STABILITY PRESERVING MAPPINGS FOR STOCHASTIC DYNAMICAL SYSTEMS

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Abstract

In the present paper we first formulate a general model for stochastic dynamical systems that is suitable in the stability analysis of invariant sets. This model is sufficiently general to include as special cases most of the stochastic systems considered in the literature. We then adapt several existing stability concepts to this model and we introduce the notion of *stability preserving mapping* of stochastic dynamical systems. Next, we establish a result which ensure that a function is a stability preserving mapping, and we use this result in proving a Comparison Stability Theorem for general stochastic dynamical systems. We apply the Comparison Stability Theorem in the stability analysis of dynamical systems determined by Ito differential equations.

1 Introduction

In this paper, we present a general model for stochastic dynamical systems and we study various stability properties of invariant sets for such systems, utilizing the notion of stability preserving mapping of stochastic dynamical systems. (Stability preserving mappings are functions that map a dynamical system into another dynamical system while preserving the stability properties of invariant sets of the systems [1].) The system model presented herein is very general and includes most of the stochastic dynamical systems encountered in the literature (see, e.g., [2]–[8]). The various stability concepts that we formulate for these systems constitute modifications of corresponding ones used by Kushner[2] and Friedman[8]. We refer to these as "stochastic stability" or "stochastic boundedness".

We first present a result (which we call the Stability Preserving Mapping Theorem) that ensures that under mappings of dynamical systems, the stability properties of invariant sets of such systems are preserved. We then use the Stability Preserving Mapping Theorem to establish a Comparison Theorem that enables us to deduce the various stochastic stability and boundedness properties of a general stochastic dynamical system (the *object of inquiry*) from the corresponding properties of a well understood stochastic dynamical system (the *comparison system*). In order to demonstrate the applicability of the developed theory, we apply the above results in the stability analysis of some specific classes of stochastic dynamical systems.

2 Stochastic Dynamical Systems

In this section, we present the definition of stochastic dynamical system and related concepts. These notions are analogous to corresponding concepts used in the qualitative analysis of deterministic dynamical systems (see, e.g., [1]).

We let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space for all the systems that we will consider, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space, and P is the probability measure. An R^n-valued random variable x with domain X is a measurable function from Ω to $X \subset R^n$. We let $R[\Omega, X]$ denote the set of all random variables. A family $\{x(t), t \in I\}$ of R^n-valued random variables with domain X is called a *stochastic process* with index set I. The class of stochastic processes defined on I into $R[\Omega, X]$ will be denoted by $R[I, R[\Omega, X]]$.

Definition 2.1 Let (X, d) be a metric space, where $X \subset R^n$, let $A \subset X$ and let $T \subset R$. For any fixed $a \in A$, $t_0 \in T$, a stochastic process $\{p(t, \omega, a, t_0) : t \in T_{a,t_0}, \omega \in \Omega\} \subset R[T_{a,t_0}, R[\Omega, X]]$ is called a *stochastic motion* if $p(t_0, \omega, a, t_0) = a$, where $T_{a,t_0} = [t_0, t_1) \cap T$, $t_1 > t_0$ is finite or infinite. □

Definition 2.2 Let S be a family of stochastic motions with domain X, i.e., $S \subset \{p(\cdot, \cdot, a, t_0) : p(t_0, \omega, a, t_0) = a, a \in A, t_0 \in T\}$. We call the four-tuple $\{T, X, A, S\}$ a *stochastic dynamical system.* □

Example 2.1 We consider the set of solution processes of the (Ito) stochastic differential equation given by

$$dx = f(x,t)dt + \sigma(x,t)dz(t) \qquad (1)$$

where $x \in R^n$, $f : R^n \times R^+ \to R^n$ and $\sigma : R^n \times R^+ \to R^{n\times m}$ are continuous, and $z(t)$ is a normalized R^m-valued Wiener process with $E\big(z(t) - z(s)\big)\big(z(t) - z(s)\big)^T =$

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$I|t-s|$, where I is the identity matrix. Under proper assumptions (see, e.g., [6]), there exist unique solutions to the above equation for given initial conditions. Let $S_{(1)}$ denote the set of all solutions with initial conditions in a specified set $A \subset R^n$. Then $\{R^+, R^n, A, S_{(1)}\}$ is a stochastic dynamical system and we will say that this system is a stochastic dynamical system determined by equation (1). We will refer to $\{R^+, R^n, A, S_{(1)}\}$ as an *Ito process.* □

There are also stochastic dynamical systems that are determined by stochastic differential (resp., difference) inequalities, stochastic delay differential equations (resp., inequalities), and the like (see, e.g., [2]-[6]).

