Canonical Forms and Orbit Identification Problems in Machine Vision

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Abstract

We introduce canonical forms for perspective dynamical systems under the action of a perspective group and illustrate their application to parameter identification with the aid of a single CCD camera. We show that the parameters in the canonical form can be identified uniquely using an Extended Kalman Filter (EKF).

1 Introduction

An important and somewhat difficult problem in machine vision is to identify parameters of motion dynamics from observing projection of feature points on the image plane observed over time. The difficulty arises from the fact that the observation function is nonlinear (perspective) and the underlying sensor (a CCD camera) is noisy. In fact, not all the parameters are identifiable and in [2], the identifiable parameters are characterized via orbits of a "perspective group". In order to identify orbits, we need to define a "canonical form" , [6] – [13], which is the main subject matter of this paper.

To illustrate the main concept we consider a linear finite dimensional dynamical system of the form

$$\dot{\mathcal{X}} = \mathcal{A}^T\mathcal{X}, \quad \mathcal{Y} = \mathcal{C}\mathcal{X}, \quad \mathcal{X}(0) = \mathcal{X}_0 \tag{1.1}$$

where $\mathcal{X}$ is the state variable of dimension n and where $\mathcal{Y}$ is the output variable of dimension m. We assume furthermore that the output is observed only up to an unknown scale factor. Thus the observation function is given by

$$\begin{array}{rccc} \mathcal{Z}: & \mathbb{R}^m - \{0\} & \longrightarrow & \mathbb{RP}^{m-1} \\ & \mathcal{Y} & \longmapsto & [\mathcal{Y}] \end{array} \tag{1.2}$$

where we assume that $m > 1$. Note that the observation function is defined for all nonzero vectors in $\mathbb{R}^m$. A problem of interest in parameter identification is described as follows.

Perspective Parameter Identification: Assume that we observe $\mathcal{Z}(t)$ in an unspecified interval $[0, T)$. The problem is to determine the extent to which the parameters $\mathcal{A}^T, \mathcal{C}$ are identifiable from the observed data.

Perspective problems have already been considered in the literature Kanatani [5] where the goal is to identify parameters of motion using a 'charged coupled device' CCD camera. Generically it is known [2] that parameters can be identified up to orbits of a perspective group. The scalings on the parameters $(\mathcal{A}^T, \mathcal{C})$ as a result of the group action is described as follows,

$$\begin{array}{lcc} \mathcal{X}_1 : GL(n) \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ \quad (P, \mathcal{A}^T, \mathcal{C}) & \longmapsto & (P\mathcal{A}^T P^{-1}, \mathcal{C}P^{-1}) \end{array} \tag{1.3}$$

$$\begin{array}{lcc} \mathcal{X}_2 : \mathbb{R} \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ \quad (\lambda, \mathcal{A}^T, \mathcal{C}) & \longmapsto & (\lambda I + \mathcal{A}^T, \mathcal{C}) \end{array} \tag{1.4}$$

$$\begin{array}{lcc} \mathcal{X}_3 : \mathbb{R}^+ \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ \quad (\mu, \mathcal{A}^T, \mathcal{C}) & \longmapsto & (\mathcal{A}^T, \mu\mathcal{C}) \end{array} \tag{1.5}$$

The perspective group is the direct sum of each of the three groups in (1.3), (1.4) and (1.5). As a result of the action of the perspective group on the parameter space, the space is split up into orbits.

It may be trivially verified that parameters in the same orbit of the perspective group cannot be observed using (1.2). Furthermore it has been shown by Ghosh et.al. [1], [2] that generically the orbits of the perspective group are indeed observable, i.e. produces distinct outputs for at least some interval of time. In order to actually carry out the problem of observing the orbits, it is essential to parameterize the orbits by describing a chart with an appropriate set of coordinates and construct an observer that computes an estimate of the coordinates.

In this paper we consider primarily the parameterization problem but also illustrate our results with observers that are derived from EKF.

