Robust output feedback stabilization of nonlinear systems

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Abstract

This paper describes a direct way to solve the robust output feedback stabilization problem for a class of uncertain nonlinear systems with nonlinear parameterization using the backstepping technique. The backstepping method is performed in a batch way rather than recursively. The paper begins with the stabilization of a system containing a series of integrators with unknown gains. The solution of the problem is then used to solve the output feedback stabilization problem of the nonlinear system.

Keywords: Robust nonlinear stabilization, Nonlinear parameterization, Backstepping.

1 Introduction

The backstepping technique gives us a tool to prove the solvability of many nonlinear control problems. Unfortunately, the procedure is lengthy and tedious due to its recursive nature. However, [1] provides a structural view point of the backstepping method; it replaces the recursive algorithm with a nonlinear transformation. The purpose of this paper is to simplify the procedure of solving the output feedback stabilization problem for a class of uncertain minimum-phase nonlinear systems with unknown parameters entering nonlinearly which was described in [2]. In section 2, we start with the robust stabilization of a series of integrators with unknown gains using the backstepping method via the technique introduced in [1]. Then, in section 3, we extend the idea to include some nonlinearities and uncertainties in order to solve the output stabilization problem posed in [2].

2 Robust Stabilization

Let us first consider a system given in the following form

$$\dot{x} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \right) \triangleq A(Nx + bu) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and each a_i is an unknown positive constant with known upper bound β_i and lower bound α_i, i.e. $0 < \alpha_i \leq a_i \leq \beta_i$. Physically, system (1) is a series of integrators with unknown gains or virtual control coefficients as described in [3]. Before proceeding, we define a vector or matrix function of x to have lower triangular dependence **LTD** (strictly lower triangular dependence **SLTD**) in x if the i-th component or row is a function of $\vec{x}_i$ ($\vec{x}_{i-1}$) only, where $\vec{x}_i \triangleq (x_1, \cdots, x_i)$, $i \leq n$ and $x \in \mathbb{R}^n$.

Firstly, define a nonlinear transformation ($x \to z$) and full state feedback

$$z = x + N^T f(x), \quad u = -b^T f(x) \tag{2}$$

where $f(x) = [f_1(x_1)\ f_2(\vec{x}_2) \cdots f_n(\vec{x}_n)]^T$. The vector field f is smooth (we will ignore the function argument of f throughout the paper for simplicity) and $f(0) = 0$. By using the fact that $f(0) = 0$, it can be shown that $x = \Phi(x, f)z$, where Φ is a $n \times n$ matrix **LTD** in x and **SLTD** in f. Then by applying transformation (2) to system (1) and considering the function $V = \frac{1}{2} z^T A^{-1} z$, we have

$$\dot{V} = z^T N z - z^T f + z^T A^{-1} N^T \nabla f A N \Phi z. \tag{3}$$

With some manipulation, the last term in (3) becomes

$$\begin{aligned} & z^T A^{-1} N^T \nabla f A N \Phi z \\ & \quad \leq \frac{1}{2} z^T A_l^{-1} N^T F N A_l^{-1} z + \frac{1}{2} \sum_{i=2}^{n} (i-1) z^T z \end{aligned} \tag{4}$$

where $F = \text{diag}\{\Delta_1, \cdots, \Delta_n\}$ with each element defined as:

$$\Delta_i = \nabla f_i A_u N \tilde{\Phi} N^T A_u (\nabla f_i)^T, \quad 1 \leq i \leq n, \tag{5}$$

and ∇f_i the i-th row of ∇f, $\tilde{\Phi} = \text{diag}\{\Phi_1^T \Phi_1, \cdots, \Phi_n^T \Phi_n\}$ (Φ_i^T is the i-th row of the matrix Φ), $A_l = \text{diag}\{\alpha_1, \cdots, \alpha_n\}$, $A_u = \text{diag}\{\beta_1, \cdots, \beta_n\}$. Then we define a function $H(x, f)$ as

$$H(x, f) = \frac{1}{2} A_l^{-1} N^T F N A_l^{-1} z + \frac{1}{2} \sum_{i=2}^{n} (i-1) z. \tag{6}$$

