An LMI Approach for Designing Sliding Mode Observers

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Abstract

This paper presents a method to design sliding mode observers for a class of uncertain systems using Linear Matrix Inequalities. The objective is to exploit the degrees of freedom available in the design which have hitherto been ignored because of the lack of a tractable solution framework. The relationship between the linear component of the sliding mode observer and a particular sub-optimal observer arising from classical Linear Quadratic Gaussian theory is demonstrated. This helps motivate how the design weighting matrices inherent in the method may be chosen in practice.

1 Introduction

Sliding mode observers differ from linear Luenberger observers in that there is a non-linear discontinuous term injected into the observer depending on the output estimation error. These observers are more robust than Luenberger observers, as the discontinuous term enables the observer to reject disturbances, and also a class of mismatch between the system and observer. The discontinuous term is designed to drive the trajectories of the observer so that the state estimation error vector is forced onto and subsequently remains on a surface in the error space. This motion is referred to as the sliding mode [10]. In most cases, the sliding surface is set to be the difference between the observer and system output which is therefore forced to be zero. When a sliding mode is achieved the system will experience a reduced-order motion which is insensitive to a class of system/plant mismatch. Utkin [10] designed a simple observer, with only the discontinuous term being fed back through an appropriate gain. Walcott and Zak [11] designed an observer which also has the output error being fed back linearly and used a Lyapunov approach to prove stability. The method in [11] invariably requires a symbolic manipulation package to solve the synthesis problem which is formulated. Edwards and Spurgeon [3, 5] proposed a canonical form for sliding mode observer design subject to certain conditions relating to the input and output distribution matrices, and also the invariant zeros of the system. Their method described in [3, 5] utilised both linear and discontinuous output error injection. A method for computing the gain associated with the linear output error injection term is presented. The solution is explicit, but does not exploit all the degrees of freedom.

This paper builds on the work of Edwards and Spurgeon [3, 5] and presents a method that will utilise Linear Matrix Inequalities (LMIs) [1] and seeks to exploit this freedom. The relationship between the linear component of the sliding mode observer and a particular sub-optimal observer arising from classical Linear Quadratic Gaussian theory is demonstrated. This helps motivate how the design weighting matrices inherent in the method may be chosen in practice.

The notation used throughout the paper is standard; in particular $\|.\|$ will represent the Euclidean norm for vectors and the induced spectral norm for matrices.

2 Preliminaries

Consider the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + D\xi(t,x,u) \quad (1)$$
$$y(t) = Cx(t) \quad (2)$$

where $A \in \mathcal{R}^{n\times n}, B \in \mathcal{R}^{n\times m}, C \in \mathcal{R}^{p\times n}$ and $D \in \mathcal{R}^{n\times q}$ where $p \geq q$. Assume that the matrices C and D are full rank and the function $\xi : \mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^q$ is unknown but bounded so that

$$\|\xi(t,x,u)\| \leq r_1\|u\| + \alpha(t,y) \quad (3)$$

where r_1 is a known scalar and $\alpha : \mathcal{R}_+ \times \mathcal{R}^p \to \mathcal{R}_+$ is a known function.

Consider an observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - G_l e_y(t) + G_n\nu \quad (4)$$
$$\hat{y}(t) = C\hat{x}(t) \quad (5)$$

where $G_l \in \mathcal{R}^{n\times p}$ and $G_n \in \mathcal{R}^{n\times p}$. The discontinuous vector ν is defined by

$$\nu = \begin{cases} -\rho(t,y,u)\|\mathcal{D}_2\|\frac{P_2 e_y}{\|P_2 e_y\|} & \text{if } e_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

where $e_y = \hat{y} - y$ and $P_2 \in \mathcal{R}^{p\times p}$ is symmetric positive definite. The matrices P_2 and $\mathcal{D}_2$ will be formally defined in §3. The scalar function $\rho : \mathcal{R}_+ \times \mathcal{R}^p \times \mathcal{R}^m \to$