2 Background and Motivation

Assume that the motion dynamics is described by the following equation

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$
$$+ \begin{pmatrix} f_1 & f_2 & f_3 & 0 & 0 & 0 \\ 0 & f_1 & 0 & f_2 & f_3 & 0 \\ 0 & 0 & f_1 & 0 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} X^2 \\ XY \\ XZ \\ Y^2 \\ YZ \\ Z^2 \end{pmatrix}. \tag{2.1}$$

Let us denote

$$A = (a_{ij}), \quad b = (b_1 \;\; b_2 \;\; b_3)^T, \quad f^T = (-f_1 \;\; -f_2 \;\; -f_3),$$

where $i, j = 1, 2, 3$. The dynamical system (2.1) is a Riccati dynamics in $\mathbb{R}^3$, which has been illustrated in [3]. It is a class of *quadratic motion models* more general than a *rigid flow* which preserves *shape*, i.e., the shape of a planar surface remain planar, although the distance between two points on the plane may not remain constant. An important question, that is of interest in machine vision is to ask the following.

Question 2.1 : Consider the dynamical system (2.1) and assume that the state vector (X, Y, Z) is observed by the function

$$x = X/Z, \quad y = Y/Z, \quad Z \neq 0, \tag{2.2}$$

which is typically the case under perspective projection, or

$$x = X, \quad y = Y \tag{2.3}$$

which occurs under orthographic projection, to what extent are the motion parameters and the initial conditions $X(0), Y(0), Z(0)$ identifiable from the observation vector $(x(t), y(t))$ in a given time interval $[0, T), T > 0$?

In order to show the connection between the dynamical system (2.1), (2.2) with that of (1.1), (1.2); note that one can homogenize the state vector (X, Y, Z) as

$$X = X_1/W_1, \quad Y = Y_1/W_1, \quad Z = Z_1/W_1 \tag{2.4}$$

and write (2.1) as

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} = \begin{pmatrix} A & b \\ f^T & 0 \end{pmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix}. \tag{2.5}$$

The observation vector (2.2) can also be written, by defining

$$x = y_1/y_3, \quad y = y_2/y_3,$$

as follows

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix}. \tag{2.6}$$

If only the first component of the observation function (2.1) is available, we can write

$$\begin{bmatrix} y_1 \\ y_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix}. \tag{2.7}$$

Note that the state vector in (2.5) and the observation vectors in (2.6) and (2.7) are elements of a projective space and the pairs (2.5), (2.6) and (2.5), (2.7) are of the form (1.1), (1.2).

Likewise if we consider orthographic projection (2.3) as the observation function, then (2.6) is changed to the following equation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix}. \tag{2.8}$$

In this paper we distinguish between two important cases, one in which the camera calibration is unknown and the other in which the camera calibration is known. If the camera calibration is known, one can in turn assume that the matrix $\mathcal{C}$ in (1.1) is known and can be chosen to take the special structure described in (2.6), (2.7) and (2.8). Otherwise the matrix $\mathcal{C}$ is assumed to be unknown.

3 The case of unknown camera calibration

For the case when the calibration parameter is not known we consider the dynamical system (1.1), (1.2) where we assume $n = 4$ and m is either 2 or 3 depending upon the choice of the observation functions (2.6) or (2.7). We assume that both $\mathcal{A}^T$ and $\mathcal{C}$ are unknown parameters.

3.1 Parameterization of the orbit under the perspective group

We begin this section by considering the case $n = 4$, $m = 2$ and assume that the associated Kronecker indices are $\kappa_1 = \kappa_2 = 2$ (see [10], [13] for a definition of Kronecker indices). It follows that via the group action described in (1.3), the pair $\mathcal{A}$, $\mathcal{B}$ would take the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & \gamma \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.1}$$

where the parameters $\alpha_i,\ \beta_i,\ i = 1, 2, 3, 4$ and γ are arbitrary. Writing $\mathcal{B}$ as $(b_1\ b_2)$ we assume that

$$\begin{aligned} \mathcal{A}^2 b_1 &= \gamma_1 b_1 + \gamma_2 b_2 + \gamma_3 \mathcal{A} b_1 + \gamma_4 \mathcal{A} b_2, \\ \mathcal{A}^2 b_2 &= \delta_1 b_1 + \delta_2 b_2 + \delta_3 \mathcal{A} b_1 + \delta_4 \mathcal{A} b_2. \end{aligned}$$