Example 2.2 Another class of stochastic dynamical systems are those that are determined by so-called *jump systems.* In this case there are parameters in the plant description that are subjected to random jumps. Such systems have variable structure and can be used to model systems subjected to sudden changes or failures in structure. As a specific example, we consider stochastic dynamical systems determined by equations of the form

$$\dot{x}(t) = F(r(t))x(t), \tag{2}$$

where $\{r(t)\}_{t\geq 0}$ is a finite state homogeneous Markov process, which is defined on the state space $\{1, 2, \cdots, s\}$ and $F(\cdot)$ is a matrix-valued function. Such systems have been studied, e.g, in [9], [10]. For any fixed initial state $x(0)=x_0$ and any initial distribution of $r(t)$, there is a solution process for (2). Let $S_{(2)}$ denote the set of all solutions, and let A denote the set of initial conditions. Then $\{R^+, R^n, A, S_{(2)}\}$ is a stochastic dynamical system. We will refer to this system as the *stochastic dynamical system determined by equation (2).* □

3 Stability Preserving Mappings

It is demonstrated in [1] that stability preserving mappings play a central role in the qualitative analysis of deterministic dynamical systems. In particular, the conventional comparison stability theory and Lyapunov theory can be derived as direct consequences of results involving stability preserving mappings. In this section, we present the concept of stability preserving mapping for stochastic dynamical systems. In doing so, we will require several additional concepts.

Definition 3.1 Let $\{T, X, A, S\}$ be a stochastic dynamical system. A set $M \subset A$ is said to be *invariant* with respect to system S if $a \in M$ implies that $P\{\omega : p(t, \omega, a, t_0) \in M \text{ for all } t \in T_{a,t_0}\} = 1$ for all $t_0 \in T$ and all $p(\cdot, \cdot, a, t_0) \in S$. □

Definition 3.2 $x_0 \in A$ is called an *equilibrium (point)* of a stochastic dynamical system $\{T, X, A, S\}$ if the set $\{x_0\}$ is invariant with respect to S. □

Definition 3.3 (S, M) is said to be *stochastically stable* if for any $\epsilon > 0$, $\rho \in (0,1)$, and $t_0 \in T$, there exists a $\delta = \delta(t_0, \epsilon, \rho) > 0$ such that for every process $x(\cdot, \cdot, a, t_0) \in S$, $P\{\omega : \sup_{t\in T_{a,t_0}} d(x(t, \omega, a, t_0), M) \geq \epsilon\} < \rho$ whenever $d(a, M) < \delta$. ($d(a, M)$ denotes the distance of the element a from the set M.) Otherwise it is said to be *stochastically unstable.* If δ is independent of t_0, then (S, M) is said to be *uniformly stochastically stable.* (S, M) is said to be *stochastically asymptotically stable* if it is stochastically stable and, moreover, for every $\rho \in (0,1)$, there exists a $\delta = \delta(t_0, \rho) > 0$ such that for any process $x(\cdot, \cdot, a, t_0) \in S$, $P\{\omega : \lim_{t\to\infty} d(x(t, \omega, a, t_0), M) = 0\} \geq 1-\rho$ whenever $d(a, M) < \delta$. (S, M) is said to be *uniformly stochastically asymptotically stable* if (S, M) is uniformly stochastically stable and for every $\epsilon > 0$ and $\rho > 0$, there exists a $\delta = \delta(\rho) > 0$, independent of ϵ, and a $\tau = \tau(\epsilon, \rho) > 0$, such that $P\{\omega : \sup_{t\in T_{a,t_0}, t>t_0+\tau} d(x(t, \omega, a, t_0), M) < \epsilon\} \geq 1-\rho$ whenever $d(a, M) < \delta$, for any process $x(\cdot, \cdot, a, t_0) \in S$. □

Definition 3.4 A stochastic motion $x(\cdot, \cdot, a, t_0) \in S$ is *stochastically bounded* if for every $\rho \in (0,1)$, there exists a $\beta = \beta(\rho) > 0$ such that $P\{\omega : \sup_t d(x(t, \omega, a, t_0), a) < \beta\} \geq 1-\rho$. A stochastic dynamical system S is *uniformly stochastically bounded* if for every $\rho \in (0,1)$, for every $\alpha > 0$ and for every $t_0 \in T$ there exists a $\beta = \beta(\alpha, \rho) > 0$ such that if $d(a, x_0) < \alpha$, then for all $x(\cdot, \cdot, a, t_0) \in S$, $P\{\omega : \sup_t d(x(t, \omega, a, t_0), x_0) < \beta\} \geq 1-\rho$, where x_0 is a fixed point in X. A stochastic dynamical system S is *uniformly stochastically ultimately bounded* if for every $\rho \in (0,1)$ there exists a $B = B(\rho) > 0$ and if for every $\alpha > 0$ and for every $t_0 \in T$ there exists a $\tau = \tau(\alpha) > 0$ such that for all $x(\cdot, \cdot, a, t_0) \in S$, $P\{\omega : \sup_{t\in T_{a,t_0}, t>t_0+\tau} d(p(t, \omega, a, t_0), x_0) < B\} \geq 1-\rho$ whenever $d(a, x_0) < \alpha$, where x_0 is a fixed point in X. □

In the above definitions, the constants β and B may in general depend on the choice of $x_0 \in X$. However, the definitions themselves are independent of the choice of x_0. Furthermore, we may replace $x_0 \in X$ by a bounded set in X.

