Asymptotic tracking of periodic trajectories for a simple mechanical system subject to non-smooth impacts

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Abstract

An infinitely rigid, fully-actuated mass is considered, which moves on a plane within a closed region delimited by an infinitely massive and rigid circular barrier. The tracking problem of a class of periodic trajectories involving an infinite number of impacts is considered. Since the jumps in the velocities at the impact times render difficult (if not impossible) to obtain the classical stability and attractivity properties for the tracking error, such properties are properly amended for the case of interest. A simple PD-like control law is proposed, giving rise to control forces that are piece-wise continuous functions of time.

1 Introduction

Modeling and control of impact phenomena is a challenging research field, which has attracted the attention of researchers from mathematics, mechanics and engineering, due to the variety of different problems involved (see [1] and the references therein). When only *elastic* impacts are considered between perfectly rigid bodies, and friction during the impact is considered negligible (such approximations can be accepted, for example, in collisions between spheres made of glass or steel), the classical Erdmann-Weierstrass corner conditions [2] are often sufficient to determine the effects of impacts on the colliding bodies; in particular, this happens in this paper, where no multiple impacts can occur. The assumption of rigidity of the impacting bodies leads to the so called *non-smooth impacts*, since the velocities of the colliding bodies are subject to jumps at the impact times.

The need for control laws capable of handling systems subject to impacts is evident especially in robotics, since a lot of the tasks, which can be assigned to a robot, involve collisions [3, 4, 5, 6]. The case in which the collisions happen during the so-called *transition phases*, which are expected to be followed by an interval in which the robot is in permanent contact with the environment, is particularly frequent [7, 8, 9, 10, 11, 2]: in such cases, the goal of the control law is often the one of guaranteeing that the transition phases actually end, and that the robot exerts the desired contact force during the permanent contact. The tracking of trajectories which intrinsically involve an infinite number of impacts, with no finite accumulation point for the impact times, leads to some technical difficulties (see the subsequent Remark 1 for a detailed discussion) and is therefore of special interest. Moreover such a control problem has many applications in robotics, *e.g.* hopping [12, 13] or walking robots [14, 15], juggling robots [19, 20], hammering tasks [21].

In this paper, an infinitely rigid mass is considered, which moves on a plane, within a region delimited by an infinitely massive and rigid circular barrier. In particular, we consider as a case study the tracking problem of periodic trajectories involving an infinite number of impacts between the mass and the barrier. In the mathematical literature, the system considered here is called a Birkhoff billiard; a lot of research has been done to study the properties of periodic trajectories on billiards of general shape [22], when no control is exerted on the moving mass.

2 Problem preliminaries

Consider a dimensionless body having unitary mass, which moves on an horizontal plane on which a Cartesian inertial frame xOy is defined. Let $\mathbf{q}(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$ denote the position of the body at time $t \geq t_0$, with $t_0 \in \mathbb{R}$, and let $\mathbf{q}(t)$ be constrained to belong to the following *admissible* region:

$$\mathcal{A} := \{\mathbf{q} \in \mathbb{R}^2 : f(\mathbf{q}) \leq 0\},$$

where $f(\mathbf{q}) := x^2 + y^2 - 1$.

The control inputs are two forces $u_x(t)$, $u_y(t) \in \mathbb{R}$, acting directly on the body, directed as the x and y axes, respectively. It is also assumed that $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ are measured. The system is completely characterized by the Lagrangian function $L := \frac{1}{2}(\dot{x}^2+\dot{y}^2)+u_x\, x+u_y\, y$ and by the inequality $x^2+y^2 \leq 1$. Assuming that the impacts are non-smooth and perfectly elastic, a model of the system can be written as described in [2]. In the following, for every function $g(\cdot)$, the following short notation will be used: $g(t^-) := \lim_{\tau \to t^-} g(\tau)$, $g(t^+) := \lim_{\tau \to t^+} g(\tau)$.

