Static output feedback stabilization of linear and nonlinear systems

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Abstract

The static output feedback stabilization problem for linear and nonlinear (affine) systems is discussed. A novel necessary and sufficient condition for linear systems is proposed. For nonlinear systems a sufficient condition is established and a (partial) converse is also discussed. The nonlinear formulation is used to derive a simple characterization of stabilizing static output feedback control laws for linear systems in terms of the intersection of two convex sets and a (generally) non-convex set. This characterization is used to establish a series of simple obstructions to the solvability of the problem for linear SISO systems. A fully worked out example complete the paper.

1 Introduction

The static output feedback (SOF) stabilization problem is probably one of the most known puzzle in system and control. The simple statement of the problem is as follows: find a static output feedback control such that the closed-loop system is asymptotically stable. This problem is important in its own right, since output static controllers are less expensive to be implemented and more reliable in practice. Moreover, it is possible to verify that in many cases the design of a dynamical output feedback controller boils down to the solution of a static output feedback control problem [1, 2]. Unfortunately, despite the simplicity of its formulation, the fundamental question of the existence (in the general case) of a stabilizing static output feedback control law is still open. Many attempts have been made in the last years, so that at this stage we can count several nontrivial contributions to the problem, both numerical and speculative, see the recent paper [3], where the state of the art is presented and the existing methods are surveyed and compared. An important characterization of SOF stabilization is related to the solution of coupled linear matrix inequalities, which also provide a parametrization of all SOF gains [4]. A few algorithms have been proposed to get through the coupling condition. Among them the most interesting seem to be the min/max procedure proposed in [5] and the cone complementary algorithm of [6]. All the contributions confirm that (generically) the SOF stabilization problem for linear time-invariant systems is intrinsically nonlinear, *i.e.* the linearity of the system to be stabilized does not yield any special advantage in finding analytic solutions or systematic procedures. This observation motivated the present paper, where the SOF stabilization problem is consider for affine nonlinear systems. Precisely a sufficient condition is provided based on the solution of a suitable Hamilton-Jacobi inequality, and a partial converse is also proved. These results allow to define a set of static output nonlinear control law for the linear system. The linear output feedback is then recovered by linearization and by exploiting the so-called rank theorem for mappings defined through diffeomorphisms between open sets [7]. The nonlinear analysis is useful to formulate a novel necessary and sufficient condition for SOF stabilization of linear systems, given in terms of the coupled solution to a linear matrix inequality and a linear equality. From this equality, in the case of single-input single-output systems, it is possible to go deeply into the structure of the problem, establishing a nested series of testable necessary conditions for its solvability.

The paper is organized as follows. In Section 2 the SOF stabilization problem for linear systems is recalled and a preliminary result is stated. This can be seen as a slight modification of the necessary and sufficient condition given in [8]. The SOF stabilization problem for nonlinear systems is tackled in Section 4, where the sufficient and the necessary conditions are provided in terms of the solution of a constrained Hamilton-Jacobi equation along with a rank condition. This sets the basis for the introduction of the novel necessary and sufficient condition provided in Section 4. Further issues and a discussion on the convexity properties for SISO systems are given in Section 5. In Section 6 a fully worked out example is presented in order to illustrate the theory developed so far. The paper ends with some comments and hints for future research.

2 Preliminary results

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu \qquad (1)$$

$$y = Cx \qquad (2)$$

where $x \in I\!R^n$, $u \in I\!R^m$, $y \in I\!R^p$ are the state, input and output vectors, respectively, and A, B, C are matrices with constant real coefficients and appropriate dimensions.

The *static output feedback stabilization* problem for system (1)-(2) consists in finding, if possible, a static control law described by

$$u = Fy \qquad (3)$$

such that the closed-loop system is asymptotically stable, *i.e.* the matrix $A+BFC$ has all its eigenvalues with negative real parts. If such an output feedback does exist, we say that the system (1)-(2) is *output stabilizable* and that F is a solution of the problem[1].

In what follows, whenever we deal with the linear system (1)-(2) we make the following standing assumptions.