The pair (3.1) can be reduced to

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \gamma_1 & \gamma_3 & \delta_1 & \delta_3 \\ 0 & 0 & 0 & 1 \\ \gamma_2 & \gamma_4 & \delta_2 & \delta_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2}$$

where we assume that the pair in (3.2) is the representation of $\mathcal{A}, \mathcal{B}$ with respect to the basis given by

$$\{\mathcal{A}b_1 - \gamma_3 b_1 - \gamma_4 b_2,\ b_1,\ \mathcal{A}b_2 - \delta_3 b_1 - \delta_4 b_2,\ b_2\}.$$

If we now consider the group action (1.4) on the pair (3.2), we can assume that the trace of the matrix $\mathcal{A}$ is 0. Thus we can replace δ_4 by $-\gamma_3$. Therefore a canonical form for the perspective system (1.1), (1.2) is given by

$$\mathcal{A}^T = \begin{pmatrix} 0 & \gamma_1 & 0 & \gamma_2 \\ 1 & \gamma_3 & 0 & \gamma_4 \\ 0 & \delta_1 & 0 & \delta_2 \\ 0 & \delta_3 & 1 & -\gamma_3 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.3}$$

that is,

$$\begin{aligned} \dot{x_1} &= \gamma_1 x_2 + \gamma_2 x_4, & \dot{x_2} &= x_1 + \gamma_3 x_2 + \gamma_4 x_4, \\ \dot{x_3} &= \delta_1 x_2 + \delta_2 x_4, & \dot{x_4} &= \delta_3 x_2 + x_3 - \gamma_3 x_4, \\ z &= x_2 / x_4. \end{aligned}$$

Therefore, for the Kronecker index pair $(2, 2)$ we have a total of 7 parameters.

If we choose $n = 4$ and $m = 2$ and assume that the Kronecker indices are $\kappa_1 = 3$ and $\kappa_2 = 1$, a suitable canonical form is given by

$$\mathcal{A}^T = \begin{pmatrix} 0 & 0 & \alpha_1 & \beta_1 \\ 1 & 0 & \alpha_2 & \beta_2 \\ 0 & 1 & \alpha_3 & \beta_3 \\ 0 & 0 & \alpha_4 & \beta_4 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix}$$

where the parameters $\alpha_i,\ \beta_i,\ i = 1, 2, 3, 4$ and γ are arbitrary. Denoting the columns of $\mathcal{B} := \mathcal{C}^T$ by $(b_1\ b_2)$, the Kronecker structure implies that the vectors b_1, b_2, $\mathcal{A}b_1$ and $\mathcal{A}^2 b_1$ are independent and

$$\begin{aligned} \mathcal{A}^3 b_1 &= \gamma_1 b_1 + \gamma_2 b_2 + \gamma_3 \mathcal{A} b_1 + \gamma_4 \mathcal{A}^2 b_1, \\ \mathcal{A} b_2 &= \delta_1 b_1 + \delta_2 b_2 \end{aligned}$$

for some scalar constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4,\ \delta_1, \delta_2$ that can be determined from the pair $\mathcal{A}$, $\mathcal{B}$. If we consider a basis given by

$$\{\mathcal{A}^2 b_1 - \gamma_4 \mathcal{A} b_1 - \gamma_3 b_1,\ \mathcal{A}b_1 - \gamma_4 b_1,\ b_1,\ b_2\}$$

it is easy to see that the pair $\mathcal{A}^T$, $\mathcal{C}$ is given by

$$\mathcal{A}^T = \begin{pmatrix} 0 & 0 & \gamma_1 & \gamma_2 \\ 1 & 0 & \gamma_3 & 0 \\ 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & \delta_1 & -\gamma_4 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.4}$$

and the perspective system is given by

$$\begin{aligned} \dot{x}_1 &= \gamma_1 x_3 + \gamma_2 x_4, & \dot{x}_2 &= x_1 + \gamma_3 x_3, \\ \dot{x}_3 &= x_2 + \gamma_4 x_3, & \dot{x}_4 &= \delta_1 x_3 - \gamma_4 x_4, \\ z &= x_3 / x_4. \end{aligned}$$

There are a total of 5 parameters for the Kronecker index pair $(3, 1)$.

