angles and the reference yaw, pitch, and roll angles as follows

$$\tilde{\alpha}_1 = \alpha_1 - \alpha_{1r} \qquad \tilde{\alpha}_2 = \alpha_2 - \alpha_{2r} \qquad \tilde{\theta}_H = \theta_H - \theta_{Hr}. \quad (62)$$

After taking the time derivative of (58), and using (55), (57), (58), (61), and (62), we can rewrite the open-loop tracking error dynamics as follows

$$\begin{aligned} \dot{\bar{w}} &= \bar{u}^T J^T \bar{z} + f_H \qquad (63) \\ \dot{\bar{z}} &= \bar{u}, \\ \dot{\bar{u}}_2 &= L_3 f_b - \dot{\omega}_r \end{aligned}$$

where the auxiliary control input $\bar{u}(t) = \begin{bmatrix} \bar{u}_1(t) & \bar{u}_2(t) \end{bmatrix}^T \in \Re^2$ is defined as

$$\bar{u} = T_H^{-1} \begin{bmatrix} f_a \\ \dot{\theta}_H \end{bmatrix} + \Pi \qquad \begin{bmatrix} f_a \\ \dot{\theta}_H \end{bmatrix} = T_H\left(\bar{u} - \Pi\right), \qquad (64)$$

the matrix $T_H(\cdot) \in \Re^{2\times 2}$ and the auxiliary vector $\Pi_H(\cdot) \in \Re^2$ are defined as follows

$$T_H = \begin{bmatrix} \dfrac{r_{\alpha 2}\sin\theta}{L_2} - \dfrac{r_{\alpha 1}\cos\theta}{L_1} & 1 \\ 1 & 0 \end{bmatrix} \qquad (65)$$

$$\Pi_H = \begin{bmatrix} -\dot{\theta}_{Hr} \\ \Pi_{H1} \end{bmatrix} \qquad (66)$$

where $\Pi_{H1} \in \mathbb{R}^{\nvdash}$ is defined as

$$\begin{aligned} \Pi_{H1} &= -f_{ar}\cos\tilde{\theta}_H - \frac{\cos\theta_H}{L_2} Mg\left(\cos\alpha_2 - \cos\alpha_{2r}\right) \quad (67) \\ &\quad + \mu_1\left(\frac{\dot{\tilde{\alpha}}_1 \sin\theta_H}{L_1} + \frac{\dot{\tilde{\alpha}}_2 \cos\theta_H}{L_2}\right), \end{aligned}$$

$f_H(\cdot) \in \Re^1$ is defined as

$$\begin{aligned} f_H = \;& 2\Big(\dot{\theta}_{Hr}\bar{z}_2 - f_{ar}\sin\tilde{\theta}_H \\ & + \mu_1\left(\frac{\dot{\tilde{\alpha}}_2 \sin\theta_H}{L_2} - \frac{\dot{\tilde{\alpha}}_1 \cos\theta_H}{L_1}\right) \\ & - \frac{\sin\theta_H}{L_2} Mg\left(\cos\alpha_2 - \cos\alpha_{2r}\right)\Big), \end{aligned} \qquad (68)$$

and J was previously defined in (22). From the form of (63), (68), (64), (65), and (66), it is straightforward that the auxiliary input vector $\left[\bar{u}_{d1}(t) \quad \bar{u}_2(t)\right]^T$ can be designed in the same manner as the proposed SV tracking controller to obtain GUUB tracking and regulation results for the TRH.

8. Conclusion

In this paper, we have designed a continuous, time-varying tracking controller for an underactuated surface vessel. Through a Lyapunov-based stability analysis, we have demonstrated that: *i*) the position/orientation tracking error is globally exponentially forced to a neighborhood about zero that can be made arbitrarily small, and *ii*) a unified framework is developed that solves the regulation problem as a simplified case of the tracking control problem. An extension was provided that illustrates that the proposed controller can be applied to other nonlinear underactuated systems subject to nonintegrable dynamics such as the twin rotor helicopter. Using similar techniques as illustrated with the twin rotor helicopter extension, additional systems with similar dynamics may be solved with the proposed controller. For example, in [17], Reyhanoglu *et al.* described a planar prismatic-prismatic-revolute (PPR) robot with an elastic joint that has similar dynamics as the examples presented in this paper. It is straightforward to illustrate that the proposed controller yields a GUUB tracking/regulation result for the PPR elastic-joint robot utilizing similar arguments presented in the twin rotor helicopter extension.

