Classification and stabilizability analysis of bimodal piecewise affine systems

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Abstract

This paper presents a classification of bimodal piecewise affine systems from the viewpoint of well-posedness. First, we address the problem of feedback equivalence to a well-posed system, called here the feedback well-posedness problem, of a general class of bimodal piecewise affine systems. Next, based on this result, we classify all feedback well-posed systems into four classes to address the control problem of piecewise affine systems in a systematic way. Finally, as its application, the stabilizability with well-posedness is discussed for each class, and several remarks on stabilizability are given.

1 Introduction

In hybrid systems with autonomous switchings and jumps, it is well-known that there exist various phenomena such as no solutions, multiple solutions, sliding motions, Zeno solutions, and multiple events by jump solutions, and so the well-posedness (existence and uniqueness of solutions) problem is crucial for discussing the properties of hybrid systems (see e.g., [1]). However, the existing standard theory (see e.g., [2]) on well-posedness of such systems is not quite satisfactory, even in the case of piecewise *linear* systems, which is the simplest one of hybrid systems. Thus several papers have recently discussed the well-posedness problem for a class of hybrid systems (e.g., [3]-[6]). We also have derived necessary and sufficient conditions for some classes of piecewise *affine* systems to be well-posed in [7]-[10]. Furthermore, in [8], we have proposed the concept of *feedback well-posedness*, which implies that the system is feedback equivalent to a well-posed system, and derived a feedback well-posedness condition for a limited class of bimodal piecewise affine systems.

This paper continues upon our papers of [7]-[10], and presents a classification of bimodal piecewise affine systems with feedback well-posedness, which will be useful for developing the control theory of a class of hybrid systems. First, we give a solution to the feedback well-posedness problem of a more general class of bimodal piecewise affine systems. Next, by using this result, all feedback well-posed systems are divided into four classes, and then a canonical form of each class of feedback well-posed systems is presented. Finally, based on a canonical form in each class, we discuss stabilizability with well-posedness, and give sufficient conditions for stabilizability in some classes and interesting remarks on stabilizability in the other classes.

In the sequel, we use the symbol $*$ representing any fixed but unspecified number or matrix. $0_{m,n}$ and 0_n denote the $m \times n$ and $n \times n$ zero matrix, respectively.

2 Definition of well-posedness

Consider the bimodal piecewise affine system

$$\Sigma_O \begin{cases} \text{mode } 1 : \dot{\bar{x}} & = & \bar{A}_1 \bar{x}, & \text{if } \bar{C}\bar{x} \geq 0 \\ \text{mode } 2 : \dot{\bar{x}} & = & \bar{A}_2 \bar{x}, & \text{if } \bar{C}\bar{x} \leq 0 \end{cases} \tag{1}$$

where

$$\bar{A}_i \triangleq \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \quad \bar{C} \triangleq \begin{bmatrix} C & c \end{bmatrix}, \quad \bar{x} \triangleq \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and $x \in \mathcal{R}^n$, $A_i \in \mathcal{R}^{n\times n}$, $a_i \in \mathcal{R}^n$, $C \in \mathcal{R}^{1\times n}$, and $c \in \mathcal{R}$. If t is a time instant at which the system switches from one mode to another mode, t is said to be an event time. A point $\hat{t}$ is called a right(left)-accumulation point of event times, if there exists a sequence $\{t_i\}$ of event times such that $t_i < (>) t_{i+1}$ and $\lim_{i\to\infty} t_i = \hat{t} < \infty$. Then a solution of this system is defined as follows.

Definition 2.1
(i) If, for a given initial state $x(t_0)$, $x(t)$ satisfies on $[t_0, t_1)$ for some $t_1 > t_0$

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau))d\tau \tag{2}$$

where $f(x)$ is the discontinuous vector field given by the right-hand side of (2), and there is no left-accumulation point of event times on $[t_0, t_1)$, then $x(t)$ is said to be a continuous-state solution (for brevity, C-solution) of Σ_0 on $[t_0, t_1)$ in the sense of Carathéodory.
(ii) If, for a given initial hybrid state $(I(t_0), x(t_0))$, where $I(t) \in \{1, 2\}$ is the label of the mode, $x(t)$ satisfies the same conditions as that of a C-solution on $[t_0, t_1)$ for some $t_1 > t_0$, then $(I(t), x(t))$ is said to be a hybrid-state solution (for brevity, H-solution) of the system Σ_O on $[t_0, t_1)$.

Definition 2.2 *The system Σ_O is said to be C-(H-)well-posed if there exists a unique C-solution on* $[0, \infty)$

for every initial state $x(0) \in \mathcal{R}^n$ (if for every initial state $x(0) \in \mathcal{R}^n$ there exists some initial discrete state $I(0) \in \{1,2\}$ such that there exists a unique H-solution on $[0,\infty)$.)

See [7]-[10] for the details of the above definition. In considering H-solutions, we assume that the discrete state of every trajectory converges to a unique value at every right-accumulation point of event times, although it is not explicitly expressed hereafter.