Finally if we now choose $n = 4$ and $m = 3$ and assume that the Kronecker indices are $\kappa_1 = 2$, $\kappa_2 = 1$, $\kappa_3 = 1$, it is possible to show that a suitable canonical form is given by

$$\mathcal{A}^T = \begin{pmatrix} 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 1 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 \\ 0 & \alpha_4 & \beta_4 & \gamma_4 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \eta_1 & 1 & 0 \\ 0 & \eta_2 & \eta_3 & 1 \end{pmatrix}$$

wherein, like before, the parameters $\alpha_i, \beta_i, \gamma_i,\ i = 1, 2, 3, 4$ and $\eta_j, j = 1, 2, 3$ are arbitrary. Denoting the columns of $\mathcal{B} := \mathcal{C}^T$ by $(b_1\ b_2\ b_3)$, the Kronecker structure implies that the vectors b_1, b_2, b_3 and $\mathcal{A}b_1$ are independent and

$$\begin{aligned} \mathcal{A}^2 b_1 &= \delta_1 b_1 + \delta_2 b_2 + \delta_3 b_3 + \delta_4 \mathcal{A} b_1, \\ \mathcal{A} b_2 &= \xi_1 b_1 + \xi_2 b_2 + \xi_3 b_3, \\ \mathcal{A} b_3 &= \zeta_1 b_1 + \zeta_2 b_2 + \zeta_3 b_3 \end{aligned}$$

for some scalar constants $\delta_1, \delta_2, \delta_3, \delta_4, \xi_1, \xi_2, \xi_3, \zeta_1, \zeta_2, \zeta_3$ that can be determined from the pair $\mathcal{A}$, $\mathcal{B}$. If we consider a basis given by

$$\{\mathcal{A}b_1 - \delta_4 b_1,\ b_1,\ b_2,\ b_3\}$$

it follows quite easily that the pair $\mathcal{A}^T$, $\mathcal{C}$ is given by

$$\mathcal{A}^T = \begin{pmatrix} 0 & \delta_1 & \delta_2 & \delta_3 \\ 1 & \delta_4 & 0 & 0 \\ 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & \zeta_1 & \zeta_2 & -\delta_4 - \xi_2 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.5}$$

and the perspective system is given by

$$\begin{aligned} \dot{x}_1 &= \delta_1 x_2 + \delta_2 x_3 + \delta_3 x_4, \quad \dot{x}_2 = x_1 + \delta_4 x_2, \\ \dot{x}_3 &= \xi_1 x_2 + \xi_2 x_3 + \xi_3 x_4, \\ \dot{x}_4 &= \zeta_1 x_2 + \zeta_2 x_3 - (\delta_4 + \xi_2) x_4, \\ z_1 &= x_2 / x_4, \quad z_2 = x_3 / x_4. \end{aligned}$$

Note that for the Kronecker index $(2,1,1)$ there are a total of 9 parameters. From the above remark one obtains the following result.

Theorem 3.1 (*Canonical form under unknown camera calibration*) : Let us consider the perspective dynamical system (1.1), (1.2) for the cases $n=4, m=2$, $n=4, m=2$ and $n=4, m=3$, for the Kronecker indices (2, 2), (3, 1) and (2, 1, 1), respectively. A canonical form for each of the three cases is given by (3.3), (3.4) and (3.5), respectively.

Remark 3.2 : Note that for each of the above three cases, the group action (1.5) can be used to scale the matrix $\mathcal{C}$ to a suitable canonical structure, viz. a matrix with unit norm. The details about this have been omitted.