If we choose $f(x) = N^T z + Cz + H(x, f)$ where C is a diagonal matrix with positive real elements i.e. $C = \text{diag}\{c_1, \cdots, c_n\}$, it implies that $\dot{V} \leq -z^T C z$. The term $N^T z + Cz + H(x, f)$ is called a model of iteration and, by observation, it has the required structure as introduced in [1], i.e. **LTD** in x and **SLTD** in f, therefore the stabilizing control, $u = -f_n(\vec{x}_n)$, can be computed recursively when implementation is required. The recursive equations generated are just the backstepping equations.

3 Robust Output feedback stabilization

In [2], a class of n-dimensional uncertain single-input single-output nonlinear systems with nonlinear parameterization can be transformed into

$$\begin{aligned}\dot{\eta} &= \Gamma(\theta) + y\beta(\theta) + y\gamma(y,\theta), \quad \eta \in \mathbb{R}^{n-1}, y \in \mathbb{R}\\ \dot{y} &= \eta_1 + \frac{d_2(\theta)}{d_1(\theta)}y + y\phi_1(y,\theta) + d_1(\theta)\xi_1\\ \dot{\xi} &= \Lambda\xi + \sigma(y)u, \quad \xi \in \mathbb{R}^{\rho-1}, u \in \mathbb{R}\end{aligned} \tag{7}$$

where θ is an unknown parameter vector belonging to a compact set $\Omega \subset \mathbb{R}^p$, $\Gamma(\theta)$ is an $(n-1)\times(n-1)$ stable matrix, β, γ are smooth vector fields, ϕ_1 is a smooth function, $d_i(\theta)$ $(i = 1, 2)$ is an unknown constant depending on θ, and $\rho > 1$ is the relative degree of the overall system. The matrix Λ is defined as

$$\Lambda = \begin{bmatrix} -\lambda_1 & 1 & & 0 \\ & -\lambda_2 & \ddots & \\ & & \ddots & 1 \\ 0 & & & -\lambda_{\rho-1} \end{bmatrix} \tag{8}$$

where each λ_i is positive real. System (7) is basically a series of stable filters augmented to a relative degree 1 system with output y, while maintaining the relative degree of the overall system unchanged. If we define a new $(\rho \times 1)$ vector $x = [y \;\; \xi]^T$ and $\bar{u} = \sigma(y)u$, system (7) becomes

$$\begin{aligned}\dot{x} &= A(Nx + b\bar{u}) + Dx + e\eta_1 + ey(\frac{d_2(\theta)}{d_1(\theta)} + \phi_1(y,\theta))\\ \dot{\eta} &= \Gamma(\theta) + y\beta(\theta) + y\gamma(y,\theta), \quad \eta \in \mathbb{R}^{n-1}, y \in \mathbb{R}\end{aligned} \tag{9}$$

where $A = \mathrm{diag}\{d_1(\theta), 1, \cdots, 1\}$, $e = [1 \;\; 0 \;\; \cdots \;\; 0]^T$ and $D = \mathrm{diag}\{0, -\lambda_1, \cdots, -\lambda_{\rho-1}\}$. System (9) is similar to (1) except with additional nonlinear terms and a zero dynamics, so we will solve the stabilization problem proposed and solved in [2] but using the same procedure as in section 2.