Definition 3.5 (S, M) is said to be *stochastically asymptotically stable in the large* if it is stochastically stable and for all $a \in A$, $x(\cdot, \cdot, a, t_0) \in S$ $P\{\omega : \lim_{t\to\infty} d(x(t, \omega, a, t_0), M) = 0\} = 1$. (S, M) is *uniformly stochastically asymptotically stable in the large* if (i) it is uniformly stochastically stable; (ii) S is uniformly stochastically bounded, and (iii) for every $\alpha > 0$, for every $\epsilon > 0$, and for every $t_0 \in T$, there exists $\tau = \tau(\epsilon, \alpha) > 0$, such that $P\{\omega : \sup_{t\in T_{a,t_0}, t>t_0+\tau} d(p(t, \omega, a, t_0), x_0) < \epsilon\} = 1$ for all $x(\cdot, \cdot, a, t_0) \in S$ whenever $d(a, M) < \alpha$. □

The above definitions for stochastic stability and boundedness are adaptations of corresponding definitions given in [2]. There are other types of definitions in the literature for stability and boundedness of stochastic dynamical systems, depending on the specific sense in which convergence is understood. For example, in [3], notions of *stability in probability, stability with probability one, and stability in the pth mean* are also employed. We address these separately elsewhere [11].

We are now in a position to introduce the concept of stability preserving mappings for stochastic dynamical systems.

Definition 3.6 Let $\{T, X_1, A_1, S_1\}$ and $\{T, X_2, A_2, S_2\}$ be two stochastic dynamical systems defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $M_1 \subset A_1$ and $M_2 \subset A_2$ be two invariant sets with respect to S_1 and S_2, respectively. We say that a function $V : X_1 \times T \to X_2$ is a *stochastic stability preserving mapping* from (S_1, M_1) to (S_2, M_2) if V satisfies the following conditions:

(i) The set S_2 is given by $S_2 \triangleq \mathcal{V}(S_1) = \{q(\cdot,\cdot,b,t_0) : q(t,\omega,b,t_0) = V(p(t,\omega,a,t_0),t)$ for each $\omega \in \Omega$, with $b = V(a,t_0)$ and $T_{b,t_0} = T_{a,t_0}\}$, where $\mathcal{V}$ is the mapping of S_1 onto S_2 induced by V.

(ii) The set M_2 is given by $M_2 \triangleq V(M_1) = \{b \in X_2 : b = V(a,t')$ for some $a \in X_1, t' \in T\}$.

(iii) the stochastic stability of (S_1, M_1) is equivalent to the stochastic stability of (S_2, M_2), i.e., (S_1, M_1) is stochastically stable if and only if (S_2, M_2) is stochastically stable.

(iv)If in addition, the stochastic asymptotic stability, the uniform stochastic stability, and the uniform stochastic asymptotic stability of (S_1, M_1) and (S_2, M_2) are equivalent, respectively, then V is said to be a *strongly stochastic stability preserving mapping.* □

4 The Stability Preserving Mapping Theorem

We will employ two classes of monotone functions. We say that a continuous function $\phi : [0,r] \to R^+$ (resp., $\phi : R^+ \to R^+$) belongs to *class K* (i.e., $\phi \in K$) if $\phi(0) = 0$ and if ϕ is strictly increasing on $[0,r]$ (resp., on R^+). We say that a continuous function $\sigma : [0,\infty) \to R^+$ belongs to *class KR* if $\sigma \in K$ and $\lim_{s\to\infty} \sigma(s) = +\infty$.

Theorem 4.1 Let $\{T, X_1, A_1, S_1\}$ and $\{T, X_2, A_2, S_2\}$ be two stochastic dynamical systems defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $M_1 \subset A_1$ be closed. Assume that $V : X_1 \times T \to X_2$ satisfies the following hypotheses:

(i) $S_2 = \mathcal{V}(S_1)$ (see Definition 3.6 (i) for the definition of $\mathcal{V}$); and

(ii) there exist $\phi_1, \phi_2 \in K$, defined on R^+, such that

$$\phi_1(d_1(x, M_1)) \le d_2(V(x,t), M_2) \le \phi_2(d_1(x, M_1)) \tag{3}$$

for all $x \in X_1$ and $t \in T$, where $M_2 = V(M_1)$ (see Definition 3.6 (ii) for the definition of $V(M_1)$), and d_1, d_2 are the metrics defined on X_1 and X_2, respectively.

Then the following statements are true:

(a) the invariance of (S_1, M_1) and the invariance of (S_2, M_2) are equivalent;

(b) V is a strongly stochastic stability preserving mapping.