The model of the system is given by the following *Euler-Lagrange equations*

$$\begin{aligned} \ddot{x}(t) + 2\,\dot{\lambda}(t)\,x(t) &= u_x(t), && \text{(1a)}\\ \ddot{y}(t) + 2\,\dot{\lambda}(t)\,y(t) &= u_y(t), && \text{(1b)}\\ 2\,\gamma(t)\,\dot{\lambda}(t) &= 0, && \text{(1c)}\\ 2\,x(t)\,\dot{x}(t) + 2\,y(t)\,\dot{y}(t) + 2\,\gamma(t)\,\dot{\gamma}(t) &= 0, && \text{(1d)} \end{aligned}$$

where $\gamma(t) \in \mathbb{R}^+$ is the Valentine variable and $\lambda(t) \in \mathbb{R}$ is the Lagrange multiplier ($\dot{\lambda}(t)$ has to be understood in the distribution sense), with the following *Erdmann-Weierstrass corner conditions*

$$\begin{aligned} \dot{x}^2(t^+) + \dot{y}^2(t^+) &= \dot{x}^2(t^-) + \dot{y}^2(t^-), && \text{(2a)}\\ \dot{x}(t^+) - \dot{x}(t^-) &= -2\,x(t)\,\big(\lambda(t^+) - \lambda(t^-)\big)\,, && \text{(2b)}\\ \dot{y}(t^+) - \dot{y}(t^-) &= -2\,y(t)\,\big(\lambda(t^+) - \lambda(t^-)\big)\,; && \text{(2c)} \end{aligned}$$

the initial conditions at the initial time t_0 are:

$$\begin{aligned} & x(t_0) = x_0, \quad y(t_0) = y_0, && \text{(3a)}\\ & \dot{x}(t_0^+) = v_{x,0}, \quad \dot{y}(t_0^+) = v_{y,0}, && \text{(3b)}\\ & \gamma(t_0) = \sqrt{-f(\mathbf{q}(t_0))}, \quad \lambda(t_0^+) = 0. && \text{(3c)} \end{aligned}$$

For later use, introduce the following set:

$$\hat{\mathcal{A}} = \{(\mathbf{q},\, \dot{\mathbf{q}}) \in \mathcal{A} \times \mathbb{R}^2 \,:\, \mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}} \leq 0 \text{ if } f(\mathbf{q}) = 0\},$$

where $\mathbf{J}(\mathbf{q})$ is the Jacobian of $f(\mathbf{q})$, *i.e.*, $\mathbf{J}(\mathbf{q}) = [\,2\,x \quad 2\,y\,]$. For each $t_0 \in \mathbb{R}$, it is required that $(\mathbf{q}(t_0),\, \dot{\mathbf{q}}(t_0^+)) \in \hat{\mathcal{A}}$ in order that the body does not leave the admissible region at the times immediately after t_0.

An impact for the controlled body occurs if, at a given time t, one has

$$f(\mathbf{q}(t)) = 0, \quad \mathbf{J}\,(\mathbf{q}(t))\;\dot{\mathbf{q}}(t^-) > 0. \tag{4}$$

It is easy to see that the Erdmann-Weierstrass corner conditions (2) can be solved uniquely in the unknowns $\dot{x}(t^+)$, $\dot{y}(t^+)$, $\lambda(t^+)$ at an impact time t, by requiring that $\mathbf{J}\,(\mathbf{q}(t))\,\dot{\mathbf{q}}(t^+) \leq 0$:

$$\begin{aligned} \dot{x}(t^+) &= \big(y^2(t) - x^2(t)\big)\,\dot{x}(t^-) - 2\,y(t)\,x(t)\,\dot{y}(t^-),\\ \dot{y}(t^+) &= \big(x^2(t) - y^2(t)\big)\,\dot{y}(t^-) - 2\,y(t)\,x(t)\,\dot{x}(t^-). \end{aligned}$$

As for the jump in the Lagrange multiplier, we have

$$\lambda(t^+) = \lambda(t^-) + x(t)\,\dot{x}(t^-) + y(t)\,\dot{y}(t^-).$$

The goal of this paper is the design of a control law such that $\mathbf{q}(t)$ asymptotically tracks the desired trajectory, $\mathbf{q}_{d,\,N}(t) := \left[\; x_{d,\,N}(t) \quad y_{d,\,N}(t) \;\right]^T$, which is defined as follows:

$$\begin{aligned} x_{d,\,N}(t) &:= x_{d,\,N,\lfloor t\rfloor} + v_{x,d,\,N,\lfloor t\rfloor}\,(t - \lfloor t\rfloor)\,, \quad t \in \mathbb{R},\\ y_{d,\,N}(t) &:= y_{d,\,N,\lfloor t\rfloor} + v_{y,d,\,N,\lfloor t\rfloor}\,(t - \lfloor t\rfloor)\,, \quad t \in \mathbb{R}, \end{aligned}$$

where $\lfloor\cdot\rfloor$ denotes the largest integer not greater than the argument and, for each $k \in \mathbb{Z}$,

$$\begin{aligned} x_{d,\,N,\,k} &:= \cos(k\,\beta),\\ y_{d,\,N,\,k} &:= \sin(k\,\beta),\\ v_{x,d,\,N,\,k} &:= 2\,\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\pi}{2} + \frac{\beta}{2} + k\,\beta\right),\\ v_{y,d,\,N,\,k} &:= 2\,\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\pi}{2} + \frac{\beta}{2} + k\,\beta\right) \end{aligned}$$

with $\beta = 2\,\pi/N$, and N being an arbitrary integer, $N \geq 2$.

In particular, for a given N, the desired trajectory is constituted by the regular polygon (having N vertices) inscribed in the circle of unitary radius delimiting the admissible region with one vertex coincident with the point $\left[\; 1 \quad 0 \;\right]^T$. We call desired trajectory both the function $\mathbf{q}_{d,\,N}(t)$, $t \in \mathbb{R}$, and the regular polygon $\mathcal{Q}_N := \{\mathbf{q} \in \mathbb{R} \,:\, \mathbf{q} = \mathbf{q}_{d,\,N}(t) \text{ for some } t \in \mathbb{R}\}$. The desired trajectory is depicted (in bold line) in Figure 1 for $N = 5$.

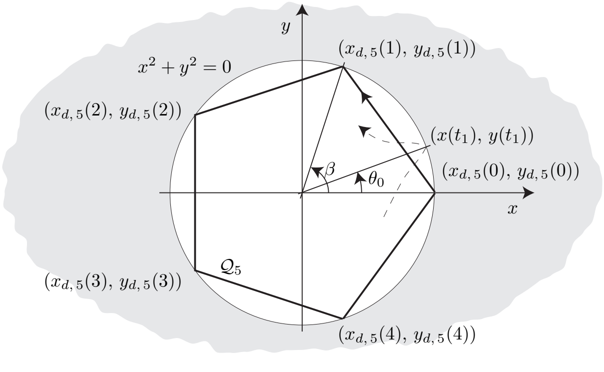


Figure 1: The desired trajectory for $N = 5$ (bold) and an actual trajectory (dashed), having an impact at $t = t_1$.

The desired trajectory is an admissible trajectory of the system, with $u_x(\cdot) = 0$ and $u_y(\cdot) = 0$, and it involves an impact between the body and the boundary of the admissible region at each time $t = k$, $k \in \mathbb{Z}$.

In the following, it will be useful to describe the desired and the actual trajectories in an unified manner; this can be achieved by considering the system as constituted by two identical bodies, the first one being the controlled one and the second one being a fictitious body, whose position at time t is given by $\mathbf{q}_d(t) := [\, x_d(t) \quad y_d(t) \,]^T$, which is not actuated and moves along the desired trajectory, as its initial position and velocity are properly chosen. In this way, the reference trajectory can be taken as the solution of the following Euler-Lagrange equations:

$$\begin{aligned} \ddot{x}_d(t) + 2\,\dot{\lambda}_d(t)\,x_d(t) &= 0, && \text{(5a)}\\ \ddot{y}_d(t) + 2\,\dot{\lambda}_d(t)\,y_d(t) &= 0, && \text{(5b)}\\ 2\,\gamma_d(t)\,\dot{\lambda}_d(t) &= 0, && \text{(5c)}\\ 2\,x_d(t)\,\dot{x}_d(t) + 2\,y_d(t)\,\dot{y}_d(t) + 2\,\gamma_d(t)\,\dot{\gamma}_d(t) &= 0, && \text{(5d)} \end{aligned}$$

with the Erdmann-Weierstrass corner conditions:

$$\begin{aligned} \dot{x}_d^2(t^+) + \dot{y}_d^2(t^+) &= \dot{x}_d^2(t^-) + \dot{y}_d^2(t^-), && \text{(6a)}\\ \dot{x}_d(t^+) - \dot{x}_d(t^-) &= -2\,x_d(t)\,\left(\lambda_d(t^+) - \lambda_d(t^-)\right), && \text{(6b)}\\ \dot{y}_d(t^+) - \dot{y}_d(t^-) &= -2\,y_d(t)\,\left(\lambda_d(t^+) - \lambda_d(t^-)\right), && \text{(6c)} \end{aligned}$$

from the initial conditions:

$$\begin{aligned} & x_d(t_0) = x_{d,\,N}(t_0), \quad y_d(t_0) = y_{d,\,N}(t_0), && \text{(7a)}\\ & \dot{x}_d(t_0^+) = \dot{x}_{d,\,N}(t_0^+), \quad \dot{y}_d(t_0^+) = \dot{y}_{d,\,N}(t_0^+), && \text{(7b)}\\ & \gamma_d(t_0) = \sqrt{-f(\mathbf{q}_{d,\,N}(t_0))}, \quad \lambda_d(t_0^+) = 0, && \text{(7c)} \end{aligned}$$

with $\gamma_d \in \mathbb{R}^+$ being the Valentine variable for the fictitious body and $\lambda_d \in \mathbb{R}$ being the corresponding Lagrange multiplier.

Finally, define the tracking error as $\mathbf{e}(t) := \begin{bmatrix} e_x(t) \\ e_y(t) \end{bmatrix}$, where $e_x(t) = x(t) - x_d(t)$, $e_y(t) = y(t) - y_d(t)$.

3 Problem definition and main result

The presence of the Erdmann-Weierstrass corner conditions (2) and (6) and of the constraints (1c), (1d), (5c) and (5d) complicates the trajectory tracking problem for the considered system as compared with the case of unconstrained mechanical systems, as shown in the following remark ($\|\cdot\|$ denotes the Euclidean norm of the vector at argument).

Remark 1 In this remark, we assume that $N = 2$, as the generalization to the case $N > 2$ is simple. The stability properties that we would like to guarantee for the closed-loop system are:

(i) for each $\varepsilon > 0$ and for each $t_0 \in \mathbb{R}$, there exists $\delta_{\varepsilon,\,t_0} > 0$ such that if $\|\mathbf{e}(t_0)\| \leq \delta_{\varepsilon,\,t_0}$ and $\|\dot{\mathbf{e}}(t_0^-)\| \leq \delta_{\varepsilon,\,t_0}$, then $\|\mathbf{e}(t)\| \leq \varepsilon$, $\|\dot{\mathbf{e}}(t^-)\| \leq \varepsilon$ and $\|\dot{\mathbf{e}}(t^+)\| \leq \varepsilon$, for all $t \geq t_0$;

(ii) for each $t_0 \in \mathbb{R}$, there exists a neighborhood Θ_{t_0} of $[\, \mathbf{q}_d^T(t_0) \quad \dot{\mathbf{q}}_d^T(t_0^-) \,]^T$ such that the following relationships hold for each $[\, \mathbf{q}^T(t_0) \quad \dot{\mathbf{q}}^T(t_0^-) \,]^T \in \Theta_{t_0} \cap \left(\mathcal{A} \times \mathbb{R}^2\right)$:

$$\begin{aligned} \lim_{t \to +\infty} \|\mathbf{e}(t)\| &= 0, && \text{(8a)}\\ \lim_{t \to +\infty} \|\dot{\mathbf{e}}(t^-)\| &= 0, && \text{(8b)}\\ \lim_{t \to +\infty} \|\dot{\mathbf{e}}(t^+)\| &= 0, && \text{(8c)} \end{aligned}$$

where the limits in equations (8) are taken with $t \in \mathbb{R}$.

