

Block Decoupling of Linear Systems

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Abstract

In this paper, the Kronecker product of matrices is used to achieve the decomposition of finite dimensional linear systems into a set of decoupled systems of smaller dimension. A class of systems is specified for which such decomposition can be carried out. Two particular cases of application are studied.

1 Introduction

The problem of system decoupling is a well known topic of research, its applications spanning spatially distributed systems as well as large scale systems. Among the many approaches to this problem, we single out the work by [1] that explored the algebraic properties of spatially invariant systems to achieve decoupling. The idea behind it is to figure out what are the subsystems a very complex system is made of and the way they are coupled. Under certain invariance assumptions it is possible to "diagonalize" this coupling structure and come up with a decomposition into systems of smaller dimension. In this paper we look at the same problem for the particular case of finite dimensional linear systems. We explore the Kronecker product as a tool to neatly express the coupling structure among subsystems of a linear system of large dimension and use its properties to achieve structural or block decoupling. In section 2.3, a decoupling basis is suggested that lead to a hierarchical structure for the flux of information in control applications.

This paper has two sections, section 2 introduces the concept, while subsections 2.2 and 2.3 present the two scenarios of application.

2 Block decoupling

Of major importance in block decoupling is the following property of the Kronecker product of matrices:

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Lemma 1 *If \mathbf{v}^l is such that $\mathbf{\Gamma}\mathbf{v}^l = \lambda_l\mathbf{v}^l$, then for any matrix \mathbf{A} and conformal \mathbf{z} , the following is true:*

$$(\mathbf{\Gamma} \otimes \mathbf{A}) (\mathbf{v}^l \otimes \mathbf{z}) = \lambda_l (\mathbf{v}^l \otimes \mathbf{A}\mathbf{z})$$

The previous equality allow us to show, among other things, the following result:

Theorem 2 (*Block decoupling*) *Consider the following linear system*

$$\dot{\mathbf{x}}(t) = \sum_{j=1}^p [\mathbf{\Gamma}_j \otimes \mathbf{A}_j] \mathbf{x}(t) + [\mathbf{\Gamma}_j \otimes \mathbf{B}_j] \mathbf{u}(t), \quad (1)$$

$$\mathbf{y}(t) = \sum_{j=1}^p [\mathbf{\Gamma}_j \otimes \mathbf{C}_j] \mathbf{x}(t) \quad (2)$$

where $\mathbf{\Gamma}_l \in \mathcal{R}^{m \times m}$ and $\mathbf{A}_j \in \mathcal{R}^{n \times n}$. If there is a set of m linearly independent vectors $\{\mathbf{v}^1, \dots, \mathbf{v}^m\}$ such that $\mathbf{\Gamma}_j \mathbf{v}^l = \lambda_{jl} \mathbf{v}^l$ for $(j, l) \in [1, p] \times [1, m]$, then it is possible to decouple (1) in the form:

$$\dot{\mathbf{x}}(t) = \sum_{l=1}^m \mathbf{v}^l \otimes \dot{\mathbf{x}}_l(t), \quad \mathbf{y}(t) = \sum_{l=1}^m \mathbf{v}^l \otimes \mathbf{y}_l(t) \quad (3)$$

$$\mathbf{u}(t) = \sum_{l=1}^m \mathbf{v}^l \otimes \mathbf{u}_l(t), \quad \mathbf{y}_l(t) = \sum_{j=1}^p \lambda_{jl} \mathbf{C}_j \mathbf{x}_l(t) \quad (4)$$

where $\mathbf{x}_l(t)$ are solution of:

$$\dot{\mathbf{x}}_l(t) = \left[\sum_{j=1}^p \lambda_{jl} \mathbf{A}_j \right] \mathbf{x}_l(t) + \left[\sum_{j=1}^p \lambda_{jl} \mathbf{B}_j \right] \mathbf{u}_l(t) \quad (5)$$

Proof: The result can be easily derived by substituting (3) in (1) and using lemma 1. ■

It can be shown that if the set $\{\mathbf{v}^1, \dots, \mathbf{v}^m\}$ is orthogonal, then $\mathbf{x}_l(t)$ can be computed from $\mathbf{x}(t)$ by means of the projection $\mathbf{x}_l(t) = [\mathbf{v}^l \otimes \mathbf{I}]^T \mathbf{x}(t)$. When designing optimal controllers for systems in the form (1), if the sets $\{\mathbf{v}^1 \otimes \mathbf{x}_1(t), \dots, \mathbf{v}^m \otimes \mathbf{x}_m(t)\}$ and $\{\mathbf{v}^1 \otimes \mathbf{u}_1(t), \dots, \mathbf{v}^m \otimes \mathbf{u}_m(t)\}$ are both orthogonal with respect to the cost function, then the optimization can also be cast as m independent problems of smaller dimension.

2.1 Spatial Invariance

One of the situations where theorem 2 proves to be useful is in the decoupling of spatially invariant systems [2].

Definition: A system described by (1) is spatially invariant if Γ_j are circulant matrices. This definition is motivated by the fact that if \mathbf{P} is a permutation matrix such that $\mathbf{P}\mathbf{e}_i = \mathbf{e}_{i+1 \bmod m}$, then for every circulant Γ_j we have $\mathbf{P}\Gamma_j\mathbf{P}^{-1} = \Gamma_j$. By inspection of (1) we see that a spatially invariant system is invariant under state transformations of the type $\tilde{\mathbf{x}}(t) = (\mathbf{P} \otimes \mathbf{I}_{n \times n}) \mathbf{x}(t)$.

