A Strict LMI Condition for H_2 Control of Descriptor Systems

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Abstract

This paper presents a new LMI condition for H_2 control of linear time-invariant descriptor systems. The condition is expressed in terms of definite LMIs with no equality constraint, which is much more tractable in numerical computation than existing conditions for descriptor systems, that is, definite LMIs with equality constraints or semidefinite LMIs. Using the results of this paper, we can analyze and design descriptor systems in the almost same way as in the case of state-space representations.

1. Introduction

In this paper, we propose a strict LMI (Linear Matrix Inequality) approach for H_2 control of linear time-invariant descriptor systems. The term "strict LMI" means a definite LMI with no equality constraint. While strict LMI approaches are popular for state-space representations [1], LMIs with equality constraints have been used extensively for descriptor systems. For example, there have been reported such conditions for stability [2], robust stabilization [3], H_2 control [4], and H_∞ control [5, 6].

LMI conditions containing equality constraints are theoretically fine, but may cause a trouble in checking the conditions numerically. Because of round-off errors in digital computation, the equality constraints are in usual not satisfied perfectly. Then, it is difficult to judge whether the constraint is really unsatisfied or satisfied but looks unsatisfied by computatioal errors.

For this reason, the authors have proposed strict LMI conditions for stability, robust stabilization, and H_∞ control of descriptor systems [7, 8]. Strict LMIs are tractable and reliable when we use recent popular softwares [9] for solving matrix inequalities. In the present paper, we consider H_2 control in the same context.

2. H_2 Norm Condition

Let us consider a linear time-invariant descriptor system

$$E\dot{x} = Ax + B_1 w, \qquad z = C_1 x \tag{1}$$

where $x \in R^n$ is the descriptor variable, $w \in R^q$ is the input, $z \in R^p$ is the output, and $E, A \in R^{n\times n}$, $B_1 \in R^{n\times q}$, $C_1 \in R^{p\times n}$ are constant matrices. The matrix E may be singular and we denote its rank by $r = \operatorname{rank} E \le n$.

The system (1) has a unique solution for any initial condition and any continuous input function if $\det(sE - A) \not\equiv 0$. In this case, (1) is said to be regular. The finite eigenvalues of the matrix pair (E, A), that is, the solutions of $\det(sE - A) = 0$, and corresponding (generalized) eigenvectors define exponential modes of (1). If all the finite eigenvalues lie in the open left half of the complex plane, the zero-input solution decays exponentially. The infinite eigenvalues of (E, A) with the eigenvectors x satisfying $Ex = 0$ determine static behaviors. The infinite eigenvalues of (E, A) with generalized eigenvectors x_k satisfying the relation $Ex_1 = 0$ and $Ex_k = Ax_{k-1}$ $(k \ge 2)$ create impulsive modes. It is known that the system (1) has no impulsive mode if and only if

$$\operatorname{rank} E = \deg \det(sE - A). \tag{2}$$

Stability of the system (1) is defined as follows.

Definition 1 *The system* (1) *is said to be stable if it is regular and has only decaying exponential modes and static behaviors.*

A Lyapunov-type stability condition has been proposed, which is expressed by a strict LMI [7]. In the following lemma, the matrices $V, U \in R^{n\times(n-r)}$ are of full column ranks and composed of bases of $\ker E$ and $\ker E^T$, respectively. Introduction of these matrices was significant in deriving strict LMI conditions for descriptor systems [7].

Lemma 1 *The system* (1) *is stable if and only if there exist a positive definite matrix $P \in R^{n\times n}$ and a matrix $S \in R^{(n-r)\times(n-r)}$ such that the LMI*

$$A(PE^T + VSU^T) + (PE^T + VSU^T)^T A^T < 0 \tag{3}$$

holds.

The H_2 norm for a stable system (1) is defined as

$$\|C_1(sE - A)^{-1}B_1\|_2$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{trace} \left\{ \left(C_1(-j\omega E - A)^{-1} B_1 \right)^T \right.\right.$$

$$\left.\left. \cdot\, C_1 (j\omega E - A)^{-1} B_1 \right\} d\omega \right]^{\frac{1}{2}}, \qquad (4)$$

which is finite if and only if

$$\lim_{s\to\infty} C_1(sE - A)^{-1} B_1 = 0. \qquad (5)$$

To ensure finiteness of the H_2 norm, we assume that the system (1) satisfies the following condition [4].

$$\ker C_1 \supseteq \ker E \qquad (6)$$

We present a strict LMI condition for the H_2 norm to be smaller than a given positive number.