3 Feedback well-posedness

Consider the system given by

$$\Sigma \left\{ \begin{array}{lcll} \text{mode } 1: \dot{\bar{x}} & = & \bar{A}_1\bar{x} + \bar{B}_1 u_1, & \text{if } \bar{C}\bar{x} \geq 0, \\ \text{mode } 2: \dot{\bar{x}} & = & \bar{A}_2\bar{x} + \bar{B}_2 u_2, & \text{if } \bar{C}\bar{x} \leq 0 \end{array} \right. \quad (3)$$

where $\bar{x} = [x^{\mathrm{T}} \quad 1]^{\mathrm{T}}$,

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \bar{C} = [\ C \quad c\ ]$$

and $x \in \mathcal{R}^n$, $A_i \in \mathcal{R}^{n\times n}$, $B_i \in \mathcal{R}^{n\times \bar{m}_i}$, $C \in \mathcal{R}^{1\times n}$, $a_i \in \mathcal{R}^n$, and $c \in \mathcal{R}$. We say that the system Σ is *feedback C-(H-)well-posed* if it is transformable into a C-(H-)well-posed system by a state feedback of the form

$$\left\{ \begin{array}{ll} u_1 = \bar{K}_1\bar{x}, & \text{if } \bar{C}\bar{x} \geq 0, \\ u_2 = \bar{K}_2\bar{x}, & \text{if } \bar{C}\bar{x} \leq 0 \end{array} \right. \quad (4)$$

where $\bar{K}_i = [K_i \quad k_i](\neq 0)$, $K_i \in \mathcal{R}^{\bar{m}_i\times n}$, and $k_i \in \mathcal{R}^{\bar{m}_i}$ $(i = 1,2)$.

Letting $\bar{A} = \bar{A}_1$ and $\bar{G}\bar{F} = \bar{A}_2 - \bar{A}_1$ where $\bar{G}$ and $\bar{F}$ are arbitrary $(n+1)\times \bar{l}$ and $\bar{l}\times(n+1)$ matrices, the system Σ of equations (3) is rewritten by

$$\Sigma \left\{ \begin{array}{lcll} \text{mode } 1: \dot{\bar{x}} & = & \bar{A}\bar{x} + \bar{B}_1 u_1, & \text{if } \bar{C}\bar{x} \geq 0, \\ \text{mode } 2: \dot{\bar{x}} & = & (\bar{A} + \bar{G}\bar{F})\bar{x} + \bar{B}_2 u_2, & \text{if } \bar{C}\bar{x} \leq 0 \end{array} \right. \quad (5)$$

where $\bar{x} = [x^{\mathrm{T}} \quad 1]^{\mathrm{T}}$,

$$\bar{A} = \begin{bmatrix} A & a \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} G \\ 0 \end{bmatrix},$$

$$\bar{C} = [\ C \quad c\ ], \quad \bar{F} = [\ F \quad f\ ],$$

and $x \in \mathcal{R}^n$, $A \in \mathcal{R}^{n\times n}$, $B_i \in \mathcal{R}^{n\times \bar{m}_i}$, $G \in \mathcal{R}^{n\times \bar{l}}$, $C \in \mathcal{R}^{1\times n}$, $F \in \mathcal{R}^{\bar{l}\times n}$, $a \in \mathcal{R}^n$, $c \in \mathcal{R}$, and $f \in \mathcal{R}^{\bar{l}}$.

Let $\bar{B}_{ij}$ $(j = 1,\cdots,\bar{m}_i)$ be the jth column vector of $\bar{B}_i$, let $\bar{G}_j$ $(j = 1,\cdots,\bar{l})$ be the jth column vector of $\bar{G}$, let $\bar{F}_j$ $(j = 1,\cdots,\bar{l})$ be the jth row vector of $\bar{F}$. Let also p_{ij} $(j = 1,\cdots,\bar{m}_i)$ and q_j $(j = 1,\cdots,\bar{l})$ be the relative degrees of $(\bar{C},\bar{A},\bar{B}_{ij})$ and $(\bar{C},\bar{A},\bar{G}_j)$, with the leading Markov parameters defined by $\mu^i_{p_{ij}} \triangleq \bar{C}\bar{A}^{p_{ij}-1}\bar{B}_{ij}$ and $\mu^G_{q_j} \triangleq \bar{C}\bar{A}^{q_j-1}\bar{G}_j$, respectively.

Remark 3.1 *A class of systems treated in the feedback well-posedness problem in [8],[10] was limited to a class of systems satisfying $\bar{B}_1 = \bar{B}_2$ in (5). This paper deals with the more general case including $\bar{B}_1 \neq \bar{B}_2$.*

Using the input transformation, without loss of generality, we assume the following.
[A1] $p_{i1} < p_{i2} < \cdots < p_{i,m_i}(\leq n)$, and $p_{i,m_i+1} = p_{i,m_i+2} = \cdots = p_{i,\bar{m}_i} = \infty$, $i = 1,2$.
[A2] $q_1 < q_2 < \cdots < q_l(\leq n)$, and $q_{l+1} = q_{l+2} = \cdots = q_{\bar{l}} = \infty$.
[A3] $p_{ij} \neq q_k$ for all $i \in \{1,2\}$, $j \in \{1,2,\cdots,m_i\}$, and $k \in \{1,2,\cdots,l\}$.
where $m_i \leq \bar{m}_i$ and $l \leq \bar{l}$. Then the following is one of main results.