- **(A1)** The pair $\{A, B\}$ is controllable and B has full column rank.
- **(A2)** The pair $\{A, C\}$ is observable and C has full row rank.
- **(A3)** $m \leq p$, *i.e.* the transfer function $W(s) = C(sI - A)^{-1}B$ is *fat*.

The above assumtions are without loss of generality. First of all, the solvability of the problem depends only upon the controllable and observable part of the system, *i.e.* on the transfer function $W(s) = C(sI - A)^{-1}B$. Then, the rank assumptions can be always enforced, by elimination of redundant control inputs or measured outputs. Finally, if $m > p$, it is possible to consider the system

$$\dot{\lambda} = A'\lambda + C'v \qquad (4)$$

$$\eta = B'\lambda \qquad (5)$$

for which Assumption (A3) holds, and observe that the static output feedback stabilization problem for system (4)-(5) is solvable if and only if it is solvable for system (1)-(2). Moreover, if Ω (resp. F) is a solution of the static output feedback stabilization problem for system (4)-(5) (resp. (1)-(2)) then $F = \Omega'$ (resp. $\Omega = F'$) is a solution of the static output feedback stabilization problem for system (1)-(2) (resp. (4)-(5)).

A simple necessary and sufficient condition for the system to be output stabilizable is stated in the following result, whose proof is reported here for the sake of completeness, even though, in slight different forms, it can be found here and there in the existing literature, see *e.g.* [8].

Theorem 1 *Consider the system (1)-(2) with Assumptions (A1), (A2) and (A3). The system is output feedback stabilizable if and only there exist a symmetric positive semidefinite matrix* $P \in I\!R^{n\times n}$ *and a matrix* $G \in I\!R^{m\times n}$ *such that*

$$0 = A'P + PA - PBB'P + C'C + G'G \qquad (6)$$

$$0 = V(A'P + PA)V, \qquad (7)$$

where[2]

$$V = I - C'(CC')^{-1}C.$$

Theorem 1 provides also a parameterization of all stabilizing static output feedback control laws, as expressed in the following statement, whose proof is easily obtained from the proof of the theorem.

Corollary 1 *Consider the system (1)-(2). The family of all output feedback gains* F *such that the matrix* $A + BFC$ *is stable is given by*

$$F = (T'G - B'P)C'(CC')^{-1}$$

where $P = P' \geq 0$ *and* G *solve (6) and (7) and* T *is any orthogonal matrix.*

It is worth noting that, despite their simplicity, the conditions in Theorem 1 cannot be recast (to the best of the authors knowledge) in an LMI framework or in a simple computational scheme, because in general the static output feedback stabilization problem is non-convex. Moreover, to obtain simpler conditions, *e.g.* conditions involving only the matrix P, it would be tempting to replace equation (6) with the matrix inequality

$$0 \geq A'P + PA - PBB'P + C'C = \mathcal{R}(P). \qquad (8)$$

Unfortunately, using simple linear arguments, it is only possible to prove that the conditions expressed by equations (8) and (7) are necessary for static output feedback stabilizability. To recover also the sufficient part

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[1] It is obvious that, if a solution exists, this is not unique.

[2] Observe that by Assumption (A2) the matrix V is well-defined.

of the statement, one has to add a further rank condition on $\mathcal{R}(P)$, *i.e.* $\mathrm{rank}(\mathcal{R}(P)) \leq m$. However, using the more general formulation and solution of the problem given in the next section, it is possible to prove that the equations (8) and (7) provide a necessary and sufficient condition for the solvability of the static output feedback stabilization problem. These conditions are obviously simpler than conditions (6) and (7), as they involve only one unknown, *i.e.* the matrix P. Moreover, it is fairly standard to recognize that equation (8) can be given an equivalent LMI formulation in the unknown P^{-1}.

3 Static output feedback stabilization for nonlinear systems

In this section we consider a nonlinear system described by equations of the form

$$\dot{x} = f(x) + g(x)u \tag{9}$$

$$y = h(x) \tag{10}$$

where $x \in I\!R^n$ denotes the state of the system, $u \in I\!R^m$ the control input, $y \in I\!R^p$ the measured output, and the mappings $f(x)$, $g(x)$ and $h(x)$ are smooth mappings defined in a neighborhood of the origin of $I\!R^n$. Moreover, we also assume that $x = 0$ is an equilibrium point, *i.e.* $f(0) = 0$ and $h(0) = 0$. In order to describe the main results of the section we need the following definitions.