3.2 Parameter Identification using an EKF

Parameter estimation problems have already been studied using an EKF by many researchers in the past, see for example Kano [4] for motion and shape estimation under perspective projection. In this section we only discuss the case of estimating the seven parameters in (3.3). Discretizing the dynamical system (3.3) we obtain the following discretized dynamical system

$$\mathcal{X}_{k+1} = \mathcal{A}_d \mathcal{X}_k, \quad z_k = x_{2k}/x_{4k} \tag{3.6}$$

$$\mathcal{A}_d = \begin{pmatrix} 1 & \gamma_1 T & 0 & \gamma_2 T \\ T & 1+\gamma_3 T & 0 & \gamma_4 T \\ 0 & \delta_1 T & 1 & \delta_2 T \\ 0 & \delta_3 T & T & 1-\gamma_3 T \end{pmatrix}$$

where $\mathcal{X}_k = (x_{1k}\ x_{2k}\ x_{3k}\ x_{4k})^T$ and where T is the sampling time. Note that the parameters to be estimated are $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1, \delta_2, \delta_3$. The three coordinates of a point observed by the camera are given by

$$\bar{x}_k = x_{1k}/x_{4k}, \quad \bar{y}_k = x_{2k}/x_{4k}, \quad \bar{z}_k = x_{3k}/x_{4k}. \tag{3.7}$$

We now assume that, at each time instant k, a set of n points are observed with coordinates $(\bar{x}_{ik}\ \bar{y}_{ik}\ \bar{z}_{ik})$ for $i = 1, \cdots, n$. The coordinates are defined as in (3.7) for each of the n points. We consider the state vector θ_k to be given by

$$\theta_k = (\gamma_1\ \gamma_2\ \gamma_3\ \gamma_4\ \delta_1\ \delta_2\ \delta_3 \\ \bar{x}_{1k}\ \bar{y}_{1k}\ \bar{z}_{1k}\ \ldots\ \bar{x}_{nk}\ \bar{y}_{nk}\ \bar{z}_{nk})^T \in \mathbb{R}^{3n+7},$$

and the associated state equation is given by

$$\theta_{k+1} = F(\theta_k). \tag{3.8}$$

In fact, it follows from (3.6) that we need only the observation values $\bar{y}_{ik}$. Let $\bar{y}_{ik}$ be observed with an additive noise, and let $\mathcal{Z}_k \in \mathbb{R}^n$ be the observation vector. Then

$$\mathcal{Z}_k = H\theta_k + w_k \tag{3.9}$$

where $\{w_k\} \in \mathbb{R}^n$ is assumed to be a Gaussian white noise with mean zero and covariance R, i.e. $\mathrm{E}\{w_k w_l^T\} = R\delta_{kl}$, δ_{kl} indicates Kronecker's delta and $H = [O_{n,7}\ \Theta_{n,3n}] \in \mathbb{R}^{n\times(3n+7)}$. Here $\Theta_{n,3n}$ is a suitable matrix and its detailed structure is omitted. The EKF is described as follows.

$$\begin{aligned} \hat{\theta}_{k+1|k+1} &= F(\hat{\theta}_{k|k}) + \mathcal{K}_{k+1}[\mathcal{Z}_{k+1} - HF(\hat{\theta}_{k|k})], \\ \mathcal{K}_{k+1} &= Q_{k+1|k}H^T[HQ_{k+1|k}H^T + R]^{-1}, \\ Q_{k+1|k} &= F_k(Q_{k|k-1} - K_k H Q_{k|k-1})F_k^T, \\ F_k &= \left.\frac{\partial F(\theta)}{\partial \theta^T}\right|_{\theta=\hat{\theta}_{k|k}}. \end{aligned} \tag{3.10}$$

Likewise an EKF for the dynamical system (3.4) and (3.5) may be described similar to that given by (3.10).