Remark 1 For every set of circulant matrices $\{\Upsilon_1, \dots, \Upsilon_k\}$, $\Upsilon_j \in \mathcal{R}^{m \times m}$, there exist a set of common orthogonal vectors¹ $\{\mathbf{v}^1, \dots, \mathbf{v}^m\}$ such that $\Upsilon_j \mathbf{v}^l = \lambda_{jl} \mathbf{v}^l$.

We can use this fact and theorem 2 to state:

Theorem 3 Spatially invariant systems can be decoupled in the form (3)-(5).

2.2 Hierarchical Decoupling

Certain systems, when expressed in the form (1), lead to matrices Γ_j for which there is more than one set of linearly independent eigenvectors $\{\mathbf{v}^1, \dots, \mathbf{v}^m\}$. In this case, one can use this freedom and choose a "handy set". For certain systems it is possible to choose a basis such that each decoupled system is associated with a node of a hierarchical structure. We illustrate this concept by means of a simple example. Assume we have an array of 8 equal SISO systems that satisfy the state equation $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u + \zeta(t)$, where $\zeta(t)$ accounts for the coupling among systems. Furthermore, consider that there is a "regional" dynamics \mathbf{F} such that the global system can be described in the form (1) as:

$$\dot{\mathbf{x}}(t) = [\mathbf{I} \otimes \mathbf{A} + \mathbf{\Gamma} \otimes \mathbf{G}] \mathbf{x}(t) + [\mathbf{I} \otimes \mathbf{B}] \mathbf{u}(t),$$

where $\mathbf{\Gamma} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, \mathbf{x} and \mathbf{u} are the vector concatenation of the individual state and control of the 8 subsystems. In this case, a possible set of eigenvectors would be the one comprising the columns of the matrix in figure 1. This choice of eigenvectors is consistent with a binary partition of the 8 subsystems which is meaningful in resource allocation problems. In this case, \mathbf{v}^1 accounts for the "common" behavior or spatial mean value of all systems, \mathbf{v}^2 for the transfer of resources between the two groups of systems $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$. In the next level of partitioning we have \mathbf{v}^3 accounting for the transfer of resources

¹Such vectors result from the Fourier series expansion for sequences of size m and can be computed as $[\mathbf{v}^l]_q = e^{i2\pi(l/q)}$.

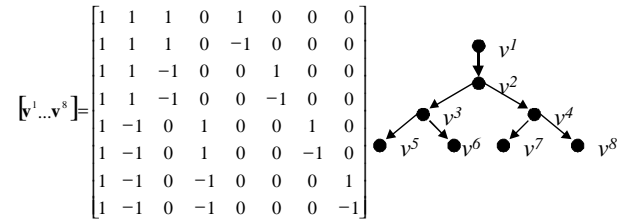


Figure 1: Definition of the eigenvectors $\{\mathbf{v}^1, \dots, \mathbf{v}^8\}$ and associated hierarchical structure.

between $\{1, 2\}$ and $\{3, 4\}$, i.e., inside one of the partitions associated with \mathbf{v}^2 . This nesting relation among the vectors of the basis is represented by the graph of figure 1. Controllers can then be designed for the 8 decoupled systems that operate in the directions \mathbf{v}^1 to \mathbf{v}^8 . In resource allocation problems, where the total amount of resources is constrained by $\sum_{i=1}^8 [\mathbf{u}(t)]_i = d(t)$, we only need to fix the solution in the direction of \mathbf{v}^1 by setting $\mathbf{u}_1(t) = \frac{1}{8}d(t)$. The remaining directions \mathbf{v}^2 to \mathbf{v}^8 are not affected by the constraint. This example is not that of an uncommon case, for there is a whole class of systems, for which it is possible to model group dynamics in a way that $\{\mathbf{v}^1, \dots, \mathbf{v}^m\}$ follows this graph structure.

This structure imposes a hierarchy on the way information flows. Assume we have a decision system (controller) for each of the decoupled systems and that we can measure directly the state $\mathbf{x}(t)$. In such case, we need to process information and get measurements of the state of each of the decoupled systems. This can always be done by means of a linear transformation of $\mathbf{x}(t)$. For most sets $\{\mathbf{v}^1, \dots, \mathbf{v}^m\}$, this means that the controllers of the decoupled systems must have access to measurements of the entire $\mathbf{x}(t)$, which, for large-scale systems, is not acceptable. But with this hierarchical structure, the state in the direction of, e.g., \mathbf{v}^8 can be computed only by measuring $[\mathbf{x}(t)]_7$ and $[\mathbf{x}(t)]_8$. Moving up, \mathbf{v}^4 only need to measure $[\mathbf{x}(t)]_7 + [\mathbf{x}(t)]_8$ and $[\mathbf{x}(t)]_5 + [\mathbf{x}(t)]_6$. This way, we may use an information structure based on the graph of figure 1, where a given node adds up his own measurements and makes it available to its upper node as a measurement. Control decisions propagate down the same structure.

References

- [1] Bamieh B., Paganini F., Dahleh M., "Distributed Control of Spatially-Invariant Systems", December 4 1999
- [2] Laughlin D., Morari M., Braatz R.D., "Robust Performance of Cross-Directional Basis-Weight Control in Paper Machines", Automatica, vol 29, pp.1395-1410, 1993