Definition 1 *The pair $\{f, h\}$ is said to be locally detectable (observable) if there exists a neighborhood U of the point $x = 0$ such that, if $x(t)$ is any integral curve of $\dot{x} = f(x)$ satisfying $x(0) \in U$, then $h(x(t))$ is defined for all $t \geq 0$ and $h(x(t)) = 0$ for all $t \geq 0$ implies $\lim_{t\to\infty} x(t) = 0$ ($x(t) = 0$ for all $t \geq 0$).*

Definition 2 *Given a smooth mapping $y = h(x)$, we denote with*[3] $\ker(h)$ *the set of all x such that $y = 0$.*

We are now ready to present the main result of this section, which is a (partial) nonlinear counterpart of Theorem 1.

Theorem 2 *Consider the system (9)-(10) and assume that the pair $\{f, h\}$ is locally detectable and that $rank\,(dh(0)) = p$. Suppose moreover that there exist a scalar function $V(x) \in C^1$, positive definite in a neighborhood of the origin, and a $m \times 1$ matrix function $G(x) \in C^1$ such that*

$$\begin{aligned} 0 = V_x(x)f(x) - \frac{1}{4}V_x(x)g(x)g'(x)V_x'(x) + \\ h'(x)h(x) + G'(x)G(x) \end{aligned} \tag{11}$$

$$0 = V_x f(x), \qquad \forall\ x \in Ker(h). \tag{12}$$

Then there exists a orthogonal matrix $T(x) \in C^1$ such that the function

$$\eta(x) = T(x)G(x) - \frac{1}{2}g'(x)V_x'(x) \tag{13}$$

is such that

(i) for all $x \in Ker(h)$

$$\eta(x) = 0; \tag{14}$$

(ii) the system $\dot{x} = f(x) + g(x)\eta(x)$ is locally asymptotically stable;

(iii) the trajectories $x(t)$ of the system $\dot{x} = f(x) + g(x)\eta(x)$ starting close to the origin are such that the output $y(t) = h(x(t))$ and the control $u(t) = \eta(x(t))$ are square integrable signals.

Moreover, if

$$p \geq k = rank(\frac{\partial \eta(x)}{\partial x}) \qquad \forall x \in \Omega \tag{15}$$

for some constant k and some neighborhood Ω of $x = 0$, then

(iv) in a neighborhood of the origin, $\eta(x)$ is a function of y, i.e. $\eta(x) = \phi(y)$ for some smooth function $\phi(\cdot)$.

The sufficient conditions in Theorem 2 are the nonlinear equivalent of the sufficient conditions in Theorem 1. They are obviously more involved, as they require the solution of a Hamilton-Jacobi equation. Moreover, in the linear case, for any fixed matrix G there exists a (unique) matrix $P = P' \geq 0$ solving the Riccati equation (6), whereas this fact is not in general true for the Hamilton-Jacobi equation (11). However, the nonlinear formulation allows to replace the Hamilton-Jacobi equation (11) with a Hamilton-Jacobi inequality involving only one unknown, as detailed in the following statement.

Corollary 2 *Assume that there exists a scalar function $W(x) \in C^1$, positive definite in a neighborhood of the origin, such that*

$$0 \geq W_x(x)f(x) - \frac{1}{4}W_x(x)g(x)g'(x)W_x'(x) + h'(x)h(x) \tag{16}$$

$$0 = W_x f(x), \quad \forall\ x \in Ker(h). \tag{17}$$

Then, there exist a (non-unique) $m \times 1$ matrix function $G(x) \in C^1$ and a scalar function $V(x) \in C^1$, positive definite in a neighborhood of the origin, such that equations (11) and (12) are satisfied.

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[3] This is also denoted with $h^{-1}(0)$, see *e.g.* [7].

Observe that Theorem 2 admits the following partial converse.