Theorem 3.1 *(Feedback C-well-posedness condition) Assume [A1] to [A3]. The system Σ of equations (5) is feedback C-well-posed if and only if one of the following statements (i) and (ii) is satisfied.*
(i) The following two conditions hold.
(a) There exist a pair $(j_1,j_2) \in \{1,2,\cdots,m_1\} \times \{1,2,\cdots,m_2\}$ satisfying $p_{1,j_1} = p_{2,j_2}$.
(b) For some integer $r \in \{0,1,2,\cdots,l\}$ satisfying $q_r < p_{2,j_2} < q_{r+1}$ where j_2 is given in (i)(a), the system

$$\hat{\Sigma} \left\{ \begin{array}{lcll} \text{mode } 1: \dot{\bar{x}} & = & \bar{A}\bar{x}, & \text{if } \bar{C}\bar{x} \geq 0, \\ \text{mode } 2: \dot{\bar{x}} & = & (\bar{A} + \hat{G}\hat{F})\bar{x}, & \text{if } \bar{C}\bar{x} \leq 0 \end{array} \right. \quad (6)$$

is C-well-posed, where $\hat{G} = [\bar{G}_1 \cdots \bar{G}_r]$ and $\hat{F} = [\bar{F}_1^{\mathrm{T}} \ \cdots \bar{F}_r^{\mathrm{T}}]^{\mathrm{T}}$, and $q_0 = 0$ and $\hat{G} = 0$ if $r = 0$.
(ii) The following two conditions hold.
(a) There exists no pair $(j_1,j_2) \in \{1,2,\cdots,m_1\} \times \{1,2,\cdots,m_2\}$ satisfying $p_{1,j_1} = p_{2,j_2}$.
(b) The system Σ with $u_1 = u_2 = 0$ is C-well-posed.

Theorem 3.1 divides the feedback C-well-posedness property into the two cases in terms of (i)(a) and (ii)(a), in other words, depending on the fact if there exists a pair of inputs for which the relative degrees to $\bar{C}\bar{x}$ are the same in both modes or not. When (i)(a) holds, the feedback C-well-posedness is characterized by condition (i)(b) that the system $\hat{\Sigma}$ with the discontinuous part $\hat{G}\hat{F}\bar{x}$, which effects on the term $\bar{C}\bar{x}$ faster than the inputs u_{i,j_i} $(i = 1,2)$ effect on it, must be C-well-posed. When (ii)(a) holds, it is characterized by condition (ii)(b) that the uncontrolled system must be C-well-posed. Note that feedback well-posedness is independent of the decomposition of $\bar{A}_2 - \bar{A}_1$ to $\bar{G}$ and $\bar{F}$.

Next, we give a condition for feedback well-posedness in the sense of H-solutions.

Theorem 3.2 *(Feedback H-well-posedness condition) Assume [A1] to [A3]. Then the system Σ of equations (5) is feedback H-well-posed if and only if for some integer $r \in \{0,1,2,\cdots,l\}$ satisfying $q_r < \min\{p_{11},p_{21}\} < q_{r+1}$, the system $\hat{\Sigma}$ given by (6) is H-well-posed, where $q_0 = 0$ and $\hat{G} = 0$ if $r = 0$.*

Condition (i)(b) in Theorem 3.1 and condition (ii) in Theorem 3.2 are characterized as follows.

Theorem 3.3 *In Theorem 3.1 and Theorem 3.2, the following relations hold.*

(i) The system $\hat{\Sigma}$ given by (6) is C-well-posed if and only if $\hat{F}_k$ $(k = 1, 2, \cdots, r)$ satisfies

$$\mu^G_{q_k}\hat{F}_k = \sum_{j=1}^{q_k+1} \alpha^k_j \bar{C}\bar{A}^{j-1}$$

where for $k = 1, 2, \cdots, r$, $\alpha^k_j \in \mathcal{R}$ $(j = 1, 2, \cdots, q_k)$ and $1 + \alpha^k_{q_k+1} > 0$.

(ii) The system $\hat{\Sigma}$ given by (6) is H-well-posed if and only if there exists an integer $s \in \{1, 2, \cdots, r\}$ such that for $\hat{F}_k$ $(k = 1, 2, \cdots, s)$ of the form

$$\mu^G_{q_k}\hat{F}_k = \sum_{j=1}^{q_k+1} \alpha^k_j \bar{C}\bar{A}^{j-1} + \beta_k[0_{1,n} \quad 1]$$

the following conditions hold.

(a) $\alpha^k_j \in \mathcal{R}$ $(j = 1, 2, \cdots, q_k)$, $1 + \alpha^k_{q_k+1} > 0$ and $\beta_k = 0$ for $k = 1, 2, \cdots, s-1$,

(b) $\alpha^s_j \in \mathcal{R}$ $(j = 1, 2, \cdots, q_s)$, and $(1+\alpha^s_{q_s+1} = 0, \beta_s = 0)$ or $(1 + \alpha^s_{q_s+1} \geq 0, \beta_s < 0)$ for $k = s$

where if $s = r$, condition (b) is replaced by (b') $\alpha^r_j \in \mathcal{R}$ $(j = 1, 2, \cdots, q_r)$, $1 + \alpha^r_{q_r+1} \geq 0$ and $\beta_r \leq 0$.

4 Classification and canonical forms

4.1 Classification of feedback well-posed systems

In the case of C-well-posedness, Theorem 3.1 provides the following two classes of feedback well-posed systems.

Class (C-1): a class of systems satisfying condition (i) in Theorem 3.1.

Class (C-2): a class of the systems satisfying condition (ii) in Theorem 3.1.

Class (C-1) implies that there exists a pair of inputs for which the relative degrees to $\bar{C}\bar{x}$ are the same in both modes, while class (C-2) implies that there exists no pair of inputs satisfying such a condition.

In the case of H-well-posedness, based on Theorem 3.2 and Theorem 3.3(ii), we divide all feedback H-well-posed systems into the following two classes.