Simulation studies on the EKFs are performed for the following parameters. For the Kronecker indices $(2,2)$ parameters in (3.3) is taken to be

$$\begin{aligned} \text{Case } (2,2) \ :\ & (\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \delta_1,\ \delta_2,\ \delta_3) \\ & = (-0.2,\ 0.1,\ 1,\ 0.5,\ 1,\ 0.3,\ 1.5). \end{aligned} \tag{3.11}$$

Moreover we set the sampling period $T = 0.01$, the number of points is chosen as $n = 3$ and

$$\begin{aligned} (\bar{x}_{10}\ \bar{y}_{10}\ \bar{z}_{10}) &= (0.05\ 0.3\ \ -0.1), \\ (\bar{x}_{20}\ \bar{y}_{20}\ \bar{z}_{20}) &= (0.3\ 0.1\ \ -0.06), \\ (\bar{x}_{30}\ \bar{y}_{30}\ \bar{z}_{30}) &= (-0.2\ 0.1\ \ -0.1). \end{aligned}$$

Let the noise variance be chosen to be $R = \sigma^2 I_3$ and $R = \sigma^2 I_6$ with $\sigma = 0.0001$, respectively. The estimates of the parameters in (3.3) have been estimated and plotted in Fig. 1. The dotted lines denote the true values in (3.11) and the solid lines represent the estimated values with the EKF. It may be noted from the figure that the initial condition for the parameters chosen during the simulation of the EKF is chosen to be close to the true value. Otherwise for various other choices of the initial conditions, the parameters do not converge to the true value but either maintains a fixed bias or diverges.

4 The case of known camera calibration

In the case when the calibration parameter is assumed to be known one could suppose that the perspective dynamical system is given by

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} = \mathcal{A}^T \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} \tag{4.1}$$

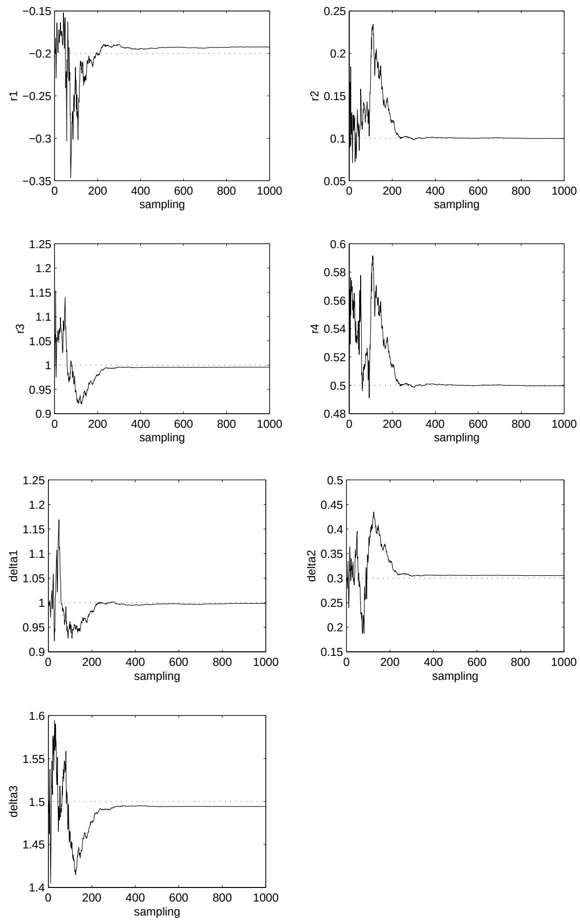

Figure 1: Kronecker index (2,2) case

with the observation function being (2.6) or (2.8) where we define

$$\mathcal{A}^T = \begin{pmatrix} A & b \\ f^T & d \end{pmatrix}.$$

4.1 Parameterization of the orbit under perspective projection

For the perspective dynamical system (4.1), (2.6), it follows from Ghosh et.al.[3] that one can consider a subgroup of the perspective group described in (1.3), (1.4) and (1.5) where P is given by

$$P^{-1} = \begin{pmatrix} p_{11} & 0 & 0 & 0 \\ 0 & p_{11} & 0 & 0 \\ 0 & 0 & p_{11} & 0 \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}. \qquad (4.2)$$

In general the action of P changes the structure of the associated matrix $\mathcal{C}$, but P of (4.2) preserves the structure. We now have the following result.