Theorem 3 *Consider the system (9)-(10) and assume that the pair $\{f, h\}$ is locally observable. Assume moreover that there exists a continuous function $\phi(y)$, with $\phi(0) = 0$, such that*

- *(a) $\dot{x} = f(x) + g(x)\phi(y)$ is locally asymptotically stable;*
- *(b) the trajectories $x(t)$ of the system $\dot{x} = f(x) + g(x)\phi(y)$ starting close to the origin are such that the output $y(t) = h(x(t))$ and the control $u(t) = \phi(y(t))$ are square integrable signals.*

Then there exist a scalar function $V(x) \in C^1$, positive definite in a neighborhood Ω of the origin, a $m \times 1$ continuous function $G(x)$, and an orthogonal matrix $T(x) \in C^1$ such that

$$
\begin{aligned}
(i) \quad & 0 = V_x(x)f(x) - \frac{1}{4}V_x(x)g(x)g(x)'V_x(x) + \\
& \quad h'(x)h(x) + G'(x)G(x) \\
(ii) \quad & 0 = V_x(x)f(x), \quad \forall\ x \in Ker(h) \\
(iii) \quad & 0 = T(x)G(x) - \frac{1}{2}g'(x)V_x(x), \quad \forall\ x \in Ker(h) \\
(iv) \quad & p \geq rank(\frac{\partial \phi}{\partial x}), \quad \forall x \in \Omega.
\end{aligned}
$$

4 A new characterization for linear systems

In this section we exploit the results established in Section 3 to derive a new necessary and sufficient condition for the solvability of the output feedback stabilizability problem for system (1)-(2). This characterization is given in term of the intersection of two convex sets subject to a (generically) non-convex coupling condition. It is worth noting that the proposed condition heavily stems from the nonlinear control design methods established in the previous section. Moreover, in general, the resulting output feedback controller is nonlinear, but (under simple regularity assumptions) a linear feedback can be computed.

Theorem 4 *Consider the system (1)-(2) with Assumptions (A1), (A2) and (A3). The system is output feedback stabilizable if and only if there exist two symmetric positive definite matrices X and P such that*

$$0 \leq \begin{bmatrix} -XA' - AX + BB' & XC' \\ CX & I \end{bmatrix} \tag{18}$$

$$0 = V(A'P + PA)V \tag{19}$$

$$I = PX. \tag{20}$$

Remark 1 Notice that the set of all X satisfying the condition (18) in Theorem 4 is convex, and it is also convex the set of all P satisfying condition (19). Unfortunately, the coupling condition (20) is not convex. We conclude that whole problem is in general non-convex.

Remark 2 The result summarized in Theorem 4 can be easily extended to the case where not only stability but also an H_∞ performance bound is taken into account. For, consider a system described by the equations

$$\dot{x} = Ax + Bu + \bar{B}w \tag{21}$$

$$z = \begin{bmatrix} Cx \\ u \end{bmatrix}. \tag{22}$$

Then it is possible to extend the results in Theorems 1 and 4 to prove that there exists a static output feedback control law $u = Fy$ such that the closed loop system is asymptotically stable and the H_∞ norm from the input w to the output z is less then a prespecified level γ if and only if assumptions (A1), (A2) and (A3) hold and there exist two symmetric positive definite matrices X and P such that

$$0 \leq \begin{bmatrix} -XA' - AX + BB' - \dfrac{\bar{B}\bar{B}'}{\gamma^2} & XC' \\ CX & I \end{bmatrix} \tag{23}$$

$$0 = V(A'P + PA + \frac{P\bar{B}\bar{B}'P}{\gamma^2})V \tag{24}$$

$$I = PX. \tag{25}$$

5 Further issues and convexity properties

In this section we restrict our interest to linear systems described by equations of the form (1)-(2) with $p = m = 1$ and satisfying Assumption (A1) and (A2). Without loss of generality it is possible to write the system in the observability canonical form, *i.e.* with

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}, \tag{26}$$

and

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{27}$$

This special form allows to get more insight into the structure of the problem, as described in the following statement, where, for convenience, we define the column vector e_i as the $i-th$ vector of the identity matrix of dimension n and the set

$$\mathcal{X} = \{X = X' \mid X_{ij} = 0 \text{ if } i + j \text{ is odd}\}$$

Theorem 5 *Consider the system (1)-(2) with A and C as in equations (26) and (27). The system is output feedback stabilizable only if there exists a positive definite matrix X satisfying the following conditions.*