Class (H-1): the system $\hat{\Sigma}$ satisfies $s < r$, or if $s = r (\geq 1)$ either of $(1 + \alpha^r_{q_r+1} = 0, \beta_r = 0)$ or $(1 + \alpha^r_{q_r+1} \geq 0, \beta_r < 0)$ in Theorem 3.2 and 3.3(ii).

Class (H-2): $r = 0$, or the system $\hat{\Sigma}$ satisfies $s = r (\geq 1)$ and $(1 + \alpha^r_{q_r+1} > 0, \beta_r = 0)$ in Theorem 3.2 and 3.3(ii).

Here each class implies the following: the system in class (H-1) is equivalent to the system that is H-well-posed for *all* u_1, u_2 of the form (4), while the system in class (H-2) is equivalent to the system that is H-well-posed for *some* u_1, u_2 of the form (4).

Note that we here use the different criterions to classify the systems in the cases of C-well-posedness and H-well-posedness. In the case of C-well-posedness, there exists no system that is C-well-posed for all u_1, u_2 of the form (4). So in this case, the criterion used in the H-well-posedness case is meaningless. On the other hand, it is important to classify the feedback C-well-posed systems in terms of the criterion whether or not there exists a pair of inputs satisfying $p_{1,j_1} = p_{2,j_2}$, although such a criterion is not so important in the case of H-well-posedness. It will be shown in the next section that the above classification is useful in considering the control problem such as the stabilization problem.

4.2 Canonical form of feedback well-posed systems

[1] Class (C-1)

Assume [A1] to [A3]. Using Theorem 3.3 (i), every feedback well-posed system in class (C-1) can be transformed into the following form.

Letting $k = p_{1,j_1}$ for simplicity of notation, and letting $T(k)$ and $d(k)$ be defined by

$$T(k) \triangleq \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix}, \quad d(k) \triangleq \begin{bmatrix} c \\ Ca \\ CAa \\ \vdots \\ CA^{k-2}a \end{bmatrix},$$

consider the following coordinate transformation:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} T(k) & d(k) \\ S & 0 \end{bmatrix} \bar{x} \tag{7}$$

where $\xi = [\xi_1 \ \xi_2 \ \cdots \xi_k]^{\mathrm{T}} \in \mathcal{R}^k$, $\eta \in \mathcal{R}^{n-k}$, and S is an $(n-k) \times n$ matrix satisfying that $[T(k)^{\mathrm{T}} \ S^{\mathrm{T}}]^{\mathrm{T}}$ is nonsingular and

$$S \begin{bmatrix} B_{11} & \cdots & B_{1,j_1} & B_{21} & \cdots & B_{2,j_2} \end{bmatrix} = 0. \tag{8}$$

Note here that such a matrix S always exists because of the property of the relative degrees.

Furthermore, consider the input transformation:

$$u_{1,j} = \begin{cases} \frac{1}{\mu^1_{p_{1j}}}\tilde{v}_{1,j}, & j = 1, \cdots, j_1 - 1, \\ \frac{1}{\mu^1_k}(\tilde{v}_{1,j_1} - \bar{C}\bar{A}^k\bar{x}), & j = j_1, \end{cases}$$
$$u_{2,j} = \begin{cases} \frac{1}{\mu^2_{p_{2j}}}\hat{v}_{1,j}, & j = 1, \cdots, j_2 - 1, \\ \frac{1}{\mu^2_k}(\hat{v}_{1,j_2} - \bar{C}\bar{A}^k\bar{x} - \bar{C}\bar{A}^{k-1}\bar{G}\bar{F}\bar{x}), & j = j_2 \end{cases} \tag{9}$$

$$\tilde{v}_1 = [\tilde{v}_{11} \ \cdots \ \tilde{v}_{1,j_1}]^{\mathrm{T}}, \quad \tilde{v}_2 = [u_{1,j_1+1} \ \cdots \ u_{1,\bar{m}_1}]^{\mathrm{T}},$$
$$\hat{v}_1 = [\hat{v}_{11} \ \cdots \ \hat{v}_{1,j_2}]^{\mathrm{T}}, \quad \hat{v}_2 = [u_{2,j_2+1} \ \cdots \ u_{2,\bar{m}_2}]^{\mathrm{T}}.$$

Then every feedback C-well-posed system in class (C-1) is given by the form

$$\text{mode 1: } \begin{cases} \dot{\xi} &= \tilde{A}_{11}\xi + \tilde{B}_1\tilde{v}_1 + \tilde{f} \\ \dot{\eta} &= \tilde{A}_{21}\xi + \tilde{A}_{22}\eta + \tilde{B}_2\tilde{v}_2 + \tilde{g} \end{cases}, \quad \xi_1 \geq 0, \tag{10}$$

$$\text{mode 2: } \begin{cases} \dot{\xi} &= \hat{A}_{11}\xi + \hat{A}_{12}\eta + \hat{B}_1\hat{v}_1 + \hat{f} \\ \dot{\eta} &= \hat{A}_{21}\xi + \hat{A}_{22}\eta + \hat{B}_2\hat{v}_2 + \hat{g} \end{cases}, \quad \xi_1 \leq 0 \tag{11}$$

where

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \in \mathcal{R}^{k \times k}$$

$$\hat{A}_{11} = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \beta_{k-1,1} & \cdots & \cdots & \beta_{k-1,k-1} & \beta_{k-1,k} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \in \mathcal{R}^{k\times k},$$