Theorem 4.1 (*Canonical form under known camera calibration, perspective projection*) : Consider the perspective dynamical system (4.1) with the observation function (2.6) and assume that $b \neq 0$. Under the perspective group action (1.3), (1.4) and (1.5) using P of (4.2), a canonical form in the case of known camera calibration is given by the subset $\mathcal{S}$ defined as follows:

$$\mathcal{S} = \left\{ \begin{pmatrix} A & b \\ f^T & d \end{pmatrix} ; \text{trace} A + d = 0,\ b^T A = 0 \text{ and } b_i = 1\ (i = 1 \text{ or } 2 \text{ or } 3) \right\} \qquad (4.3)$$

Proof of Theorem 4.1 : From the assumption $b \neq 0$ we assume that $b_3 \neq 0$ without any loss of generality. Then we may choose $p_{11} = b_3$ and $p_{44} = 1$. The scaling on the matrix $\mathcal{A}^T$ described by the group action (1.3) is given by

$$\begin{aligned} A &\longmapsto A^* \triangleq A + b^* \rho, \\ b &\longmapsto b^* \triangleq (b_1/b_3\ b_2/b_3\ 1)^T, \\ f^T &\longmapsto b_3 f^T - \rho A - (\rho b^* - d)\rho, \\ d &\longmapsto -(\rho b^* - d), \end{aligned}$$

where ρ denotes the vector $(p_{41}\ p_{42}\ p_{43})$. Here we can choose $p_{4i} = -b^{*T} A_i / \|b^*\|^2$, $i = 1, 2, 3$ where A_i is the i^{th} column vector of A. It follows that the vector ρ is unique and $b^{*T} A^* = 0$. Furthermore in order to get $\text{trace} A^* - (\rho b^* - d)$ to be 0 we apply the group in (1.4) with $\lambda = (-a^*_{11} - a^*_{22} - a^*_{33} + (\rho b^* - d))/4$ where a^*_{ii}, $i = 1, 2, 3$ denotes a diagonal element of A^*. The cases $b_1 \neq 0$ or $b_2 \neq 0$ can be described likewise. (Q.E.D.)

The canonical form (4.3) when $b_3 = 1$ is given by

$$\mathcal{A}^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ -b_1 a_{11} - b_2 a_{21} & -b_1 a_{12} - b_2 a_{22} & -b_1 a_{13} - b_2 a_{23} & 1 \\ -f_1 & -f_2 & -f_3 & -a_{11} - a_{22} + b_1 a_{13} + b_2 a_{23} \end{pmatrix} \qquad (4.4)$$

where $\mathcal{C}$ is given in (4.6).

4.2 Parameterization of the orbit under orthographic projection

Finally we consider the perspective dynamical system (4.1), (2.8). The associated subgroup of the perspective group (described in (1.3), (1.4) and (1.5)) is given by the following special structure of P

$$P^{-1} = \begin{pmatrix} p_{11} & 0 & 0 & 0 \\ 0 & p_{11} & 0 & 0 \\ p_{31} & p_{32} & p_{33} & p_{34} \\ 0 & 0 & 0 & p_{11} \end{pmatrix}. \qquad (4.5)$$

We now describe the following.

Theorem 4.2 (*Canonical form under known camera calibration, orthographic projection*) : Consider the perspective dynamical system (4.1) with the observation function (2.8) and assume that the vector c defined as $c = (a_{13}\ a_{23}\ -f_3)^T \neq 0$. Under the perspective group action (1.3), (1.4) and (1.5) using P of (4.5), a canonical form is described as follows:

$$\mathcal{S} = \left\{ \begin{pmatrix} A & b \\ f^T & d \end{pmatrix} ; \text{trace}A + d = 0, \right.$$
$$c^T \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ -f_1 & -f_2 & d \end{pmatrix} = 0 \text{ and}$$
$$\left. c_i = 1 \ (i = 1 \text{ or } 2 \text{ or } 3) \right\}$$

where c_i is the i^{th} element of c.