Proof: (a) We first note that the assumption that M_1 is closed, together with relation (3), implies that $x \in M_1$ if and only if $V(x,t) \in M_2$ for any $t \in T$.

Assume that (S_1, M_1) is invariant. For any $b \in M_2$, it follows from (i) that for any $q(\cdot,\cdot,b,t_0) \in S_2$, there exists a $p(\cdot,\cdot,a,t_0) \in S_1$ such that $q(t,\omega,b,t_0) = V(p(t,\omega,a,t_0),t)$ with $b = V(a,t_0)$. $b \in M_2$ implies that $a \in M_1$. By assumption, $P\{\omega : p(t,\omega,a,t_0) \in M_1$ for all $t \in T_{a,t_0}\} = 1$. Thus, from the definition of M_2, $P\{\omega : q(t,\omega,b,t_0) \in M_2$ for all $t \in T_{a,t_0}\} = P\{\omega : p(t,\omega,a,t_0) \in M_1$ for all $t \in T_{a,t_0}\} = 1$. This implies that (S_2, M_2) is invariant. Conversely, assume that (S_2, M_2) is invariant. If follows from (i) that for any $a \in M_1$, and any $p(\cdot,\cdot,a,t_0) \in S_1$, we have $q(\cdot,\cdot,b,t_0) = V(p(\cdot,\cdot,a,t_0),t) \in S_2$ where $b = V(a,t_0) \in M_2$. By the invariance of (S_2, M_2) we have that $P\{\omega : q(t,\omega,b,t_0) \in M_2$ for all $t \in T_{a,t_0}\} = 1$. Thus, $P\{\omega : p(t,\omega,a,t_0) \in M_1$ for all $t \in T_{a,t_0}\} = P\{\omega : q(t,\omega,b,t_0) \in M_2$ for all $t \in T_{a,t_0}\} = 1$. Therefore, (S_1, M_1) is invariant. This proves that the invariance of (S_1, M_1) and the invariance of (S_2, M_2) are equivalent.

(b) Let us first prove that the stochastic stability of (S_1, M_1) and the stochastic stability of (S_2, M_2) are equivalent. Assume that (S_1, M_1) is stochastically stable. Then for any $\epsilon > 0$, $t_0 \in T$, and $\rho > 0$, there exists a $\delta = \delta(t_0, \epsilon, \rho) > 0$ such that $P\{\omega : \sup_{t\in T_{a,t_0}} d(p(t,\omega,a,t_0), M_1) \ge \phi_2^{-1}(\epsilon)\} < \rho$ whenever $d(a, M) < \phi_1^{-1}(\delta)$. We now prove that (S_2, M_2) is stochastically stable. By assumption (i), there exists for any $q(\cdot,\cdot,b,t_0) \in S_2$ a $p(\cdot,\cdot,a,t_0) \in S_1$ such that $q(t,\omega,b,t_0) = V(p(t,\omega,a,t_0),t)$ with $b = V(a,t_0)$. When $d(b, M_2) < \delta$, we have $d(a, M_1) < \phi_1^{-1}(\delta)$ and

$$\begin{aligned} &P\{\omega : \sup_{t\in T_{b,t_0}} d(q(t,\omega,b,t_0), M_2) \ge \epsilon\} \\ \le &P\{\omega : \sup_{t\in T_{a,t_0}} d(p(t,\omega,a,t_0), M_1) \ge \phi_2^{-1}(\epsilon)\} < \rho. \end{aligned}$$

Therefore, (S_2, M_2) is stochastically stable. Conversely, assume that (S_2, M_2) is stochastically stable. Then for any $\epsilon > 0$, $t_0 \in T$, and $\rho > 0$, there exists a $\delta = \delta(t_0, \epsilon, \rho) > 0$ such that $P\{\omega :$

$\sup_{t \in \mathcal{T}_{b,t_0}} d(q(t,\omega,b,t_0),M_2) \geq \phi_1(\epsilon)\} < \rho$ whenever $d(b,M_2) < \phi_2(\delta)$. For any $p(\cdot,\cdot,a,t_0) \in S_1$ such that $d(a,M_1) < \delta$, we have that $d(V(a,t_0),M_2) < \phi_2(\delta)$, and that $P\{\omega : \sup_{t \in \mathcal{T}_{a,t_0}} d(p(t,\omega,a,t_0),M_1) \geq \epsilon\} \leq P\{\omega : \sup_{t \in \mathcal{T}_{b,t_0}} d(q(t,\omega,V(a,t_0),t_0),M_2) \geq \phi_1(\epsilon)\} < \rho$. Therefore, (S_1,M_1) is stochastically stable.

The proof of the equivalence of the uniform stochastic stability of (S_1,M_1) and (S_2,M_2) follows readily from the proof of the equivalence of the stochastic stability of (S_1,M_1) and (S_2,M_2) given above, choosing δ to be independent of t_0.