Theorem 1 *For a given positive number γ, the descriptor system* (1) *with* (6) *is stable and satisfies*

$$\|C_1(sE - A)^{-1} B_1\|_2 < \gamma \qquad (7)$$

if and only if there are a positive definite matrix $P \in R^{n\times n}$ and a matrix $S \in R^{(n-r)\times(n-r)}$ such that the LMIs

$$A(PE^T + VSU^T) + (PE^T + VSU^T)^T A^T + B_1 B_1^T < 0 \qquad (8)$$

$$\mathrm{trace}\left\{ C_1 P C_1^T \right\} < \gamma^2 \qquad (9)$$

hold.

Proof. *Necessity:* When the system (1) is stable, there are nonsingular matrices M and N such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix} \qquad (10)$$

where $A_1 \in R^{r\times r}$ is a stable matrix [10]. Using such M and N, we set

$$MB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad C_1 N = \begin{bmatrix} C_{11} & 0 \end{bmatrix}, \qquad (11)$$

where the assumption (6) and the structure of MEN in (10) imply that the last $n-r$ columns of C_1N are zero. Then,

$$C_1(sE - A)^{-1} B_1 = C_{11}(sI_r - A_1)^{-1} B_{11} \qquad (12)$$

and we see from the H_2 norm condition for state-space representations [11] that (7) holds if and only if there is a positive definite matrix P_1 such that

$$A_1 P_1 + P_1 A_1^T + B_{11} B_{11}^T < 0 \qquad (13)$$

$$\mathrm{trace}\left\{ C_{11} P_1 C_{11}^T \right\} < \gamma^2. \qquad (14)$$

We note that (13) implies

$$\begin{bmatrix} A_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ B_{12}B_{11}^T & Z \end{bmatrix} + \begin{bmatrix} P_1 & B_{11}B_{12}^T \\ 0 & Z \end{bmatrix} \begin{bmatrix} A_1^T & 0 \\ 0 & -I_{n-r} \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} \begin{bmatrix} B_{11}^T & B_{12}^T \end{bmatrix} < 0 \qquad (15)$$

where Z is a symmetric matrix defined by

$$Z = \frac{1}{2}\left(B_{12}B_{12}^T + \alpha I_{n-r} \right) \qquad (16)$$

and $\alpha > 0$. We note also that we can have the following expression.

$$\begin{bmatrix} P_1 & 0 \\ B_{12}B_{11}^T & Z \end{bmatrix}$$

$$= \begin{bmatrix} P_1 & -M_1M_2^T(M_2M_2^T)^{-1}Z + B_{11}B_{12}^T \\ -Z(M_2M_2^T)^{-1}M_2M_1^T + B_{12}B_{11}^T & \mu I_{n-r} \end{bmatrix} \cdot \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} Z(M_2M_2^T)^{-1}M_2M^T$$

$$= N^{-1}N \begin{bmatrix} P_1 & -M_1M_2^T(M_2M_2^T)^{-1}Z + B_{11}B_{12}^T \\ -Z(M_2M_2^T)^{-1}M_2M_1^T + B_{12}B_{11}^T & \mu I_{n-r} \end{bmatrix} \cdot N^T E^T M^T + N^{-1}VG_N Z(M_2M_2^T)^{-1}G_M^T U^T M^T \qquad (17)$$

Here, $\mu > 0$, and M_1, M_2 are the upper r and lower $n-r$ rows of M, respectively. In addition, G_N, G_M are nonsingular matrices such that $N_2 = VG_N$ and $M_2^T = UG_M$, where N_2 is the last $n-r$ columns of N. The existence of such G_N and G_M is guaranteed by the fact that V is a basis matrix of $\ker E$, N_2 is of full column rank in $\ker E$, U is a basis matrix of $\ker E^T$, and M_2^T is of full column rank in $\ker E^T$ [7].