$\mathcal{R}_+$ satisfies

$$\rho(t, y, u) \geq r_1\|u\| + \alpha(t, y) + \gamma_0 \tag{7}$$

where γ_0 is a positive scalar. If the state estimation error $e := \hat{x} - x$, then it is straightforward to show from equations (1) and (4), (2) and (5) that

$$\dot{e}(t) = A_0 e(t) + G_n \nu - D\xi(t, x, u) \tag{8}$$

where $A_0 = A - G_l C$. In §6.3 in [5], Edwards *et al.* show that necessary and sufficient conditions for existence of a stable sliding motion on $S = \{e \in \mathcal{R}^n : e_y = 0\}$ that is independent of ξ are

1. rank (CD)=q
2. invariant zeros of (A, D, C) lie in the open LHP

Edwards *et al.*[5] suggest an explicit method for computing the gains G_l and G_n. However, the method described in [5] does not exploit all degrees of freedom available. The next section considers the design of the matrices G_l, G_n and P_2 so that a sliding motion takes place on S. This paper assumes that conditions 1 and 2 are satisfied and sets up a new design framework which seeks to exploit all of the available design freedom.

Edwards *et al.*prove in Lemma 6.1 of [5] that if (CD) is full rank, then there exists a linear change of coordinates T_0 so that the system can be written as :-

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{D}\xi(t, x, u) \tag{9}$$

$$y(t) = \bar{C}\bar{x}(t) \tag{10}$$

where $(\bar{A}, \bar{D}, \bar{C})$ has the following structure -

- The system matrix can be written as

$$\bar{A} = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{12} \\ \hline \bar{A}_{211} & \\ \bar{A}_{212} & \bar{A}_{22} \end{array}\right] \tag{11}$$

  where $\bar{A}_{11} \in \mathcal{R}^{(n-p)\times(n-p)}, \bar{A}_{211} \in \mathcal{R}^{(p-q)\times(n-p)}$ and when partitioned have the structure

$$\bar{A}_{11} = \begin{bmatrix} \bar{A}_{11}^o & \bar{A}_{12}^o \\ 0 & \bar{A}_{22}^o \end{bmatrix} \quad \text{and} \quad \bar{A}_{211} = \begin{bmatrix} 0 & \bar{A}_{21}^o \end{bmatrix} \tag{12}$$

  where $\bar{A}_{11}^o \in \mathcal{R}^{r\times r}$ and $\bar{A}_{21}^o \in \mathcal{R}^{(p-q)\times(n-p-r)}$ for some $r \geq 0$ and the pair $(\bar{A}_{22}^o, \bar{A}_{21}^o)$ is completely observable. Furthermore, the eigenvalues of $\bar{A}_{11}^o$ are the invariant zeros of (A, D, C).

- The disturbance distribution matrix has the form

$$\bar{D} = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \tag{13}$$

  where $D_2 \in \mathcal{R}^{p\times q}$ has the structure

$$D_2 = \begin{bmatrix} 0 \\ \bar{D}_2 \end{bmatrix} \tag{14}$$

  and $\bar{D}_2 \in \mathcal{R}^{q\times q}$ is nonsingular.

- The output distribution matrix has the form

$$\bar{C} = \begin{bmatrix} 0 & T \end{bmatrix} \tag{15}$$

  where $T \in \mathcal{R}^{p\times p}$ and is orthogonal.

3 A Design Framework

Applying the linear change of co-ordinates T_0 to the observer system (4) and (5),

$$\dot{\hat{\bar{x}}}(t) = \bar{A}\hat{\bar{x}}(t) + \bar{B}u(t) - \bar{G}_l e_y(t) + \bar{G}_n \nu \tag{16}$$

$$\hat{y}(t) = \bar{C}\hat{\bar{x}}(t) \tag{17}$$

and define $\bar{A}_0 = \bar{A} - \bar{G}_l\bar{C}$. The gain matrix $\bar{G}_l$ is to be determined but assume

$$\bar{G}_n = \begin{bmatrix} -\bar{L}T^T \\ T^T \end{bmatrix} \tag{18}$$

where $\bar{L} \in \mathcal{R}^{(n-p)\times p}$ and $\bar{L} = \begin{bmatrix} L & 0 \end{bmatrix}$ with $L \in \mathcal{R}^{(n-p)\times(p-q)}$. The orthogonal matrix T is part of the output distribution matrix $\bar{C}$ from (15).