Next, we prove the equivalence of the stochastic asymptotic stability of (S_1,M_1) and (S_2,M_2). In doing so, we first note that for any $\omega \in \Omega$, $\lim_{t\to\infty} d(p(t,\omega,a,t_0),M_1) = 0$ if and only if $\lim_{t\to\infty} d(V(p(t,\omega,a,t_0),t),M_2) = 0$. Assume that (S_1,M_1) is stochastically asymptotically stable, i.e., (S_1,M_1) is stochastically stable, and for every $\rho \in (0,1)$, there exists a $\delta = \delta(t_0,\rho) > 0$ such that $P\{\omega : \lim_{t\to\infty} d(p(t,\omega,a,t_0),M_1) = 0\} \geq 1-\rho$ for any process $p(\cdot,\cdot,a,t_0) \in S_1$, whenever $d(a,M_1) < \phi_1^{-1}(\delta)$. Then, (S_2,M_2) is stochastically stable. By (i), for any $q(\cdot,\cdot,b,t_0) \in S_2$, there exists a $p(\cdot,\cdot,a,t_0) \in S_1$ such that $q(t,\omega,b,t_0) = V(p(t,\omega,a,t_0),t)$ with $b = V(a,t_0)$. Thus, whenever $d(b,M_2) < \delta$, we have that $d(a,M_1) < \phi_1^{-1}(\delta)$, and that $P\{\omega : \lim_{t\to\infty} d(q(t,\omega,b,t_0),M_2) = 0\} = P\{\omega : \lim_{t\to\infty} d(p(t,\omega,a,t_0),M_1) = 0\} \geq 1-\rho$. Therefore, (S_2,M_2) is stochastically asymptotically stable. Assume that (S_2,M_2) is stochastically asymptotically stable. To prove that (S_1,M_1) is stochastically asymptotically stable, we proceed similarly as in the proof given above, using the facts that for every process $p(\cdot,\cdot,a,t_0) \in S_1$, $V(p(\cdot,\cdot,a,t_0),t) \in S_2$ and for every $\omega \in \Omega$, $\lim_{t\to\infty} d(p(t,\omega,a,t_0),M_1) = 0$ if and only if $\lim_{t\to\infty} d(V(p(t,\omega,a,t_0),t),M_2) = 0$. This proves the equivalence of the stochastic asymptotic stability of (S_1,M_1) and (S_2,M_2).

Finally, we show that the uniform stochastic asymptotic stability of (S_1,M_1) and that of (S_2,M_2) are equivalent. Assume that (S_1,M_1) is uniformly stochastically asymptotically stable. Then (S_1,M_1) is uniformly stochastically stable and for every $\epsilon > 0$ and $\rho > 0$, there exists a $\delta = \delta(\rho) > 0$, independent of ϵ, and a $\tau = \tau(\epsilon,\rho) > 0$, such that $P\{\omega : \sup_{t \in \mathcal{T}_{a,t_0}, t > t_0+\tau} d(p(t,\omega,a,t_0),M_1) < \epsilon\} \geq 1-\rho$ whenever $d(a,M_1) < \delta$. By (i), for every $q(\cdot,\cdot,b,t_0) \in S_2$, there exists a $p(\cdot,\cdot,a,t_0) \in S_1$ such that $q(t,\omega,b,t_0) = V(p(t,\omega,a,t_0),t)$ with $b = V(a,t_0)$. Thus, whenever $d(b,M_2) < \phi_1(\delta)$, we have that $d(a,M_1) < \delta$, and that $P\{\omega : \sup_{t \in \mathcal{T}_{b,t_0}, t > t_0+\tau} d(q(t,\omega,b,t_0),M_1) < \phi_2(\epsilon)\} \geq 1-\rho$. Therefore, (S_2,M_2) is uniformly stochastically asymptotically stable. Conversely, assume that (S_2,M_2) is uniformly stochastically asymptotically stable. Then (S_2,M_2) is uniformly stochastically stable and for every $\epsilon > 0$ and $\rho > 0$, there exists a $\delta = \delta(\rho) > 0$, independent of ϵ, and a $\tau = \tau(\epsilon,\rho) > 0$, such that $P\{\omega : \sup_{t \in \mathcal{T}_{b,t_0}, t > t_0+\tau} d(q(t,\omega,b,t_0),M_1) < \phi_1(\epsilon)\} \geq 1-\rho$ for all $q(\cdot,\cdot,b,t_0) \in S_2$, whenever $d(b,M_2) < \phi_2(\delta)$. For any process $p(\cdot,\cdot,a,t_0) \in S_1$ such that $d(a,M_1) < \delta$, it follows from (i) that $q(t,\omega,b,t_0) = V(p(t,\omega,a,t_0),t) \in S_2$ with $b = V(a,t_0)$ and $d(b,M_2) < \phi_2(\delta)$. Hence, $P\{\omega : \sup_{t \in \mathcal{T}_{a,t_0}, t > t_0+\tau} d(p(t,\omega,a,t_0),M_1) < \epsilon\} \geq 1-\rho$. We have already proved that (S_1,M_1) is uniformly stochastically stable. Therefore, (S_1,M_1) is uniformly stochastically asymptotically stable.