$$\begin{aligned} \tilde{B}_1 &= \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 & \cdots & \tilde{b}_{j_1} \end{bmatrix} \in \mathcal{R}^{k\times j_1}, \\ \hat{B}_1 &= \begin{bmatrix} \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_{j_2} \end{bmatrix} \in \mathcal{R}^{k\times j_2}, \end{aligned}$$

and the p_{1j}th (p_{2j}th) element of $\tilde{b}_j$ ($\hat{b}_j$) is the first nonzero real number and is given by 1, and $\hat{A}_{12} = 0$, $\tilde{f} = 0$, $\hat{f} = 0$, $\tilde{g} = -\tilde{A}_{21}d + Sa$, $\hat{g} = -\hat{A}_{21}d + S(a+Gf)$. Also $\beta_{i,i+1} > 0$ $(i = 1, 2, \cdots, k-1)$, the other β_{ij} is arbitrary, and the other matrices such as $\tilde{A}_{21}$ are given according to the above coordinate and input transformations.

[2] Class (C-2)

Assume [A1] to [A3]. Define the following four sets for any given s and the pair $(\bar{C}, \bar{A})$:

$$\begin{aligned} &\mathcal{P}_0^s \triangleq \{ (\bar{C},\bar{A}) \mid \bar{C}\bar{A}^{s-1} \in \bar{\mathcal{O}}_{s-1} \}, \\ &\mathcal{P}_1^s \triangleq \{ (\bar{C},\bar{A}) \mid \bar{C}\bar{A}^{s-1} \notin \bar{\mathcal{O}}_{s-1},\ CA^{s-1} \notin \mathcal{O}_{s-1} \}, \\ &\mathcal{P}_{2+}^s \triangleq \{ (\bar{C},\bar{A}) \mid \bar{C}\bar{A}^{s-1} \notin \bar{\mathcal{O}}_{s-1},\ CA^{s-1} \in \mathcal{O}_{s-1}, \\ &\qquad \bar{C}\bar{A}^{s-1}\bar{x} > 0, \forall x \in \mathcal{N}_{s-1} \}, \\ &\mathcal{P}_{2-}^s \triangleq \{ (\bar{C},\bar{A}) \mid \bar{C}\bar{A}^{s-1} \notin \bar{\mathcal{O}}_{s-1},\ CA^{s-1} \in \mathcal{O}_{s-1}, \\ &\qquad \bar{C}\bar{A}^{s-1}\bar{x} < 0, \forall x \in \mathcal{N}_{s-1} \} \end{aligned}$$

where $\bar{\mathcal{O}}_i \triangleq \mathrm{span}\{\bar{C}, \bar{C}\bar{A}, \cdots, \bar{C}\bar{A}^{i-1}\}$ with $\dim \bar{\mathcal{O}}_i = i$, $\mathcal{O}_i \triangleq \mathrm{span}\{C, CA, \cdots, CA^{i-1}\}$ with $\dim \mathcal{O}_i = i$, and $\mathcal{N}_i \triangleq \{ x \in \mathcal{R}^n \mid \bar{C}\bar{x} = \bar{C}\bar{A}\bar{x} = \cdots = \bar{C}\bar{A}^{i-1}\bar{x} = 0, \bar{x} = [x^{\mathrm{T}} \ \ 1]^{\mathrm{T}} \}$ with $\dim \mathcal{N}_i = n - i$.

So let κ be the miminum value of $\{q_l, q_l+1, \cdots, n\}$ satisfying either one of $(\bar{C},\bar{A}) \in \mathcal{P}_0^{\kappa+1}$, $(\bar{C},\bar{A}) \in \mathcal{P}_{2+}^{\kappa+1}$, and $(\bar{C},\bar{A}) \in \mathcal{P}_{2-}^{\kappa+1}$. Note that $p_{i,m_i} \le \kappa$ for all i. Consider the coordinate transformation of (7) with $k = \kappa$, $j_1 = m_1$, and $j_2 = m_2$, and the input transformation of (9), where u_{1,j_1} and u_{2,j_2} are replaced by

$$\begin{cases} \text{mode 1} \quad u_{1,m_1} = \dfrac{1}{\mu^1_{p_{1,m_1}}}\tilde{v}_{1,m_1}, \\ \text{mode 2} \quad u_{2,m_2} = \dfrac{1}{\mu^2_{p_{2,m_2}}}\hat{v}_{1,m_2}. \end{cases} \tag{12}$$

Then from Theorem 3.3 (i), every feedback C-well-posed system in class (C-2) is given by the form of (10) and (11) with $k = \kappa$, where $\hat{A}_{12} = 0$,

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \alpha_1 & \cdots & \cdots & \cdots & \alpha_\kappa \end{bmatrix} \in \mathcal{R}^{\kappa\times\kappa},$$

$$\hat{A}_{11} = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \beta_{\kappa-1,1} & \cdots & \cdots & \beta_{\kappa-1,\kappa-1} & \beta_{\kappa-1,\kappa} \\ \beta_{\kappa,1} & \cdots & \cdots & \cdots & \beta_{\kappa,\kappa} \end{bmatrix} \in \mathcal{R}^{\kappa\times\kappa},$$

$$\tilde{f} = Ta - \tilde{A}_{11}d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tilde{f}_\kappa \end{bmatrix}, \hat{f} = T(a+Gf) - \hat{A}_{11}d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hat{f}_\kappa \end{bmatrix},$$