(i)

$$0 \leq \left[\begin{array}{cc} -XA' - AX + BB' & XC' \\ CX & I \end{array} \right]; \qquad (28)$$

(ii)

$$e_i' X e_n = 0, \quad i = n-1,\ n-3,\ n-5,\ \cdots \qquad (29)$$

(iii) $X \in \mathcal{X}$*;*

(iv) the polynomial

$$e_n' X e_n \lambda^{n-1} + e_{n-2}' X e_n \lambda^{n-3} + e_{n-4}' X e_n \lambda^{n-5} + \cdots \qquad (30)$$

has distinct roots all on the imaginary axis.

Remark 3 Observe that condition (iii) of Theorem 5 implies condition (ii). They have been both included since they express obstructions of diverse complexity and nature.

Theorem 5 provides an obstruction to the solvability of the static output feedback stabilization problem. It can be used in steps of increasing complexity. First condition (i) is checked. If it is feasible, then conditions (i) and (ii) can be simultaneously checked. If they are feasible, condition (iii) can be added, and finally conditions (i) to (iv) has to be checked together. If this procedure fails, then no solution exists. However, if the procedure works, no conclusion can be drawn.

It is interesting to note that for low order systems the obstruction expressed in Theorem 5 can be easily checked, as formalized in the following statement.

Corollary 3 *Consider the system (1)-(2) with the matrices A and C as in equations (26) and (27).*

If $n \leq 2$ the set of all $X > 0$ satisfying conditions (i) to (iv) of Theorem 5 is convex and conditions (i) to (iv) of Theorem 5 are also sufficient.

If $n \leq 3$ and $B = C'$ the set of all $X > 0$ satisfying conditions (i) to (iv) of Theorem 5 is convex and conditions (i) to (iv) of Theorem 5 are also sufficient.

If $n \leq 4$ the set of all $X > 0$ satisfying conditions (i) to (iv) of Theorem 5 is convex.

If $n = 5$ or $n = 6$ the conditions (i) to (iv) of Theorem 5 can be recast in a convex optimization problem, i.e. *as a quadratic optimization with linear constraints.*

6 An illustrative example

To illustrate the theory we discuss in detail a simple example. Consider a linear 2-dimensional system in observability canonical form with

$$A = \left[\begin{array}{cc} 0 & 4 \\ 1 & -3 \end{array} \right], \quad B = \left[\begin{array}{c} -4 \\ 1 \end{array} \right], \quad C = \left[\begin{array}{cc} 0 & 1 \end{array} \right]. \qquad (31)$$

Observe that

$$V = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right],$$

and that the necessary conditions of Theorem 5 require the matrix X to be diagonal, *i.e.*

$$X = \left[\begin{array}{cc} X_1 & 0 \\ 0 & X_3 \end{array} \right].$$

This implies that the sufficient conditions of Theorem 4 has to be tested with X and P diagonal. Simple calculations show that the set of all positive definite

$$P = \left[\begin{array}{cc} P_1 & 0 \\ 0 & P_3 \end{array} \right] = \left[\begin{array}{cc} 1/X_1 & 0 \\ 0 & 1/X_3 \end{array} \right]$$

satisfying condition (18) is described by

$$\begin{array}{ll} 0 < & 64P_1^2P_3 - 32P_1^2 - P_3^2 - 8P_1P_3 - 8P_1P_3^2 \\ 0 < & P_1 \\ 0 < & P_3, \end{array}$$

whereas condition (19) is trivially fulfilled if P is a diagonal matrix. The set of all feasible P, *i.e.* the set of all P fulfilling condition (18) (with $X = P^{-1}$) and condition (19), is illustrated in Figure 1. Note that in this example not only the set of all feasible pairs (X_1, X_3) is convex, but also the set of all feasible pairs (P_1, P_3) is convex.