Unfortunately, it is too difficult (if not impossible) to guarantee both properties (i) and (ii), as shown hereafter. Consider statement (i). Let $0 < \varepsilon < 1$. Let the initial time t_0 be negative and very close to 0 (so that there is no impact in the interval $(t_0, 0)$), with the initial conditions $\mathbf{q}_d(t_0)$, $\dot{\mathbf{q}}_d(t_0^+)$, $\dot{\mathbf{q}}(t_0^+)$ and $\mathbf{q}(t_0)$ chosen so that $\mathbf{q}_d(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\dot{\mathbf{q}}_d(0^-) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\dot{\mathbf{q}}(0^-) = \dot{\mathbf{q}}_d(0^-)$ and $\mathbf{q}(0) = (1-\varepsilon)\,\mathbf{q}_d(0)$ (due to these choices, as $\mathbf{q}(0)$ is an interior point of $\mathcal{A}$, $t = 0$ is not an impact time for the controlled body); this can always be done, by a back integration of the Euler-Lagrange equations (1a), (1b), (5a) and (5b), with the final conditions $\mathbf{q}_d(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\dot{\mathbf{q}}_d(0^-) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\dot{\mathbf{q}}(0^-) = \dot{\mathbf{q}}_d(0^-)$ and $\mathbf{q}(0) = (1-\varepsilon)\,\mathbf{q}_d(0)$. This implies that $t = 0$ is an impact time for the fictitious body characterizing the reference trajectory, whence $\dot{\mathbf{q}}_d(0^+) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. As t_0 is negative and very close to 0, $\|\dot{\mathbf{e}}(t_0^+)\| \simeq \|\dot{\mathbf{e}}(0^-)\| = 0$ and $\|\mathbf{e}(t_0^+)\| \simeq \|\mathbf{e}(0)\| = \varepsilon$. Taking into account the expression of $\dot{\mathbf{q}}_d(0^+)$, we have $\dot{\mathbf{e}}(0^+) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, and $\|\dot{\mathbf{e}}(0^+)\| = 4$; this implies that $\|\dot{\mathbf{e}}(t^-)\|$ and $\|\dot{\mathbf{e}}(t^+)\|$ are greater than ε for all $t \neq 0$ belonging to a short interval starting from 0, and property (i) is violated. Notice that as the control inputs are not impulsive, they do not influence the previous reasoning, which therefore shows that property (i) cannot be satisfied for any piece-wise continuous and bounded control law (the case of impulsive control laws is not considered, being out the scope of this paper).

As far as property (ii) is concerned, for each arbitrarily high real T, if $\|\mathbf{e}(T)\| + \|\dot{\mathbf{e}}(T^-)\| \neq 0$ (this is not the case when either $\|\mathbf{e}(t_0)\| + \|\dot{\mathbf{e}}(t_0^-)\| = 0$ or when the considered control law has a dead-beat property; this second case can happen only in the nominal case, and, therefore, can be neglected in real applications), then it is generically true that there exists an integer $k > T$ that is not an impact time for the controlled body;

for such a k, even if $\|\dot{\mathbf{e}}(k^-)\|$ is almost zero, we have $\dot{\mathbf{e}}(k^+) = \dot{\mathbf{e}}(k^-) + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ if k is even, and $\dot{\mathbf{e}}(k^+) = \dot{\mathbf{e}}(k^-) - \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ if k is odd, whence $\|\dot{\mathbf{e}}(k^+)\|$ is almost equal to 4, which (by the arbitrariness of T) means that property (ii) does not hold. Therefore, it seems that, by requiring property (ii), one actually requires that for sufficiently high times, the impact times of the controlled body coincide exactly with the impact times $k \in \mathbb{Z}$ of the desired trajectory, which seems to be difficult (if not impossible) to be guaranteed in practice for a significant set of initial conditions.

In the following, properties (i) and (ii) (*i.e.*, the classical asymptotic stability property) will be properly amended in order to overcome the difficulties previously discussed. □

The control problem solved in this paper can be stated as follows.

Problem 1 *Find a piece-wise continuous control law (where* $\dot{\mathbf{q}}(t)$ *is to be understood as* $\dot{\mathbf{q}}(t^-)$ *at the impact times):*

$$\begin{aligned} u_x(t) &= \varphi_x(\mathbf{q}(t), \dot{\mathbf{q}}(t), t), && \text{(9a)} \\ u_y(t) &= \varphi_y(\mathbf{q}(t), \dot{\mathbf{q}}(t), t), && \text{(9b)} \end{aligned}$$

such that the following properties hold for the closed-loop system:

(a) for each $\varepsilon > 0$*, for each* $t_0 \in \mathbb{R}$ *and for each* $\gamma \in (0, 1/2)$*, there exists* $\delta_{\varepsilon, t_0, \gamma} > 0$ *such that if* $[\, \mathbf{q}^T(t_0) \quad \dot{\mathbf{q}}^T(t_0^+) \,]^T \in \hat{\mathcal{A}}$*,* $\|\mathbf{e}(t_0)\| < \delta_{\varepsilon, t_0, \gamma}$ *and* $\|\dot{\mathbf{e}}(t_0^+)\| < \delta_{\varepsilon, t_0, \gamma}$*, then*

$$\begin{aligned} \|\mathbf{e}(t)\| &< \varepsilon, \quad \forall t \in \mathbb{R},\ t \geq t_0, && \text{(10a)} \\ \|\dot{\mathbf{e}}(t^-)\| &< \varepsilon, \quad \forall t \in \mathbb{R},\ t > t_0,\ |t - [t]| > \gamma, && \text{(10b)} \\ \|\dot{\mathbf{e}}(t^+)\| &< \varepsilon, \quad \forall t \in \mathbb{R},\ t > t_0,\ |t - [t]| > \gamma, && \text{(10c)} \end{aligned}$$

where $[t]$ *denotes the integer nearest to* t*;*

(b) for each $t_0 \in \mathbb{R}$*, there exists a neighborhood* Θ_{t_0} *of* $[\, \mathbf{q}_d^T(t_0) \quad \dot{\mathbf{q}}_d^T(t_0^+) \,]^T$ *such that the following relationships hold for each* $[\, \mathbf{q}^T(t_0) \quad \dot{\mathbf{q}}^T(t_0^+) \,]^T \in \Theta_{t_0} \cap \hat{\mathcal{A}}$*:*

$$\begin{aligned} \lim_{t \to +\infty} \|\mathbf{e}(t)\| &= 0, && \text{(11a)} \\ \lim_{k \to +\infty} \|\dot{\mathbf{e}}((k+\tau)^-)\| &= 0, \quad \forall \tau \in (0, 1), && \text{(11b)} \\ \lim_{k \to +\infty} \|\dot{\mathbf{e}}((k+\tau)^+)\| &= 0, \quad \forall \tau \in (0, 1), && \text{(11c)} \end{aligned}$$

where the limits in equations (11b) *and* (11c) *are taken with* k *being integer, whereas the limit in equation* (11a) *is taken with* t *being real.*

Remark 2 Properties (a) and (b) resemble closely to the usual definitions of stability and attractivity of the motion constituted by the desired trajectory. Sometimes, when dealing with the stability of trajectories, which are not equilibrium points, the definitions of *orbitally stable motion* and *orbitally attractive motion* are used [24, Section V], instead of the stronger ones of stable motion and attractive motion, respectively. Consider the regular polygon $\mathcal{Q}_N$ defined in the previous section, and define the distance $\varrho(\mathbf{q}, \mathcal{Q}_N)$, between $\mathbf{q} \in \mathbb{R}^2$ and $\mathcal{Q}_N$ as:

$$\varrho(\mathbf{q}, \mathcal{Q}_N) = \inf_{\mathbf{p} \in \mathcal{Q}_N} \|\mathbf{q} - \mathbf{p}\|.$$

If the desired trajectory is an orbitally stable motion, for a given $t_0 \in \mathbb{R}$, then, for each $\epsilon > 0$, it is possible to define a neighborhood $\Xi_{t_0, \epsilon}$ of $[\, \mathbf{q}_d^T(t_0) \quad \dot{\mathbf{q}}_d^T(t_0^+) \,]^T$, such that, for every $[\, \mathbf{q}^T(t_0) \quad \dot{\mathbf{q}}^T(t_0^+) \,]^T \in \Xi_{t_0, \epsilon} \cap \hat{\mathcal{A}}$, $\varrho(\mathbf{q}(t), \mathcal{Q}_N) < \epsilon$, for all $t \geq t_0$ (actually the definition of orbital stability implies a stronger property involving also the actual and desired velocities, but this is irrelevant here). It can be easily seen, by means of purely geometric reasoning, that the desired trajectory is not an orbitally stable motion, nor an orbitally attractive motion, of the open-loop system. The study of the orbital stability of billiards of a more general class (*i.e.*, billiards for which the admissible region is delimited by closed curves other than a circle) is carried out in [22, Section II].