Proposition 1 - *If there exists a positive definite Lyapunov matrix $\bar{P}$, that satisfies $\bar{P}\bar{A}_0 + \bar{A}_0^T\bar{P} < 0$, with the structure*

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_1\bar{L} \\ \bar{L}^T\bar{P}_1 & \bar{P}_2 + \bar{L}^T\bar{P}_1\bar{L} \end{bmatrix} > 0 \tag{19}$$

where $\bar{P}_1 \in \mathcal{R}^{(n-p)\times(n-p)}$ and $\bar{P}_2 \in \mathcal{R}^{p\times p}$, then the error system in equation (8) is quadratically stable.

Proof : Consider the quadratic form given by

$$V(\bar{e}) = \bar{e}^T\bar{P}\bar{e} \tag{20}$$

as a candidate Lyapunov function where $\bar{e} := T_o e$. Notice that if $\bar{P}_1, \bar{P}_2 > 0$ then $\bar{P} > 0$ from the Schur expansion. From (8) the derivative along the system trajectory :-

$$\dot{V} = \bar{e}^T(\bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0)\bar{e} + 2\bar{e}^T\bar{P}\bar{G}_n\nu - 2\bar{e}^T\bar{P}\bar{D}\xi \tag{21}$$

From the definitions in (15), (18) and (19):

$$\bar{P}\bar{G}_n = \begin{bmatrix} 0 \\ \bar{P}_2 T^T \end{bmatrix} = \bar{C}^T P_2 \tag{22}$$

where $P_2 := T\bar{P}_2 T^T$.

Using the special structures of $\bar{L}$ and D_2, $\bar{L}D_2 = 0$ and

$$\bar{P}\bar{D} = \begin{bmatrix} 0 \\ \bar{P}_2 D_2 \end{bmatrix} = \bar{C}^T P_2 \mathcal{D}_2 \tag{23}$$

where $\mathcal{D}_2 := TD_2$. Note this implies $\|\mathcal{D}_2\| = \|D_2\|$.

Consequently, (21) becomes :-

$$\begin{aligned}\dot{V} &= \bar{e}^T(\bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0)\bar{e} + 2{e_y}^T P_2\nu - 2{e_y}^T P_2\mathcal{D}_2\xi \\ &\le \bar{e}^T(\bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0)\bar{e} - 2\rho\|\mathcal{D}_2\|\|P_2e_y\| - 2{e_y}^T P_2\mathcal{D}_2\xi\end{aligned}$$

Using the bounds for ξ in equations (3) and (7)

$$\begin{aligned}\dot{V} &\le \bar{e}^T(\bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0)\bar{e} + 2\|\mathcal{D}_2\|\,[r_1\|u\| + \alpha(y)]\,\|P_2e_y\| \\ &\qquad -2\rho\|\mathcal{D}_2\|\|P_2e_y\| \\ &\le \bar{e}^T(\bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0)\bar{e} - 2\gamma_0\|\mathcal{D}_2\|\|P_2e_y\|\end{aligned}$$

Since $(\bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0) < 0$ it follows that $\dot{V} < 0$ for all $\bar{e} \neq 0$. ■

Corollary 1 - An ideal sliding motion takes place on S in finite time. Furthermore the sliding dynamics are given by the system matrix $\bar{A}_{11} + L\bar{A}_{211}$.