Now, we define

$$P = N \begin{bmatrix} P_1 & -M_1M_2^T(M_2M_2^T)^{-1}Z + B_{11}B_{12}^T \\ -Z(M_2M_2^T)^{-1}M_2M_1^T + B_{12}B_{11}^T & \mu I_{n-r} \end{bmatrix} N^T \qquad (18)$$

$$S = G_N Z (M_2M_2^T)^{-1} G_M^T \qquad (19)$$

and see that (15) is written as

$$M\left\{ A(PE^T + VSU^T) + (PE^T + VSU^T)^T A^T + B_1B_1^T \right\} M^T < 0 \qquad (20)$$

Obviously, (20) is equivalent to (8) and there always exists a μ in (18) for any α in Z of (16) such that P becomes positive definite. In addition, since the structure of C_1N in (11) implies

$$C_1 P C_1^T = C_{11} P_1 C_{11}^T, \qquad (21)$$

we conclude (9) from (14).

Suffiency: We first note that (8) implies (3) and stability of the system (1). Therefore, there exist nonsingular matrices M and N which transform E, A, B_1, and C_1 to the forms of (10) and (11). Then, (20), which is equivalent to (8), is rewritten as

$$\begin{bmatrix} A_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix} \left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} G_N^{-1} S U^T \begin{bmatrix} M_1^T & M_2^T \end{bmatrix} \right)$$

$$+\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\right.$$
$$\left. + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} US^T (G_N^{-1})^T \begin{bmatrix} 0 & I_{n-r} \end{bmatrix}\right)\begin{bmatrix} A_1^T & 0 \\ 0 & -I_{n-r} \end{bmatrix}$$
$$+\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}\begin{bmatrix} B_{11}^T & B_{12}^T \end{bmatrix} < 0 \tag{22}$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = N^{-1}P(N^{-1})^T.$$

The $(1,1)$ block of this inequality is (13) and (9) is equivalent to (14) from (11), if P_1 is replaced with P_{11} in (13) and (14). Then, the result for state-space representations [11] implies

$$\|C_{11}(sI_r - A_1)^{-1}B_{11}\|_2 < \gamma, \tag{23}$$

and we conclude (7) by (12). The proof is completed.

By considering the dual system of (1), that is,

$$E^T\dot{\tilde{x}} = A^T\tilde{x} + C_1^T\tilde{w}, \qquad \tilde{z} = B_1^T\tilde{x}, \tag{24}$$

we can obtain a corollary of Theorem 1 under the condition

$$\text{range}\, B_1 \subseteq \text{range}\, E. \tag{25}$$

We note that the H_2 norms of (1) and (24) are identical.

Corollary 1 *For a given positive number γ, the descriptor system* (1) *with* (25) *is stable and satisfies*

$$\|C_1(sE - A)^{-1}B_1\|_2 < \gamma \tag{26}$$

if and only if there are a positive definite matrix $Q \in R^{n\times n}$ and a matrix $R \in R^{(n-r)\times(n-r)}$ such that the LMIs

$$A^T(QE + URV^T) + (QE + URV^T)^T A + C_1^T C_1 < 0 \tag{27}$$

$$\text{trace}\left\{B_1^T Q B_1\right\} < \gamma^2 \tag{28}$$

hold.

3. H_2 Control by Descriptor Variable Feedback

We now consider H_2 control of the descriptor system

$$E\dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x \tag{29}$$

where $x \in R^n$ is the descriptor variable, $w \in R^q$ is the disturbance, $u \in R^m$ is the control input, and $z \in R^p$ is the controlled output. We assume that the descriptor variable can be measured. In (29), the coefficient matrices are constant and $\text{rank}\, E = r \le n$. While this description does not have a direct transmission path from u to z explicitly, it is known that systems having such a path can always be rewritten as (29) by augmenting the descriptor variable [5].