On the other hand, it can be seen that, if properties (a) and (b) hold, then the desired trajectory is an orbitally stable and attractive motion. Hence, properties (a) and (b) do not hold for the open-loop system. □

The goal of this section is to show that the following PD-like control law (where $\dot{e}_x(t)$ and $\dot{e}_y(t)$ are to be understood as $\dot{e}_x(t^-)$ and $\dot{e}_y(t^-)$ at the impact times both of the controlled body and of the desired trajectory):

$$\begin{aligned} u_x(t) &= -K_V\, \dot{e}_x(t) - K_P\, e_x(t), \quad t \geq t_0, && \text{(12a)} \\ u_y(t) &= -K_V\, \dot{e}_y(t) - K_P\, e_y(t), \quad t \geq t_0, && \text{(12b)} \end{aligned}$$

where $K_V = 2\,\alpha,\ K_P = \alpha^2, \quad \alpha \in \mathbb{R}^+$, succeeds in obtaining "asymptotic tracking" of the desired trajectory, for sufficiently high values of the parameter α. Notice that if $\|\mathbf{e}(t_0)\| + \|\dot{\mathbf{e}}(t_0^+)\| \neq 0$, then $u_x(t)$ and $u_y(t)$ are piece-wise continuous functions of time, with finite jumps at the impact timesand also at the integer times (due to the jumps in the desired velocity).

By substituting such a control law into equations (1a) and (1b), we have (omitting the dependence on t):

$$\begin{aligned} &\ddot{x} + K_V\ (\dot{x} - \dot{x}_d) + K_P\ (x - x_d) + 2\,\dot{\lambda}\, x = 0, \\ &\ddot{y} + K_V\ (\dot{y} - \dot{y}_d) + K_P\ (y - y_d) + 2\,\dot{\lambda}\, y = 0; \end{aligned}$$

these equations and equations (1c), (1d), (5) have to be solved from the initial conditions (3), (7), with the help of the relevant Erdmann-Weierstrass corner conditions (2) and (6).

Theorem 1 *For every $N \geq 2$, there exists an $\alpha_N \in \mathbb{R}^+$ such that the control law* (12) *is a solution of Problem 1, for any $\alpha \geq \alpha_N$.*

Hint for the proof. Consider the N-periodic discrete-time system S_D obtained from the closed-loop system by sampling the tracking error at the times $t = k - 1/2$ for $k \in \mathbb{Z}^+$; denote its state vector at time $k \in \mathbb{Z}^+$ by:

$$\mathbf{e}_D(k) = \begin{bmatrix} \mathbf{e}(k-1/2) \\ \dot{\mathbf{e}}(k-1/2) \end{bmatrix}; \tag{13}$$

notice that, for system S_D to be *well posed*, it is needed that none of the times $k - 1/2$, $k \in \mathbb{Z}^+$, is an impact time for the actual trajectory $(x(t),\ y(t))$.

Consider the function $W(\cdot,\ \cdot) : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^+$:

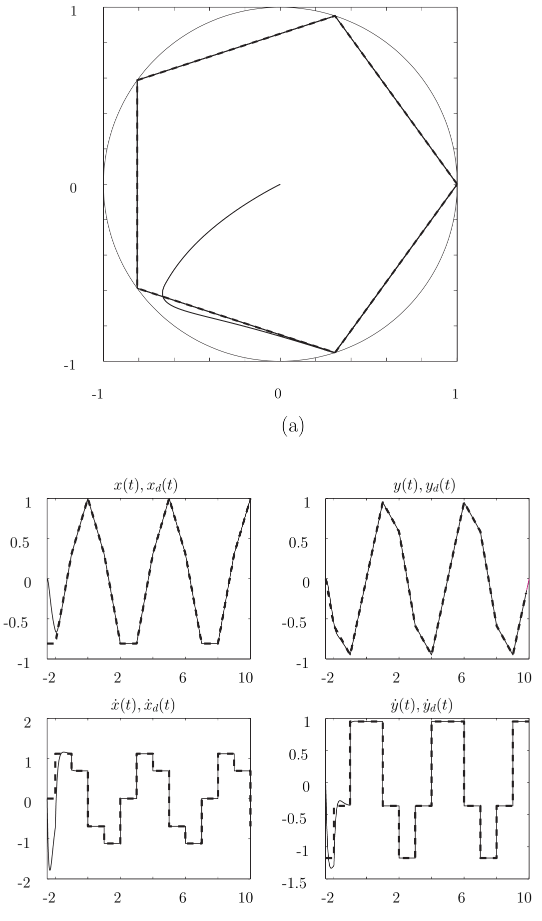
$$\begin{aligned} W(t,\, \mathbf{z}) \quad = \quad & \frac{1}{2}\left(\dot{x} - \dot{x}_d(t)\right)^2 + \frac{1}{2}\left(\dot{y} - \dot{y}_d(t)\right)^2 + \\ & \frac{1}{2} K_P \left(x - x_d(t)\right)^2 + \frac{1}{2} K_P \left(y - y_d(t)\right)^2, \end{aligned}$$