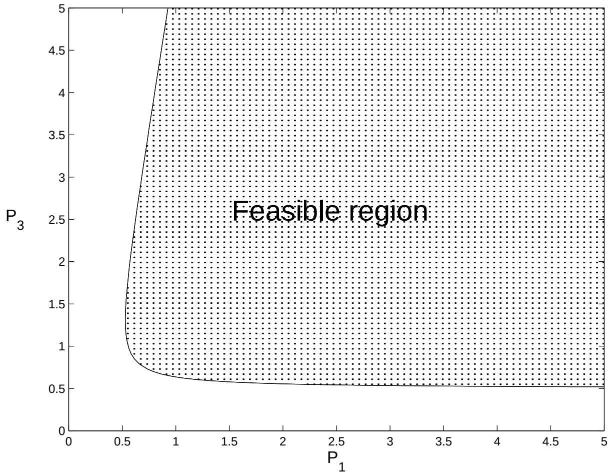

Figure 1: The set of all pair $\{P_1, P_3\}$ satisfying condition (18) (with $X = P^{-1}$) and condition (19).

To construct the stabilizing gain F we follow the procedure outlined in the proof of Theorem 2. For, we select an admissible pair $\{P_1, P_3\}$, *e.g.* $\{1, 1\}$, construct the function $G(x)$, and the scalar $T(x)$. As a result, we obtain the nonlinear *state feedback* control law

$$\eta(x) = G(x) - 4x_1 + x_2$$

yielding local asymptotic stability for the closed loop system. To compute the output feedback gain we have to differentiate the function $\eta(x)$ at some point close to the origin and such that $Cx = 0$. If the pair $\{P_1, P_3\} = (1, 1)$ is selected the corresponding output feedback gain is $F = 1.45$, which is indeed a stabilizing gain.

It is worth noting that, in this example, it is possible to perform the calculations of the stabilizing gain in a parameteric form, *i.e.* without selecting a priori a value for the pair $\{P_1, P_3\}$. This general procedure yields a family of stabilizing gains, which is described by the equation

$$F = F(P_3) = \sqrt{6P_3 - 1 + P_3^2} - P_3,$$

where, as can be seen from Figure 1 and verified with very simple calculations, $P_3 \in (1/2, \infty)$. For $P_3 \in (1/2, \infty)$, the function $F(P_3)$ takes value in the set[4] $(1, 3)$, and a very simple root locus analysis (see Figure 2) reveals that these are indeed all the static output feedback gains yielding closed loop stability.

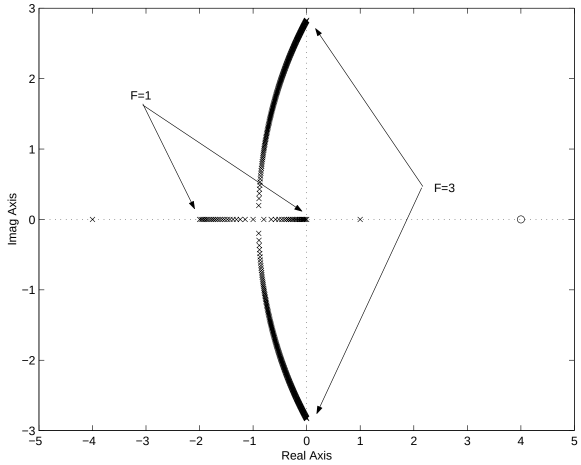

Figure 2: The root locus of the system (31) for $F \in (1, 3)$. Observe that for $F = 1$ and $F = 3$ the eigenvalues of the closed loop system are on the imaginary axis.

7 Concluding remarks

The problem of static output feedback stabilization for linear and nonlinear systems has been studied. For linear systems a new necessary and sufficient condition has been proposed. Moreover, for single-input single-output linear systems a set of simple to test necessary conditions have been derived. These necessary conditions, which can be recast in terms of *convex* conditions for low dimensional systems, can be exploited to decide the unsolvability of the problem.

A sufficient condition for nonlinear control affine systems has also been developed, together with a partial converse. It is worth noting that the proof of the sufficient condition for nonlinear systems is instrumental to develop the new characterization for linear systems.

The theoretical part is complemented with a worked out example. Future work will be directed toward the study of multi-input multi-output linear systems, to establish to what extent the necessary condition in Theorem 5 is also sufficient, to provide a description of all stabilizing gains, and to work out some nonlinear phisically motivated examples.

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[4] Note that $\lim_{P_3 \to \infty} F(P_3) = 3$.