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9. Appendix A: Calculation of $\dot{u}_{d1}(t)$

To calculate $\dot{u}_{d1}(t)$, we take the time derivative of (26) and then substitute for the time derivative of $u_a(t)$ defined in (27) to obtain the following expression

$$\begin{aligned}\dot{u}_{d1} = \; & -\left(\frac{k_1\dot{w}+\dot{f}}{\delta_d^2}\right)z_{d2}+2\left(\frac{(k_1 w+f)\,\dot{\delta}_d}{\delta_d^3}\right)z_{d2} \\ & +\dot{\Omega}_1 z_{d2}+\Omega_1\dot{z}_{d2}+\left(\frac{k_1 w+f}{\delta_d^2}\right)\dot{z}_{d2}-k_2\dot{z}_1\end{aligned} \tag{69}$$

where the time derivatives of $\Omega_1(t)$ and $f(t)$ are explicitly given by the following expressions

$$\dot{\Omega}_1=\frac{\ddot{\delta}_d}{\delta_d}-\frac{\dot{\delta}_d^2}{\delta_d^2}+\frac{(2k_1 w+f)\,\dot{w}+w\dot{f}}{\delta_d^2}-2\frac{\left(k_1 w^2+wf\right)\dot{\delta}_d}{\delta_d^3} \tag{70}$$

and

$$\begin{aligned}\dot{f} = \; & 2\Big(v_{3r}u_2+\dot{v}_{3r}z_2-\dot{F}_{1r}\sin z_1-F_{1r}u_1\cos z_1\Big) \\ & +2\mu\left(\left(\dot{\tilde{x}}\cos\theta+\dot{\tilde{y}}\sin\theta\right)v_3\right) \\ & -2\mu\left(\frac{Y_{v2}}{m}\left(\dot{y}_c\cos\theta-\dot{x}_c\sin\theta\right)+\tfrac{Y_{v3}v_3}{m}\right) \\ & -\ddot{x}_{rc}\sin\theta+\ddot{y}_{rc}\cos\theta \\ & +\frac{2Y_{v2}}{m}\left(\ddot{y}_{rc}\cos\theta_r-\ddot{x}_{rc}\sin\theta_r\right. \\ & \left.-\dot{\theta}_r\left(\dot{y}_{rc}\sin\theta_r-\dot{x}_{rc}\cos\theta_r\right)\right)\cos z_1 \\ & -\frac{2Y_{v2}}{m}\left(\left(\dot{y}_{rc}\cos\theta_r-\dot{x}_{rc}\sin\theta_r\right)u_1\sin z_1\right) \\ & +\frac{2Y_{v2}}{m}\left(\left(\tfrac{1}{m}\left(Y_{v2}v_2+Y_{v3}v_3\right)-v_1v_3\right)\right) \\ & +2\left(\frac{Y_{v3}\dot{v}_{3r}}{m}\left(\cos z_1-1\right)-u_1\frac{Y_{v3}v_{3r}}{m}\sin z_1\right)\end{aligned} \tag{71}$$

where (10), (11), (13), (18), and the second time derivative of (17) was utilized. Based on the definition of $\delta_d(t)$ given in (30), the fact that $z(t)$, $\dot{z}(t)$, $\dot{\tilde{\theta}}(t)$, $u(t)$, $\dot{\tilde{x}}(t)$, $\dot{\tilde{y}}(t)$, $\dot{f}(t)$, $\dot{\Omega}_1(t)$, $\dot{w}(t)$, $\dot{z}_d(t)$, $w(t)$, $f(t)$, $z_d(t)$, $u_{d1}(t)$, $\eta(t)\in\mathcal{L}_\infty$ (see *Theorem 1*), and the fact that the reference trajectory is selected so that $x_{cr}(t)$, $y_{cr}(t)$, $\theta_r(t)$, $\dot{x}_{cr}(t)$, $\dot{y}_{cr}(t)$, $\dot{\theta}_r(t)$, $\ddot{x}_{cr}(t)$, $\ddot{y}_{cr}(t)$, $\ddot{\theta}_r(t)$, $\dot{v}_{3r}(t)\in\mathcal{L}_\infty$, it is straightforward to see from (69), (70), and (71) that $\dot{u}_{d1}(t)\in\mathcal{L}_\infty$.