and $\tilde{B}_2 = \hat{B}_2 = 0$, and $\tilde{B}_1$, $\hat{B}_1$, $\tilde{g}$, and $\hat{g}$ have the same form as those in case (C-1). Moreover, α_i $(i = 1, 2, \cdots, \kappa)$ is arbitrary, $\beta_{i,i+1} > 0$ $(i = 1, 2, \cdots, \kappa-1)$, the other β_{ij} is arbitrary, and

$$\begin{cases} \tilde{f}_\kappa = 0, & \hat{f}_\kappa = 0, & \text{if } (\bar{C},\bar{A}) \in \mathcal{P}_0^{\kappa+1}, \\ \tilde{f}_\kappa > 0, & \hat{f}_\kappa > 0, & \text{if } (\bar{C},\bar{A}) \in \mathcal{P}_{2+}^{\kappa+1}, \quad (\kappa > q_l) \\ \tilde{f}_\kappa < 0, & \hat{f}_\kappa < 0, & \text{if } (\bar{C},\bar{A}) \in \mathcal{P}_{2-}^{\kappa+1}, \end{cases}$$

$$\begin{cases} \tilde{f}_{q_l} = 0, & \hat{f}_{q_l} \ge 0, & \text{if } (\bar{C},\bar{A}) \in \mathcal{P}_0^{q_l+1}, \\ \tilde{f}_{q_l} > 0, & \hat{f}_{q_l} > 0, & \text{if } (\bar{C},\bar{A}) \in \mathcal{P}_{2+}^{q_l+1}, \quad (\kappa = q_l) \\ \tilde{f}_{q_l} < 0, & \hat{f}_{q_l} \le 0, & \text{if } (\bar{C},\bar{A}) \in \mathcal{P}_{2-}^{q_l+1}. \end{cases}$$

The other matrices such as $\tilde{A}_{21}$ are defined according to the above coordinate and input transformations.

[3] Class (H-1)

In addition to [A1] to [A3], we assume $p_{11} \ge p_{21}$ without loss of generality. Consider the coordinate transformation of (7) with $k = p_{11}$, $j_1 = 1$ and j_2 satisfying $p_{2,j_2} \le p_{11} < p_{2,j_2+1}$, and the input transformation

$$\begin{cases} u_{11} = \frac{1}{\mu^1_{p_{11}}}(\tilde{v}_{11} - \bar{C}\bar{A}^{p_{11}}\bar{x}), \\ u_{21} = \frac{1}{\mu^2_{p_{21}}}(\hat{v}_{21} - \bar{C}\bar{A}^{p_{21}}\bar{x} - \bar{C}\bar{A}^{p_{21}-1}\bar{G}\bar{F}\bar{x}), \\ u_{2j} = \frac{1}{\mu^2_{p_{2j}}}\hat{v}_{2j}, \qquad j = 2, 3, \cdots, j_2. \end{cases} \tag{13}$$

Then from Theorem 3.3 (ii), every feedback H-well-posed system in class (H-1) can be transformed into the form of (10) and (11) with $k = p_{11}$, where for some $s \in \{1, 2, \cdots, l\}$ satisfying $q_s < p_{21}$

$$\hat{A}_{11} = \left[\begin{array}{cccccccc} \beta_{11} & \beta_{12} & 0 & \cdots & & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \beta_{q_s,1} & \cdots & \cdots & \beta_{q_s,q_s+1} & 0 & \cdots & & 0 \\ \beta_{q_s+1,1} & \cdots & \cdots & \cdots & & \cdots & \cdots & \beta_{q_s+1,k} \\ \vdots & & & & & & & \vdots \\ \beta_{p_{21}-1,1} & \cdots & \cdots & \cdots & & \cdots & \cdots & \beta_{p_{21}-1,k} \\ 0 & \cdots & \cdots & \cdots & & \cdots & \cdots & 0 \\ \hline * & \cdots & \cdots & \cdots & & \cdots & \cdots & * \\ \vdots & & & & & & & \vdots \\ * & \cdots & \cdots & \cdots & & \cdots & \cdots & * \end{array}\right],$$

$$\hat{A}_{12} = \left[\begin{array}{c} 0_{q_s, n-p_{11}} \\ \hline * \end{array}\right] \in \mathcal{R}^{p_{11}\times(n-p_{11})}$$

and the q_sth element of $\hat{f}$ is the first nonzero number and is denoted by $\hat{f}_{q_s}$. Also $\beta_{i,i+1} > 0$ $(i = 1, 2, \cdots, q_s - 1)$, and ($\beta_{q_s,q_s+1} = 0,\ \hat{f}_{q_s} = 0$) or ($\beta_{q_s,q_s+1} \geq 0,\ \hat{f}_{q_s} < 0$), while the other β_{ij} is arbitrary. The conditions on the other matrices and vectors such as $\tilde{A}_{11}$ and $\tilde{f}$ are the same as those of class (C-1).