Proof : Using Proposition 1, a modification to Corollary 6.1 in [4] shows that sliding takes place on S in finite time. Using the concept of equivalent output error injection, the sliding motion is governed by

$$(I - \bar{G}_n(\bar{C}\bar{G}_n)^{-1}\bar{C})\bar{A}_0 = \begin{bmatrix} \bar{A}_{11} + L\bar{A}_{211} & \bar{A}_{12} + \bar{L}\bar{A}_{22} \\ 0 & 0 \end{bmatrix}$$

Hence the reduced order sliding motion is governed by $\bar{A}_{11} + L\bar{A}_{211}$ as claimed. ■

Remarks: If a further linear change of co-ordinates

$$T_L = \begin{bmatrix} I_{n-p} & \bar{L} \\ 0 & T \end{bmatrix} \tag{24}$$

is applied to the triple $(\bar{A}, \bar{D}, \bar{C})$ and its Lyapunov matrix $\bar{P}$, the system matrix, disturbance distribution matrix and the output distribution matrix will be in the form

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \qquad \mathcal{D} = \begin{bmatrix} 0 \\ \mathcal{D}_2 \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{25}$$

where $\mathcal{A}_{11} = \bar{A}_{11} + L\bar{A}_{211}$ and $\mathcal{D}_2 = TD_2$. In the new co-ordinate system, the Lyapunov matrix will be $\mathcal{P} = ({T_L}^{-1})^T\bar{P}({T_L}^{-1})$ and hence

$$\mathcal{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & T\bar{P}_2T^T \end{bmatrix} \tag{26}$$

As argued in [4], the fact that $\mathcal{P}$ is a block diagonal Lyapunov matrix for $\mathcal{A}_0 = \mathcal{A} - \mathcal{G}_l\mathcal{C}$ implies that $\mathcal{A}_{11}$ is stable and hence the sliding motion is stable. Furthermore

$$T_L\bar{G}_n = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

and the canonical observer form from [5] is recovered.

Finally note that equation (23) is (effectively) the 'structural constraint' employed by Walcott & Zak [11].

The remainder of this paper focuses on *design methods* to synthesise the gain $\bar{G}_l$ and the Lyapunov matrix $\bar{P}$ for $\bar{A}_0$ which has the structure given in (19). The problems will be posed in such a way that Linear Matrix Inequalities (LMIs) can be used to numerically synthesise the required matrices.

4 Synthesis Procedure for Matrices $\bar{P}$ and $\bar{G}_l$

In this section, $\bar{P}$ and $\bar{G}_l$ will be chosen so that the matrix inequality

$$\bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0 < -\bar{P}W\bar{P} - \bar{P}\bar{G}_lV\bar{G}_l^T\bar{P} \tag{27}$$

is satisfied, where the design weighting matrices W and V are assumed to be symmetric positive definite, and $\bar{P}$ has the structure in (19). The rationale for the matrix inequality (27) will be given in the sequel. Substituting for $\bar{A}_0$, the inequality (27) can be written as :-

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} - (\bar{Y}\bar{C})^T - \bar{Y}\bar{C} + \bar{P}W\bar{P} + \bar{Y}V\bar{Y}^T < 0 \tag{28}$$

where $\bar{Y} := \bar{P}\bar{G}_l$. Using standard matrix manipulations, inequality (28) is identical to

$$\begin{aligned}\bar{P}\bar{A} + \bar{A}^T\bar{P} + (\bar{Y}^T - V^{-1}\bar{C})^TV(\bar{Y}^T - V^{-1}\bar{C}) \\ -\bar{C}^TV^{-1}\bar{C} + \bar{P}W\bar{P} < 0\end{aligned} \tag{29}$$

Using inequality (29), the necessary and sufficient condition for (28) to hold is that $\bar{P}$ satisfies

$$\bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{C}^TV^{-1}\bar{C} + \bar{P}W\bar{P} < 0 \tag{30}$$

since choosing

$$\bar{Y}^T = V^{-1}\bar{C} \tag{31}$$

eliminates the third term in (29). The problem considered here is one of minimising $trace(\bar{P}^{-1})$ subject to $\bar{P}$ satisfying inequality (30). The observer gain $\bar{G}_l$ can then be directly calculated as $\bar{G}_l = \bar{P}^{-1}\bar{C}^TV^{-1}$ which follows from equation (31) and the definition of $\bar{Y}$. The matrix inequality in (30) is equivalent to