After applying transformation (2) to the $\dot{x}$ equation in system (9), we then define a candidate Lyapunov function as $V = \frac{1}{2}z^T A^{-1} z + \eta^T P(\theta)\eta$, since we assume $d_1(\theta) > 0$. The matrix $P(\theta)^T = P(\theta) > 0$ is the solution of the matrix equation $P(\theta)\Gamma(\theta) + \Gamma^T(\theta)P(\theta) \leq -aI$ (since $\Gamma(\theta)$ is stable). Therefore,

$$\begin{aligned}\dot{V} &= z^T N z - z^T f + z^T A^{-1} Dx + \frac{d}{dt}\eta^T P(\theta)\eta\\ &\quad + z^T A^{-1} N^T \nabla f (ANx + Dx + e\eta_1 + ey(\frac{d_2(\theta)}{d_1(\theta)} + \phi_1(y,\theta)))\\ &\quad + z^T A^{-1} ey(\frac{d_2(\theta)}{d_1(\theta)} + \phi_1(y,\theta)) + z^T A^{-1} e\eta_1.\end{aligned} \tag{10}$$

Various terms in (10) are then manipulated and we have

$$\begin{aligned} z^T A^{-1} Dx &= z^T Dx\\ z^T A^{-1} ey(\frac{d_2(\theta)}{d_1(\theta)} + \phi_1(y,\theta)) &\\ &\leq z^T ey(k_2 + k_1\alpha_1(y))\\ z^T A^{-1} e\eta_1 &\leq \frac{1}{4}z^T eyk_1^2 + \eta^T\eta\\ z^T A^{-1} N^T \nabla f ANx &\leq \frac{1}{2} z^T A_l^{-1} N^T F N A_l^{-1} z\\ &\quad + \frac{1}{2}\sum_{i=2}^{n}(i-1)z^T z\\ z^T A^{-1} N^T \nabla f Dx &= z^T N^T \nabla f Dx\\ z^T A^{-1} N^T \nabla f e\eta_1 &\leq \frac{1}{4} z^T \Delta z + (\rho - 1)\eta^T\eta\\ z^T A^{-1} N^T \nabla f ey(\frac{d_2(\theta)}{d_1(\theta)} + \phi_1(y,\theta))) &\\ &\leq \frac{1}{2}z^T \Delta z + z^T ey(\rho - 1)(k_3 + \alpha_1^2(y))\\ \frac{d}{dt}\eta^T P(\theta)\eta &\leq -(a-2)\eta^T\eta + z^T ey(k_4 + k_5\alpha_2^2(y))\end{aligned} \tag{11}$$

where, for all θ, the inequalities: $k_1 \geq d_1^{-1}(\theta)$; $k_2 \geq |d_2(\theta)|/|d_1^2(\theta)|$; $k_3 \geq d_2^2(\theta)/d_1^2(\theta)$; $k_4 \geq ||P(\theta)||^2||\beta(\theta)||^2$; $k_5 \geq ||P(\theta)||^2$; $\alpha_1(y) \geq |\phi_1(y,\theta)|$; $\alpha_2(y) \geq ||\gamma(y,\theta)||$ hold and $\Delta = \mathrm{diag}\{0, (\partial f_1/\partial x_1)^2, \cdots, (\partial f_{\rho-1}/\partial x_1)^2\}$, $a > 0$. Next, we collect and group terms from the right-hand side of (11) and also define a new function $H(x, f)$ similar to (6). If we choose $f = N^T z + Cz + H(x, f)$ ($C = \mathrm{diag}\{c_1, \cdots, c_\rho\} > 0$) and $a > (2 + \rho)$, then (10) becomes $\dot{V} \leq -z^T Cz - \varepsilon\eta^T\eta$. Finally, since the model of iteration, i.e. $N^T z + Cz + H(x, f)$, is **LTD** in x and **SLTD** in f, a stabilizing control $\bar{u} = -f_n(\vec{x}_n)$ can be solved recursively. Therefore, we have introduced an alternative, clearer way to prove Theorem 4.1 in [2].

4 Conclusions

In this paper, we have provided a clearer way to solve the robust stabilization problem for a class of uncertain nonlinear systems with nonlinear parameterization using the backstepping method. The method presented is purely structural and the existence of a solution is guaranteed by checking whether the model of iteration has the right structure or not. This approach will allow us to solve for more complicated nonlinear control problems.

References

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