This completes the proof of the theorem. ☐

Remark 4.1 Clearly, under the conditions of Theorem 4.1, the *stochastic instability* of (S_1,M_1) and (S_2,M_2) are equivalent. ☐

The next result is concerned with mappings which preserve stochastic boundedness.

Theorem 4.2 Assume that hypotheses (i) and (ii) of Theorem 4.1 are satisfied with $\phi_1,\phi_2 \in KR$ and that the sets M_1 and M_2 are bounded. Then

(a) the uniform boundedness of S_1 and the uniform boundedness of S_2 are equivalent.

(b) the uniform ultimate boundedness of S_1 and the uniform ultimate boundedness of S_2 are equivalent.

Proof: (a) Assume that S_1 is uniformly stochastically bounded. Then for every $\rho \in (0,1)$, there exists for every $\alpha > 0$ and for every $t_0 \in T$ a $\beta = \beta(\alpha,\rho) > 0$ such that if $d(a,M_1) < \alpha$, then for all $p(\cdot,\cdot,a,t_0) \in S_1$, $P\{\omega : \sup_t d(p(t,\omega,a,t_0),M_1) < \beta\} \geq 1-\rho$. By the assumption that $S_2 \subset \mathcal{V}(S_1)$, there exists for every $q(\cdot,\cdot,b,t_0) \in S_2$ a $p(\cdot,\cdot,a,t_0) \in S_1$ such that $q(t,\omega,b,t_0) = V(p(t,\omega,a,t_0),t)$ with $b = V(a,t_0)$. Thus, when $d(b,M_2) < \phi_1(\alpha)$, we have that $d(a,M_1) < \alpha$, and that $P\{\omega : \sup_t d(q(t,\omega,b,t_0),M_2) < \phi_2(\beta)\} \geq P\{\omega : \sup_t d(p(t,\omega,a,t_0),M_1) < \beta\} \geq 1-\rho$. Therefore, S_2 is uniformly stochastically bounded.

Conversely, assume that S_2 is uniformly stochastically bounded. We can proceed similarly as in the proof above to show that S_1 is uniformly stochastically bounded.

(b) Assume that S_1 is uniformly stochastically ultimately bounded. By definition, for every $\rho \in (0,1)$ there exists a $B = B(\rho) > 0$ and for every $\alpha > 0$ and for every $t_0 \in T$ there exists a $\tau = \tau(\alpha) > 0$ such that for all $p(\cdot,\cdot,a,t_0) \in S_1$, $P\{\omega : \sup_{t \in \mathcal{T}_{a,t_0}, t > t_0+\tau} d(p(t,\omega,a,t_0),M_1) < B\} \geq 1-\rho$ whenever $d(a,M_1) < \alpha$. By the assumption that $S_2 \subset \mathcal{V}(S_1)$, there exists for every $q(\cdot,\cdot,b,t_0) \in S_2$ a $p(\cdot,\cdot,a,t_0) \in S_1$ such that $q(t,\omega,b,t_0) =$

$V(p(t,\omega,a,t_0),t)$ with $b = V(a,t_0)$. Thus, when $d(b,M_2) < \phi_1(\alpha)$, we have that $d(a,M_1) < \alpha$, and that $P\{\omega : \sup_{t\in T_{b,t_0}, t>t_0+\tau} d(q(t,\omega,b,t_0),M_1) < \phi_2(B)\} \geq P\{\omega : \sup_{t\in T_{a,t_0}, t>t_0+\tau} d(p(t,\omega,a,t_0),M_1) < B\} \geq 1-\rho$. Therefore, S_2 is uniformly stochastically ultimately bounded.

The converse can be proved in a similar manner by using the relation $\mathcal{V}(S_1) \subset S_2$. □

Theorem 4.3 Assume that hypotheses (i) and (ii) of Theorem 4.1 are satisfied with $\phi_1, \phi_2 \in KR$ and that the sets M_1 and M_2 are bounded. Then the uniform stochastic asymptotic stability in the large of (S_1, M_1) and the uniform stochastic asymptotic stability in the large of (S_2, M_2) are equivalent.