$$\begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{C}^TV^{-1}\bar{C} & \bar{P} \\ \bar{P} & -W^{-1} \end{bmatrix} < 0 \tag{32}$$

by using the Schur complement. If $\bar{X} \in \mathcal{R}^{n\times n}$ is symmetric positive definite then (again using the Schur complement) the LMI

$$\begin{bmatrix} -\bar{P} & I \\ I & -\bar{X} \end{bmatrix} < 0 \tag{33}$$

is equivalent to $\bar{X} > \bar{P}^{-1}$. Thus minimising $trace(\bar{P}^{-1})$ subject to (30) is equivalent to minimising $trace(\bar{X})$ subject to the LMIs (32) and (33). Writing $\bar{P}$ from (19) as

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{34}$$

where $P_{11} \in \mathcal{R}^{(n-p)\times(n-p)}, P_{22} \in \mathcal{R}^{p\times p}$ and $P_{12} := \begin{bmatrix} P_{121} & 0 \end{bmatrix}$ with $P_{121} \in \mathcal{R}^{(n-p)\times(p-q)}$, it follows there is a one-to-one correspondence between the variables $(P_{11}, P_{121}, P_{22})$ and $(\bar{P}_1, \bar{L}, \bar{P}_2)$ since

$$P_{11} = \bar{P}_1 \quad (35)$$

$$\bar{L} = P_{11}^{-1} P_{12} \quad (36)$$

$$\bar{P}_2 = P_{22} - P_{12}^T P_{11}^{-1} P_{12} \quad (37)$$

It follows that the constrained minimisation problem is convex with regard to P_{11}, P_{121}, P_{22} and $\bar{X}$. Standard LMI software, such as [6], can be employed to synthesise numerically $\bar{P}$ and $\bar{X}$.

4.1 Connection with the ARE

The motivation for the choice of the inequality posed in (27), and the minimisation of $trace(\bar{P}^{-1})$ subject to (30) and (33), is that in the absence of uncertainty and as $\gamma_0 \to 0$ the observer tends to a sub-optimal Linear Quadratic Gaussian formulation. This will be demonstrated using an argument similar to that on page 115 of [1]. Defining $\bar{Q} = \bar{P}^{-1}$, then pre and post multiplying inequality (30) by $\bar{Q}$, the following inequality can be obtained :

$$\bar{A}\bar{Q} + \bar{Q}\bar{A}^T - \bar{Q}\bar{C}^T V^{-1} \bar{C}\bar{Q} + W < 0 \quad (38)$$

and $\bar{G}_l = \bar{Q}\bar{C}^T V^{-1}$. The objective is thus to minimise $trace(\bar{Q})$ subject to (38).

The Linear Quadratic Gaussian [9] optimal observer design method uses the stabilising solution $\bar{Q}_{are}$ to the Algebraic Ricatti Equation (ARE)

$$\bar{A}\bar{Q}_{are} + \bar{Q}_{are}\bar{A}^T - \bar{Q}_{are}\bar{C}^T V^{-1} \bar{C}\bar{Q}_{are} + W = 0 \quad (39)$$

to calculate the optimal observer gain as

$$\bar{G}_{lare} := \bar{Q}_{are}\bar{C}^T V^{-1}$$

The associated optimal cost is given by $trace(\bar{Q}_{are})$.

It is argued in the Appendix that if $\bar{Q} > 0$ is any matrix satisfying (38) then $\bar{Q} > \bar{Q}_{are}$. Hence the requirement of minimising $trace(\bar{Q})$ follows from the desire to approach the true minimal cost given by $trace(\bar{Q}_{are})$. Of course a particular sub-optimal cost is enforced by the requirement that $\bar{P} := \bar{Q}^{-1}$ is in the structure of (19).

In inequality (38), W is the performance weighting matrix for the observer, and V is the co-variance matrix of the system's sensor noise. As in classical Linear Quadratic Gaussian theory the choice of W and V trades-off performance and noise amplification.