[4] Class (H-2)

In addition to [A1] to [A3], we assume $p_{11} \geq p_{21}$ without loss of generality. Consider the coordinate transformation given by (7) with $k = p_{11}$, $j_1 = 1$ and j_2 satisfying $p_{2,j_2} \leq p_{11} < p_{2,j_2+1}$, and the input transformation given by (13). Then from Theorem 3.3 (ii), every feedback H-well-posed system in class (H-2) can be transformed into the form of (10) and (11) with $k = p_{11}$, where

$$\hat{A}_{11} = \left[\begin{array}{cccccc|ccc} \beta_{11} & \beta_{12} & 0 & \cdots & & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & & \vdots & & & \\ \vdots & & \ddots & \ddots & & 0 & & 0 & \\ \beta_{p_{21}-1,1} & \cdots & \cdots & \cdots & & \beta_{p_{21}-1,p_{21}} & & & \\ 0 & \cdots & \cdots & \cdots & & 0 & & & \\ \hline * & \cdots & \cdots & \cdots & \cdots & * & * & \cdots & * \\ \vdots & & & & & \vdots & \vdots & & \vdots \\ * & \cdots & \cdots & \cdots & \cdots & * & * & \cdots & * \end{array}\right],$$

$$\hat{A}_{12} = \left[\begin{array}{c} 0_{p_{21},n-p_{11}} \\ \hline * \end{array}\right] \in \mathcal{R}^{p_{11}\times(n-p_{11})}, \quad \hat{f} = \left[\begin{array}{c} 0_{p_{21}} \\ \hline * \end{array}\right] \in \mathcal{R}^{p_{11}}$$

where $\beta_{i,i+1} > 0$ $(i = 1, 2, \cdots, p_{21} - 1)$ and the other β_{ij} is arbitrary. The conditions on the other matrices and vectors such as $\tilde{A}_{11}$ and $\tilde{f}$ are the same as those of class (C-1).

5 Stabilizability with well-posedness

In section 4, we have classified all feedback well-posed systems into four classes. For each class, we discuss here the stabilizability.

Consider the system of (5) and the control input $u_i = [u_{i1}, u_{i2}, \cdots, u_{i,m_i}]^{\mathrm{T}}$ of the form (4). Throughout this section, we assume that $a = 0$ and $c > 0$ in $\bar{A}$ and $\bar{C}$, respectively. This implies that the origin, i.e., $x = 0$, is in the interior of the set $\{\ x \in \mathcal{R}^n \mid Cx + c \geq 0\ \}$, namely the location invariant of mode 1. Thus for this system, we will consider the stabilization at $x = 0$. Furthermore, throughout this section, we assume [A1] to [A3] in section 3 without loss of generality.

5.1 Class (C-1)

For this class of systems, we have the following result.

Theorem 5.1 *Suppose that the pair $(\tilde{A}_{22}, \tilde{B}_2)$ is stabilizable. Then every feedback well-posed system Σ in class (C-1) of (5) is stabilizable with C-well-posedness.*

5.2 Class (C-2)

For this class of systems, all the admissible feedback control inputs that keep the corresponding closed loop system C-well-posed are given by, from Theorem 3.3(i),

$$\tilde{v}_1 = \tilde{K}\xi + \tilde{k}, \quad \hat{v}_1 = \hat{K}\xi + \hat{k} \tag{14}$$

where the ith row vectors $\tilde{K}_i$ and $\hat{K}_i$ of $\tilde{K}$ and $\hat{K}$ are given by the forms $\tilde{K}_i = [\tilde{K}_{i,1}\ \tilde{K}_{i,2}\ \cdots\ \tilde{K}_{i,p_{1i}+1}\ 0\ \cdots\ 0]$, $\hat{K}_i = [\hat{K}_{i,1}\ \hat{K}_{i,2}\ \cdots\ \hat{K}_{i,p_{2i}+1}\ 0\ \cdots\ 0]$, respectively, and $\tilde{k} = \hat{k} = 0$ if $p_{1,m_1} < \kappa$ and $p_{2,m_2} < \kappa$, $\tilde{k} = [0\ 0\ \cdots\ 0\ \delta]^{\mathrm{T}}$ and $\hat{k} = 0$ if $p_{1,m_1} = \kappa$, and $\tilde{k} = 0$ and $\hat{k} = [0\ 0\ \cdots\ 0\ \delta]^{\mathrm{T}}$ if $p_{2,m_2} = \kappa$ (δ is some constant). Furthermore, to make the closed loop system stable at $x = 0$, it is required that $\tilde{K}_{i,1} = 0$ $(i = 1, 2, \cdots, m_1 - 1)$ and $\tilde{K}_{m_1,1}c + \delta = 0$.

Thus from (14), we see that the stabilization problem in this case in general is reduced to a kind of the stabilization problem via static output feedback. It does not seem easy to solve this kind of problem, although we may apply various approaches to stabilization via static output feedback developed in the previous literature to this case. One way to avoid this difficulty may be to exploit observer-based controllers, as well-known in the continuous linear systems theory. This topic will be explored in a future paper.

5.3 Class (H-1)

Noting that we consider the stabilization problem at the origin in mode 1, we divide the problem into the two cases of $p_{11} \geq p_{21}$ and $p_{11} < p_{21}$.

First, we consider the case $p_{11} \geq p_{21}$. Since the corresponding closed loop system in this case is H-well-posed for *any* feedback controller, we do not need to take well-posedness into explicit consideration. Thus when this claim is described by using the terminology of the Lyapunov's stability theorem, we come to the following result. For convenience of notation, let us write the above system as

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u$$

and the feedback controller of (4) as $u = k(\hat{x})$, where $\hat{x} = [(\xi - d)^{\mathrm{T}}\ \ \eta^{\mathrm{T}}]^{\mathrm{T}}$.