We use linear descriptor variable feedback

$$u = Kx \tag{30}$$

where $K \in R^{m\times n}$ is a constant matrix. The resultant closed-loop system is written as

$$E\dot{x} = (A + B_2K)x + B_1 w, \qquad z = C_1 x. \tag{31}$$

Under the assumption (6), we obtain the following:

Theorem 2 *For a given positive number γ, there is a feedback gain K such that the closed-loop system* (31) *is stable and satisfies*

$$\|C_1(sE - A - B_2K)^{-1}B_1\|_2 < \gamma \tag{32}$$

if and only if there exist a positive definite matrix $P \in R^{n\times n}$ and matrices $S \in R^{(n-r)\times(n-r)}$, $L \in R^{m\times n}$, $H \in R^{m\times(n-r)}$ which satisfy the LMIs

$$\begin{aligned} &A(PE^T + VSU^T) + (PE^T + VSU^T)^T A^T \\ &\quad + B_2(LE^T + HU^T) + (LE^T + HU^T)^T B_2^T \\ &\quad + B_1 B_1^T < 0 \end{aligned} \tag{33}$$

$$\text{trace}\left\{C_1 P C_1^T\right\} < \gamma^2. \tag{34}$$

A desired feedback gain is given by

$$K = (LE^T + HU^T)(PE^T + VSU^T)^{-1}. \tag{35}$$

Proof. *Necessity:* If the closed-loop system (31) is stable and satisfies (32), then from Theorem 1, there exist a positive definite matrix $P \in R^{n\times n}$ and a matrix $S \in R^{(n-r)\times(n-r)}$ such that

$$\begin{aligned} &(A + B_2K)(PE^T + VSU^T) \\ &\quad + (PE^T + VSU^T)^T(A + B_2K)^T + B_1B_1^T < 0 \end{aligned} \tag{36}$$

$$\text{trace}\left\{C_1 P C_1^T\right\} < \gamma^2 \tag{37}$$

hold. By setting $L = KP$ and $H = KVS$ in (36), we obtain (33), and (37) is identical to (34).

Sufficiency: When (33) holds, we can assume without loss of generality that the matrix S is nonsingular. If it is not, we introduce a small perturbation in S to make it nonsingular without violating (33). Then, $PE^T + VSU^T$ is nonsingular [7], and we can define K of (35). By substituting $LE^T + HU^T = K(PE^T + VSU^T)$ into (33), we obtain (36), which, with (34), implies that the closed-loop system (31) is stable and satisfies (32).

4. Extension to Robust H_2 Control

The result of the previous section can be extended to robust H_2 control for uncertain systems with polytopic coefficient matrices. Here, we assume that E, A, B_1, and B_2 in (29) are constant and respectively belong to the classes

$$E = E_L\left(\sum_{i=1}^{N_I}\theta_i\Omega_i\right)E_R^T, \qquad \theta_i \ge 0, \quad \sum_{i=1}^{N_I}\theta_i = 1$$
$$A = \sum_{j=1}^{N_J}\mu_j A_j, \quad \mu_j \ge 0, \quad \sum_{j=1}^{N_J}\mu_j = 1$$
$$B_1 = \sum_{k=1}^{N_K}\eta_k B_{1k}, \quad \eta_k \ge 0, \quad \sum_{k=1}^{N_K}\eta_k = 1$$
$$B_2 = \sum_{\ell=1}^{N_L}\xi_\ell B_{2\ell}, \quad \xi_\ell \ge 0, \quad \sum_{\ell=1}^{N_L}\xi_\ell = 1 \tag{38}$$

where $E_L, E_R \in R^{n\times r}$ are of full column ranks, $\Omega_i \in R^{r\times r}$ are nonsingular matrices, $A_j \in R^{n\times n}$, $B_{1k} \in R^{n\times q}$, $B_{2\ell} \in R^{n\times m}$ are any matrices, θ_i, μ_j, η_k, ξ_ℓ

are scalars, and N_I, N_J, N_K, $N_L$are positive integers. In (38), the class of E may look restrictive, but it is not so when we consider actual systems.