Proof: By Theorem 4.1, the uniform stochastic stability of (S_1, M_1) and the uniform stochastic stability of (S_2, M_2) are equivalent. By Theorem 4.2, the uniform stochastic boundedness of (S_1, M_1) and the uniform stochastic boundedness of (S_2, M_2) are equivalent. It suffices to prove that the global uniform attractivity (condition (iii) in the definition of uniform stochastic asymptotic stability in the large) of (S_1, M_1) and the global uniform attractivity of (S_2, M_2) are equivalent. The proof is similar to the proof of the uniform stochastic asymptotic stability in Theorem 4.1. We omit the details. □

5 The Comparison Theorem

In the results of the previous section we assume that $S_2 = \mathcal{V}(S_1)$, which in general is extremely difficult to verify in applications. However, in the proofs of Theorems 4.1, 4.2, and 4.3, we actually proved that when $\mathcal{V}(S_1) \subset S_2$, the qualitative properties of (S_2, M_2) imply the corresponding qualitative properties of (S_1, M_1). This fact can be used as the basis of an easily applied comparison theory. The purpose of such a theory is to deduce the qualitative properties of a dynamical system, say S_1 (the *object of inquiry*), from the corresponding qualitative properties of a dynamical system, say S_2 (the *comparison system*). From Theorem 4.1, the next result follows immediately.

Theorem 5.1 (Comparison Theorem) Let $\{T, X_1, A_1, S_1\}$ and $\{T, X_2, A_2, S_2\}$ be two stochastic dynamical systems defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $M_1 \subset A_1$ be closed. Assume that $V : X_1 \times T \to X_2$ satisfies (i) $\mathcal{V}(S_1) \subset S_2$; and (ii) there exist $\phi_1, \phi_2 \in K$, defined on R^+, such that $\phi_1(d_1(x, M_1)) \leq d_2(V(x,t), M_2) \leq \phi_2(d_1(x, M_1))$ for all $x \in X_1$ and $t \in T$, where $M_2 = V(M_1)$, and d_1, d_2 are the metrics on X_1 and X_2, respectively.

Then the following statements are true: (a) the invariance of (S_2, M_2) implies the invariance of (S_1, M_1); (b) the stochastic stability, uniform stochastic stability, stochastic asymptotic stability, and uniform stochastic asymptotic stability of (S_2, M_2) imply the same corresponding types of stability of (S_1, M_1). □

As a *specific example*, we apply Theorem 5.1 to establish a Lyapunov stability result for higher order (Ito) stochastic differential equations given by

$$dx = F(x)dt + G(x)dz(t) \tag{4}$$

where $x \in R^n$, $F : R^n \to R^n$, $G : R^n \to R^n$, $F(0) = G(0) = 0$, and $z(t)$ is a normalized scalar Wiener process. We assume that F and G possess the appropriate properties which ensure the existence and uniqueness of solutions of (4) for given initial conditions (see, e.g., [6]). The comparison system which we will employ is the stochastic dynamical system $S_{(5)}$ comprised of nonnegative processes that are determined by the scalar stochastic differential equation

$$dr = f(r,t)dt + g(r,t)dz(t), \tag{5}$$

where $f, g : R^+ \times R^+ \to R^+$ with $f(0,t) = g(0,t) = 0$. We assume that f and g possess the appropriate properties which ensure the existence and uniqueness of solutions of (5) for given initial conditions (see, e.g., [6]). The following theorem provides sufficient conditions for uniform stochastic asymptotic stability of the trivial solution of system $S_{(5)}$.

Theorem 5.2 Let $S_{(5)} = \{r(t)\}$ denote the set of continuous nonnegative processes that satisfy equation (5). Suppose that $|f(r,t)| + |g(r,t)| \neq 0$ for any $r \neq 0$. If there exists $\alpha \in (0,1)$ such that $\frac{2f(r,t)}{g^2(r,t)} \leq \frac{\alpha}{r}$ is true for all $0 < r \leq \delta$ and $t \in R^+$, where $0 < \delta \leq +\infty$, then the trivial solution $r(t) = 0$ of $S_{(5)}$ is uniformly stochastically asymptotically stable.

Proof: Let $V : [0, \delta] \to R^+$ such that $V(r) = r^\beta$, where $0 < \beta < 1 - \alpha$. Since, by assumption, $\frac{2f(r,t)}{g^2(r,t)} \leq \frac{\alpha}{r}$ for all $0 < r \leq \delta$ and $t \in R^+$, we have that

$$\begin{aligned}
\mathcal{L}V(r) &\triangleq V'(r)f(r,t) + \frac{1}{2}V''(r)g^2(r,t) \\
&= r^{(\beta-2)}\Big(\beta r f(r,t) + \frac{1}{2}\beta(\beta-1)g^2(r,t)\Big) \\
&\leq min\Big\{\frac{1}{2}r^{(\beta-2)}g^2(r,t)\beta(\alpha+\beta-1), \\
&\qquad r^{(\beta-1)}f(r,t)\beta(1+(\beta-1)/\alpha)\Big\},
\end{aligned}$$

where $\mathcal{L}$ denotes the weak infinitesimal operator (see, e.g., [2]). Since $\beta < 1 - \alpha$ and $|f(r,t)| + |g(r,t)| \neq 0$ when $r \neq 0$, $\mathcal{L}V$ is negative definite. It follows from existing results (see, e.g, [5]) that $r(t) = 0$ of $S_{(5)}$ is uniformly stochastically asymptotically stable. □

Next, we apply Theorems 5.1 and 5.2 to study the stability properties of nth order stochastic differential equations (4).