5 Design of the Sliding Motion System Matrix

A consequence of the design procedure proposed in §4 is that the dynamics of the sliding motion, although guaranteed to be stable, are designed somewhat implicitly. This section considers the sliding motion design problem and shows how additional LMI constraints can be augmented with those in §4 to tune the sliding mode performance. Using the new co-ordinates obtained after applying the transformation T_L in equation (24), the matrix inequality (30) becomes

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} - \mathcal{C}^T V^{-1}\mathcal{C} + \mathcal{P}\mathcal{W}\mathcal{P} < 0 \quad (40)$$

where $\mathcal{W} := T_L W {T_L}^T$. The top left block of (40) is

$$P_1\mathcal{A}_{11} + \mathcal{A}_{11}^T P_1 + P_1\mathcal{W}_1 P_1 < 0 \quad (41)$$

where $\mathcal{W}_1 \in \mathcal{R}^{(n-p)\times(n-p)} > 0$ is the top left sub-block of $\mathcal{W}$ and $\mathcal{A}_{11} = \bar{A}_{11} + L\bar{A}_{211}$. If the weighting matrix W is partitioned as

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix} \quad (42)$$

where $W_{11} \in \mathcal{R}^{(n-p)\times(n-p)}$ and W_{12} is the null space of $\bar{L}^T$ for all L, then (41) is

$$P_1\mathcal{A}_{11} + \mathcal{A}_{11}^T P_1 + P_1 W_{11} P_1 + P_1\bar{L}^T W_{22}\bar{L}P_1 < 0 \quad (43)$$

This is identical in structure to inequality (27) and hence W_{11} and W_{22} may be interpreted as playing the roles of performance and noise attenuation matrices in a Linear Quadratic Gaussian sense for the observer problem associated with the pair $(\bar{A}_{11}, \bar{A}_{211})$. Thus the choice of W_{11} and W_{22} can be used to tune the sliding motion. However since

$$P_1\mathcal{A}_{11} = P_{11}\bar{A}_{11} + P_{121}\bar{A}_{211} \quad (44)$$

and this is linear with respect to the LMI optimisation variables P_{11} and P_{121}, additional LMIs can be employed together with (32) and (33) to tune the sliding mode performance. One approach is to use Root Clustering [7] methods to achieve pole placement of $\mathcal{A}_{11}$ in regions of the complex plane. Typically the poles may be required to lie in

- conic sector centred at (0,0) with inner angle θ_a
- disc of radius r_a and centre $(q_a, 0)$
- vertical strip $a_a < x < b_a$

Chilali and Gahinet [2] have proven that the following inequalities will describe these regions

$$\begin{bmatrix} (\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1)s & -(\bar{P}_1\mathcal{A}_{11} - \mathcal{A}_{11}^T\bar{P}_1)c \\ (\bar{P}_1\mathcal{A}_{11} - \mathcal{A}_{11}^T\bar{P}_1)c & (\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1)s \end{bmatrix} < 0 \quad (45)$$

$$\begin{bmatrix} -r_a\bar{P}_1 & \bar{P}_1\mathcal{A}_{11} - q_a\bar{P}_1 \\ \mathcal{A}_{11}^T\bar{P}_1 - q_a\bar{P}_1 & -r_a\bar{P}_1 \end{bmatrix} < 0 \quad (46)$$

$$\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1 - 2b_a\bar{P}_1 < 0 \quad (47)$$

$$-(\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1) + 2a_a\bar{P}_1 < 0 \quad (48)$$

where $s = \sin\frac{1}{2}\theta_a$ and $c = \cos\frac{1}{2}\theta_a$. To obtain a convex optimization problem, write $\bar{P}_1\mathcal{A}_{11} = P_{11}\bar{A}_{11} + P_{121}\bar{A}_{211}$ and substitute into (45) - (48). This results in a well-posed convex problem as the inequalities (45) - (48) are affine in P_{11} and P_{121}. Thus the new optimization problem can be stated as : Minimize $trace(\bar{X})$ subject to the LMIs (32), (33) and (45) - (48).