which is defined only in the sense of distributions at the impact times $k \in \mathbb{Z}$ of the desired trajectory, and, if computed along the trajectories of the system, *i.e.*, obtaining $W(t,\, \mathbf{z}(t))$, at the actual impact times.

It can be seen that the function $v(t) = W(t,\, \mathbf{z}(t))$ is subject to jumps at the impact times, which can be either positive or negative. Due to the presence of positive jumps, $W(t,\, \mathbf{z})$ cannot be taken as a Liapunov function for the continuous-time closed-loop system; however, such a function is of help in finding a Liapunov function for the discrete-time system S_D. As a matter of fact, defining the function $V(\cdot,\ \cdot) : \mathbb{Z} \times \mathbb{R}^4 \to \mathbb{R}^+$ as follows:

$$\begin{aligned} V(k,\, \mathbf{e}_D) \quad := \quad & W(k - 1/2,\, \mathbf{z}_d(k - 1/2) + \mathbf{e}_D), \\ & \forall k \in \mathbb{Z}^+,\ \forall \mathbf{e}_D \in \mathbb{R}^4; \end{aligned}$$

$V(k,\, \mathbf{e}_D)$ can be taken as a candidate Liapunov function for system S_D when it is well-posed. Moreover, it can be shown that, if there is exactly one impact time in every interval $(k - 1/2, k + 1/2)$, $k \in \mathbb{Z}$, then for each $N \geq 2$ there exists $\alpha_N \in \mathbb{R}^+$ such that, for each $\alpha > \alpha_N$, we have that the forward difference $V(k+1,\, \mathbf{e}_D(k+1)) - V(k,\, \mathbf{e}_D(k))$, computed along the solutions of the system, is a negative function of $\mathbf{e}_D(k) \neq \mathbf{0}$. □



(a)

(b)

Figure 2: The desired (dashed) and actual (solid) trajectories in the xy-plane (a) and time behaviour of the desired (dashed) and actual (solid) trajectories (b), for $N = 5$.

4 An example and concluding remarks

We report an example in order to illustrate the efficacy of the proposed control law. It is related to the case of $N = 5$, with $\alpha = 6$. Starting from the initial condition $x(-2.5) = 0$, $y(-2.5) = 0$, $\dot{x}(-2.5) = 0$, $\dot{y}(-2.5) = 0$ at the initial time $t_0 = -2.5$, the behaviour of the closed-loop system during the first 12.5 seconds of motion can be observed in Figure 2.

In this paper, we have assumed that the mass of the body and the radius of the circle delimiting the admissible region are both unitary, and that the period of the trajectory is integer and equal to the number N of the sides of the regular polygon described by the trajectory. Such simplifying assumptions can be easily removed by properly scaling the variables involved. On the other hand, the assumption that the coefficient of restitution is unitary is crucial: if the coefficient of restitution is less than 1, the proposed controller is not a solution of Problem 1 (in such a case, any non-impulsive control cannot be a solution of Problem 1).

The last remark is made in order to emphasize that the simple proposed control law, properly amended, could be applied, in principle, to more general classes of fully actuated mechanical systems, when a desired trajectory involving an infinite number of impacts is to be tracked, provided that the length of the time intervals between two successive impacts is bounded from below by some strictly positive constant.

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