Theorem 5.3 Assume that there exists a function $V : R^n \to R^+$ such that $V(x)$ is positive definite, and twice continuously differentiable in the open set $B_r = \{x : V(x) < r\}$. Also assume that there exists $\alpha \in (0,1)$ such that

$$\frac{\mathcal{L}V}{\left(\sum_i \frac{\partial V}{\partial x_i} G_i(x)\right)^2} \leq \frac{\alpha}{V}, \qquad (6)$$

is true along the solution processes of (4), where $\mathcal{L}V = \sum_i \frac{\partial V}{\partial x_i} F_i(x) + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j} S_{ij}(x)$ and $S(x) = G(x)^T G(x) = \{S_{ij}(x)\}$, and $|\mathcal{L}V| + |\sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j} S_{ij}(x)| \neq 0$ for all $x \neq 0$. Then the trivial solution of (4) is stochastically asymptotically stable.

Proof: By the assumption that V is positive definite, there exist $\phi_1, \phi_2 \in K$ such that $\phi(|x|) \leq V(x) \leq \phi_2(|x|)$ for all $x \in B_r$ for some $r > 0$, where $|\cdot|$ denotes any norm on R^n.

Now let $x(t)$ be any solution of (4). Then

$$dV(x(t)) = \mathcal{L}V(x(t))dt + \sum_i \frac{\partial V}{\partial x_i} G_i(x(t))dz(t). \qquad (7)$$

This is a scalar stochastic differential equation (of the form (5)). $V = 0$ is an equilibrium of the stochastic dynamical system $S_{(7)}$ determined by (7). Define $S_2 \triangleq S_{(7)}$. When (6) is satisfied, it follows from Theorem 5.2 that (S_2, M_2), $M_2 = \{0\} \subset R^+$, is uniformly stochastically asymptotically stable. Let $S_1 \triangleq S_{(4)}$ denote the stochastic dynamical system determined by (4), and let $M_1 = \{0\} \subset R^n$. Then we have $M_2 = V(M_1)$ and $\mathcal{V}(S_1) \subset S_2$. By Theorem 5.1, (S_1, M_1) is uniformly stochastically asymptotically stable. □

The following result utilizes quadratic form V-functions in the analysis of nth order linear stochastic differential equations.

Corollary 5.1 We consider linear stochastic differential equations of the form

$$dx = Axdt + Bxdz(t) \qquad (8)$$

where $x \in R^n$, and A and B are constant matrices. Let $P = P^T$ be a positive definite matrix, and let $Q(P,x) = x^T PAP^{-1}x - (x^T PBP^{-1}x)^2 + (\frac{1}{2})x^T P^{-1} B^T P^2 BP^{-1}x$, and $c_1 = \sup_{\|x\|=1} Q(P,x)$. Then the trivial solution $x = 0$ of (8) is uniformly stochastically asymptotically stable if $c_1 < 0$.

Proof: Let $V(x) = x^T P^2 x$, and let λ_{min} and λ_{max} denote the smallest and largest eigenvalues of P, respectively. Then, it is true that $\lambda_{min}^2 \|x\|^2 \leq V(x) \leq \lambda_{max}^2 \|x\|^2$ for all $x \in R^n$, where $\|\cdot\|$ denotes the Euclidean norm. Along the solutions of (8), $\mathcal{L}V(x(t)) = 2x^T P^2 Ax + x^T B^T P^2 Bx$, and $\sum_i \frac{\partial V}{\partial x_i} G_i(x(t)) = 2x^T P^2 Bx$. Let $y = (1/\|Px\|)Px$. Then

$$\begin{aligned} &2x^T P^2 Ax + x^T B^T P^2 Bx) \\ &= 2(y^T PAP^{-1}y + \frac{1}{2} y^T P^{-1} B^T P^2 BP^{-1} y)V(x), \end{aligned}$$

$$(2x^T P^2 Bx)^2 = 4(y^T PBP^{-1}y)^2 V^2(x).$$

When $c_1 < 0$ and when $x^T P^2 Bx \neq 0$, we have $\frac{\mathcal{L}V}{\left(\sum_i \frac{\partial V}{\partial x_i} G_i(x)\right)^2} \leq \left(\frac{c_1}{\|PBP^{-1}}\| + 1\right) \frac{1}{V(x)}$, and when $x^T P^2 Bx = 0$, we have $\frac{2(2x^T P^2 Ax + x^T B^T P^2 Bx)}{(2x^T P^2 Bx)^2} = -\infty$. It follows from Theorem 5.3 that $x = 0$ is uniformly stochastically asymptotically stable when $c_1 < 0$. □

Remark 5.1 Similar results as Corollary 5.1 have been reported in [7] for $P = I$, where I is the identity matrix. Thus, Corollary 5.1 generalizes these results.

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