Under the assumption (6), we obtain the following:

Theorem 3 *For a given positive number γ, there is a feedback gain K such that the closed-loop system* (31) *with coefficient matrices of* (38) *is stable and satisfies*

$$\|C_1(sE - A - B_2K)^{-1}B_1\|_2 < \gamma \qquad (39)$$

if there exist a positive definite matrix $P \in R^{n\times n}$ and matrices $S \in R^{(n-r)\times(n-r)}$, $L \in R^{m\times n}$, $H \in R^{m\times(n-r)}$ which satisfy the LMIs

$$\begin{bmatrix} \Phi_{ij\ell} & B_{1k} \\ B_{1k}^T & -I \end{bmatrix} < 0 \qquad (40)$$

$$\begin{aligned}\Phi_{ij\ell} &= A_j(PE_i^T + VSU^T) + (PE_i^T + VSU^T)^T A_j^T \\ &\quad + B_{2\ell}(LE_i^T + HU^T) + (LE_i^T + HU^T)^T B_{2\ell}^T\end{aligned}$$

$$i = 1, ..., N_I, \quad j = 1, ..., N_J,$$
$$k = 1, ..., N_K, \quad \ell = 1, ..., N_L$$

$$\text{trace}\left\{C_1PC_1^T\right\} < \gamma^2 \qquad (41)$$

where $E_i = E_L\Omega_iE_R^T$. A desired feedback gain is given by

$$\begin{aligned}K &= LE_R(E_R^TPE_R)^{-1}E_R^T \\ &\quad + HS^{-1}(V^TV)^{-1}V^T \\ &\quad \cdot\{I - PE_R(E_R^TPE_R)^{-1}E_R^T\}. \qquad (42)\end{aligned}$$

Proof. As in the proof of Theorem 2, we can assume without loss of generality that the matrix S is nonsingular. Then, since

$$[\, E_L \quad U \,]^{-1} = \begin{bmatrix} (E_L^TE_L)^{-1}E_L^T \\ (U^TU)^{-1}U^T \end{bmatrix}$$

$$E_L(E_L^TE_L)^{-1}E_L^T + U(U^TU)^{-1}U^T = I_n \qquad (43)$$

hold, we see [7] that $PE_i^T + VSU^T$ is nonsingular and its inverse can be written as

$$\begin{aligned}(PE_i^T + VSU^T)^{-1} &= (PE_R\Omega_i^TE_L^T + VSU^T)^{-1} \\ &= E_L(E_L^TE_L)^{-1}(\Omega_i^T)^{-1}(E_R^TPE_R)^{-1}E_R^T \\ &\quad + U(U^TU)^{-1}S^{-1}(V^TV)^{-1}V^T \\ &\quad \cdot\{I_n - PE_R(E_R^TPE_R)^{-1}E_R^T\}. \qquad (44)\end{aligned}$$

This implies

$$(LE_i^T + HU^T)(PE_i^T + VSU^T)^{-1} = K \qquad (45)$$

for any $E_i = E_L\Omega_iE_R^T$, and $\Phi_{ij\ell}$ in (40) is reduced to

$$\begin{aligned}\Phi_{ij\ell} &= (A_j + B_{2\ell}K)(PE_i^T + VSU^T) \\ &\quad + (PE_i^T + VSU^T)^T(A_j + B_{2\ell}K)^T \\ &\qquad \text{for all } i, j, \ell. \qquad (46)\end{aligned}$$

Therefore, multiplying (40) by $\theta_i\mu_j\eta_k\xi_\ell$, and taking the total sum for all i, j, k, ℓ, we obtain

$$\begin{bmatrix} \Phi & B_1 \\ B_1^T & -I \end{bmatrix} < 0 \qquad (47)$$

where

$$\begin{aligned}\Phi &= (A + B_2K)(PE^T + VSU^T) \\ &\quad + (PE^T + VSU^T)^T(A + B_2K)^T. \qquad (48)\end{aligned}$$

Thus,

$$\begin{aligned}&(A + B_2K)(PE^T + VSU^T) \\ &\quad + (PE^T + VSU^T)^T(A + B_2K)^T + B_1B_1^T < 0 \quad (49)\end{aligned}$$

holds. From Theorem 1, this together with (41) imply stability of the closed-loop system and (39).

5. Concluding Remarks

A strict LMI condition has been presented for H_2 control of linear time-invariant descriptor systems. The control law considered in this paper is static feedback of the descriptor variable. Robust H_2 feedback control gain has been obtained for systems with polytopic coefficient matrices. The results of this paper can be extended to the case of dynamic controllers.

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