6 An Example

The new design method proposed in this paper will now be demonstrated by an example. This example is a 7th order aircraft model taken from [8]. The states are :-

$$x = \begin{bmatrix} \phi \\ r \\ p \\ \delta \\ x_7 \\ \delta_r \\ \delta_a \end{bmatrix} \begin{array}{l} \text{bank angle } (rad) \\ \text{yaw rate } (rad/s) \\ \text{roll rate } (rad/s) \\ \text{sideslip angle } (rad) \\ \text{washout filter state} \\ \text{rudder deflection } (rad) \\ \text{aileron deflection } (rad) \end{array}$$

the inputs

$$u = \begin{bmatrix} \delta_{rc} \\ \delta_{ac} \end{bmatrix} \begin{array}{l} \text{rudder command}(rad) \\ \text{aileron command}(rad) \end{array}$$

and the outputs

$$y = \begin{bmatrix} r_a \\ p_a \\ \phi \\ x_7 \end{bmatrix} \begin{array}{l} \text{roll acceleration } (rads/s^2) \\ \text{yaw acceleration}(rads/s^2) \\ \text{bank angle } (rad) \\ \text{washout filter state} \end{array}$$

The system matrices are given in (49)-(50).

6.1 Regions and weighting matrices

The eigenvalues of the sliding motion represented by the system matrix $\mathcal{A}_{11}$ were required to lie in the intersection of the following regions -

- a circle of centre (0,0) and radius 5
- a vertical upper bound at $x = -2$
- a conic sector symmetric about the real axis, with inner angle $\theta = 80°$

The weighting matrices W and V associated with the Ricatti-like inequality (30) were chosen respectively as $W = 0.01I_7$ and $V = 0.25I_4$.

6.2 Synthesis results

The following matrices were obtained for the canonical form described in §2 -

$$\bar{A} = \begin{bmatrix} -2.0722 & 5.0994 & 1.6893 & * & * & * & * \\ 0.0000 & -0.0000 & -0.0000 & * & * & * & * \\ 0.0000 & 0.0000 & 0.0000 & * & * & * & * \\ -0.0000 & 0.4962 & 0.0226 & * & * & * & * \\ 0.0000 & 0.0122 & -0.9159 & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{bmatrix}$$

$$\bar{D}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.0000 & 16.3353 \\ 0 & 0 & 0 & 0 & 0 & 25.8356 & -10.8242 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -0.4126 & -0.9109 \\ 0 & 0 & 0 & 0 & 0 & -0.9109 & 0.4126 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

From this representation

$$\bar{A}_{11} = \begin{bmatrix} A^o_{11} & A^o_{12} \\ 0 & A^o_{22} \end{bmatrix} = \left[\begin{array}{c|cc} -2.0722 & 5.0994 & 1.6893 \\ \hline 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{array}\right]$$

and

$$\bar{A}_{211} = \begin{bmatrix} 0 & A^o_{21} \end{bmatrix} = \left[\begin{array}{c|cc} 0.0000 & 0.4962 & 0.0226 \\ 0.0000 & 0.0122 & -0.9159 \end{array}\right]$$

Notice the system has an invariant zero at -2.0722.

Following the synthesis procedure in §4 and the imposing of the constraints in §6.1, the following matrices were obtained -

$$L = \begin{bmatrix} -9.5598 & 1.5465 \\ -4.0743 & -0.0894 \\ 0.5161 & 2.1753 \end{bmatrix}$$

$$\bar{P}_1 = \begin{bmatrix} 7.2296 & -15.6675 & -5.7861 \\ -15.6675 & 49.1586 & 13.6032 \\ -5.7861 & 13.6032 & 5.1419 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1.000 & 0 & 0 & 0 & 0 \\ 0 & -0.154 & -0.004 & 1.540 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1.000 & -5.200 & 0 & 0.337 & -1.120 \\ 0.0386 & -0.996 & -0.000 & -2.117 & 0 & 0.020 & 0 \\ 0 & 0.500 & 0 & 0 & -4.000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20.000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25.000 \end{bmatrix} \quad B^T = D^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 25 \end{bmatrix} \tag{49}$$