Theorem 5.2 *Suppose that $p_{11} \geq p_{21}$. A feedback well-posed system Σ in class (H-1) of equations (5) is stabilizable with H-well-posedness, if there exists a feedback controller $u = k(\hat{x})$ such that, for a piecewise smooth positive definite function $V(\hat{x})$,*

$$\dot{V} = \frac{\partial V}{\partial \hat{x}}\{f(\hat{x}) + g(\hat{x})k(\hat{x})\} < 0, \quad \forall \hat{x} \in \mathcal{R}^n/\{0\}.$$

Theorem 5.2 implies that the stabilization schemes such as the LMI techniques developed in [11], where the well-posedness of the closed loop system is taken into no consideration, can be applied to this class of systems. In the case of $p_{11} < p_{21}$, the same result as Theorem 5.2 is obtained.

5.4 Class (H-2)

Consider the case $p_{11} \geq p_{21}$. We start with the stabilization problem of the following interesting example.

$$\begin{cases} \text{mode 1:} \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1, \ \text{if } x_1 + c \geq 0, \\ \text{mode 2:} \quad \dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2, \ \text{if } x_1 + c \leq 0. \end{cases}$$

Since $p_{11} = 2$ and $p_{21} = 1(\leq q_1)$, it follows from Theorem 3.2 that this system is feedback H-well-posed, and belongs to class (H-2). So the system can be transformed into the following system, by the coordinate transformation $\xi_1 = x_1 + c$ and $\xi_2 = x_2$.

$$\begin{cases} \text{mode 1:} \quad \dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1, \ \text{if } x_1 + c \geq 0, \\ \text{mode 2:} \quad \dot{\xi} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \xi + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2, \ \text{if } x_1 + c \leq 0. \end{cases}$$

Now this system intuitively seems to be stabilizable if we use $u_1 = \alpha_1 x_1 + \alpha_2 x_2$ and $u_2 = \beta_1 x_1$ as a stabilizing controller, where $\alpha_i < 0$ $(i = 1, 2)$ and $\beta_1 < 0$. However, from Theorem 3.3(ii), it is verified that the resulting closed loop system is not H-well-posed (consider the solution at $x(0) = (c, -1)$). In this way, constructing a stabilizing controller intuitively and taking well-posedness into no consideration do not necessarily yield a well-posed system. This system is in fact not stabilizable by any state feedback of the form (4), as shown below.

By taking account of both stability and well-posedness of the closed loop system, every admissible input is given by

$$\begin{cases} u_1 &= \alpha_1 \xi_1 + \alpha_2 \xi_2 - \alpha_1 c, \\ u_2 &= \beta_1 \xi_1 + \beta_2 \xi_2 + \gamma_2 \end{cases}$$

where $\alpha_1 < 0$, $\alpha_2 < 0$, $\beta_2 \geq 0$ and $\gamma_2 \leq 0$. So let us see again the resulting closed loop system in the original x-coordinates:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix} x, & \text{if } x_1 + c \geq 0, \\ \dot{x} = \begin{bmatrix} \beta_1 & \beta_2 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} \beta_1 c + \gamma_2 \\ 0 \end{bmatrix}, & \text{if } x_1 + c \leq 0. \end{cases}$$

When $\beta_1 < 0$, the equilibrium x_{1e} of x_1 in mode 2 is given by $x_{1e} = -c - \frac{\gamma_2}{\beta_1} (\leq -c)$, which implies that the mode of the system may not be transformed into mode 1 from some $x(0)$. When $\beta_1 \geq 0$, on the other hand, $\dot{x}_1(0) = \beta_1(x_1(0) + c) + \gamma_2 \leq 0$ for $(x_1(0), x_2(0))$ satisfying $x_1(0) + c \leq 0$ and $x_2(0) = 0$, which also leads to a similar situation. Therefore, we obtain the above claim.

As in the above example, some of systems in class (H-2) with $p_{11} \geq p_{21}$ are not stabilizable even when the system in mode 1 is stabilizable and the system in mode 2 is controllable with respect to ξ_1. Thus it seems difficult to characterize the stabilizability in this case. However, the following class of systems in (H-2) can be reduced to that in class (H-1).

Theorem 5.3 *Suppose that $p_{11} \geq p_{21}$ and $j_2 \geq 2$. If $\hat{v}_{11}$ is given by $\hat{v}_{11} = \sum_{i=1}^{p_{21}+1} \beta_{p_{21},i} + \gamma$, where $\beta_{p_{21},i}$ $(i = 1, 2, \cdots, p_{21})$ is arbitrary, and $(\beta_{p_{21},p_{21}+1} = 0, \gamma = 0)$ or $(\beta_{p_{21},p_{21}+1} \geq 0, \gamma < 0)$, then the resulting system with the control inputs $(\tilde{u}_1, \tilde{u}_2)$ in mode 1 and $([\hat{u}_{12} \ \hat{u}_{13} \ \cdots \ \hat{u}_{1,j_2}], \hat{u}_2)$ in mode 2 is reduced to that of class (H-1).*

For the case $p_{11} < p_{21}$, we can show that if some stabilizabilty condition similar to that in Theorem 5.1 is satisfied, every feedback well-posed system in class (H-2) of (5) is semi-globally stabilizable with H-well-posedness. The details will be presented somewhere.

6 Conclusion

We have derived a series of results from feedback well-posedness to stabilizability for general bimodal piecewise affine systems. There are many open problem we should address. For systems in class (C-1), we will have to discuss the stabilizability in the case that the pair $(\tilde{A}_{22}, \tilde{B}_2)$ is not stabilizable. For classes (C-2) and (H-1), deeper discussion will be required, based on several interesting remarks we presented.

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