$$C = \begin{bmatrix} 0 & -0.154 & -0.004 & 1.540 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1.000 & -5.200 & 0 & 0.337 & -1.120 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{50}$$

$$\bar{P}_2 = \begin{bmatrix} 476.7394 & -1.5752 & -5.9434 & -12.3227 \\ -1.5752 & 1.6130 & -0.5957 & -0.7175 \\ -5.9434 & -0.5957 & 1.4526 & 2.0924 \\ -12.3227 & -0.7175 & 2.0924 & 3.6798 \end{bmatrix}$$

Hence the eigenvalues of $\mathcal{A}_0$ are $\{-44.7043, -21.1353, -1.4057 \pm 0.8888i, -4.3773, -1.4793, -4.0039\}$ and the eigenvalues of $\mathcal{A}_{11}$ which govern the sliding motion are $\{-2.0722, -2.0016 \pm 0.0490i\}$

7 Conclusion

This paper demonstrates how Linear Matrix Inequalities can be used to synthesize the gains of a sliding mode observer. A formulation has been presented in a way that the linear component of the observer resembles a sub-optimal version of the classical Linear Quadratic Gaussian observer and two design weighting matrices allow a trade-off between performance and sensor noise. Additional LMIs can be augmented to facilitate the design of the reduced order sliding motion whilst still retaining a convex optimization problem.

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8 Appendix

In §4 is was argued that the design problem should be one of minimising $trace(\bar{Q})$ subject to the inequality given in (38). The motivation for this comes from the following:

Lemma 1: Let $\bar{Q}_{lmi}$ be any symmetric positive definite matrix satisfying

$$\bar{A}\bar{Q}_{lmi} + \bar{Q}_{lmi}\bar{A}^T - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C}\bar{Q}_{lmi} + W < 0 \quad (51)$$

and let $\bar{Q}_{are}$ be the stabilising solution to the ARE (39). Then $\bar{Q}_{lmi} > \bar{Q}_{are}$ and $trace(\bar{Q}_{lmi}) > trace(\bar{Q}_{are})$.

Proof
Inequality (51) can be expressed as:-

$$\bar{A}\bar{Q}_{lmi} + \bar{Q}_{lmi}\bar{A}^T - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C}\bar{Q}_{lmi} + W + \triangle = 0 \quad (52)$$

for some symmetric positive definite matrix $\triangle$. Subtracting (39) from (52) and defining $\tilde{Q} = \bar{Q}_{lmi} - \bar{Q}_{are}$ implies

$$\bar{A}\tilde{Q} + \tilde{Q}\bar{A}^T - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C}\bar{Q}_{lmi} + \bar{Q}_{are}\bar{C}^T V^{-1}\bar{C}\bar{Q}_{are} + \triangle = 0 \quad (53)$$

Substituting $\bar{Q}_{are} = \bar{Q}_{lmi} - \tilde{Q}$ into (53) yields

$$(\bar{A} - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C})\tilde{Q} + \tilde{Q}(\bar{A} - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C})^T + \triangle + \tilde{Q}\bar{C}^T V^{-1}\bar{C}\tilde{Q} = 0 \quad (54)$$

In equality (51) can be written as

$$(\bar{A} - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C})\bar{Q}_{lmi} + \bar{Q}_{lmi}(\bar{A} - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C})^T + \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C}\bar{Q}_{lmi} + W < 0$$

and hence, since $\bar{Q}_{lmi} > 0$, it follows that the matrix $(\bar{A} - \bar{Q}_{lmi}\bar{C}^T V^{-1}\bar{C})$ is stable. Therefore, as argued in Lemma 3 in [13], equation (54) implies $\tilde{Q} > 0$ and hence $\bar{Q}_{lmi} > \bar{Q}_{are}$ as claimed. The fact that $trace(\bar{Q}_{lmi}) > trace(\bar{Q}_{are})$ follows from the properties of the *trace* operator. ■