4.3 Parameter Identification using an EKF

In here we consider parameter identification under perspective projection for the dynamical system (4.1), (2.6). We assume (4.4) as the structure of the matrix $\mathcal{A}^T$ and (2.6) for the matrix $\mathcal{C}$. The dynamics is discretized with sampling time T as

$$\mathcal{X}_{k+1} = \mathcal{A}_d \mathcal{X}_k,\ z_{1k} = x_{1k}/x_{3k},\ z_{2k} = x_{2k}/x_{3k} \quad (4.6)$$

where $\mathcal{X}_k = (x_{1k}\ x_{2k}\ x_{3k}\ x_{4k})^T$. $\mathcal{A}_d$ is a suitable discretized matrix of (4.4) and its detailed structure is omitted. We define the three observation variables given by

$$\bar{x}_k = x_{1k}/x_{3k}, \quad \bar{y}_k = x_{2k}/x_{3k}, \quad \bar{z}_k = x_{4k}/x_{3k}.$$

The parameters to be estimated are $a_{11}, \cdots, a_{23}, b_1, b_2, -f_1, -f_2, -f_3$. We consider the state vector θ_k to be given by

$$\theta_k = (a_{11}\ \ldots\ a_{23}\ b_1\ b_2\ -f_1\ -f_2\ -f_3$$
$$\bar{x}_{1k}\ \bar{y}_{1k}\ \bar{z}_{1k}\ \ldots\ \bar{x}_{nk}\ \bar{y}_{nk}\ \bar{z}_{nk})^T \in \mathbb{R}^{3n+11}$$

where a set of n points are observed with coordinates $(\bar{x}_{ik}\ \bar{y}_{ik}\ \bar{z}_{ik})$ for $i = 1, \cdots, n$. It follows from (4.6) that the observation values are $\bar{x}_{ik}$ and $\bar{y}_{ik}$. Let $\bar{x}_{ik}$ and $\bar{y}_{ik}$ be observed with an additive noise, and let $\mathcal{Z}_k \in \mathbb{R}^{2n}$ be the observation vector. Then we define the dynamical system as (3.8), (3.9) where $\{w_k\} \in \mathbb{R}^{2n}$ and $H = [O_{2n,11}\ \Theta_{2n,3n}] \in \mathbb{R}^{2n\times(3n+11)}$. $\Theta_{2n,3n}$ is described by $0, 1$ and its structure is also omitted. An an EKF for the dynamical system (4.4) is obtained similar to that given by (3.10).

Simulation is performed for the following parameters. Let parameters in (4.4) be given by

$$(a_{11},\ a_{12},\ a_{13},\ a_{21},\ a_{22},\ a_{23},\ b_1,\ b_2,\ -f_1,\ -f_2,\ -f_3)$$
$$= (-0.2,\ 0.1,\ 0.3,\ 0.5,\ 0.4,\ -0.3,$$
$$-0.1,\ -0.5,\ -0.2,\ -0.1,\ -0.4).$$

We set the sampling period $T = 0.001$, and the number of points is taken as $n = 3$ and

$$(\bar{x}_{10}\ \bar{y}_{10}\ \bar{z}_{10}) = (0.2\ 0.1\ -0.2),$$
$$(\bar{x}_{20}\ \bar{y}_{20}\ \bar{z}_{20}) = (0.3\ 0.2\ 0.1),$$
$$(\bar{x}_{30}\ \bar{y}_{30}\ \bar{z}_{30}) = (-0.1\ 0.3\ -0.1).$$

Let the noise variance be chosen to be $R = \sigma^2 I_6$ with $\sigma = 0.0001$, respectively. Now using the EKF denoted by (3.10), parameters in (4.4) have been estimated and plotted in Fig. 2. Let the dotted lines denote the true values and let the solid lines indicate the estimated values. The simulation of the EKF is shown to be close to the true value. It follows from Fig. 2 that parameters are estimated with quite an accuracy.

5 Conclusion

In this paper, we have introduced canonical forms for perspective dynamical systems under the action of a perspective group assuming both unknown camera calibration case and known camera calibration case. In the former case we show that the problem is related to the classical Kronecker indices. We also observe in this case that because the calibration parameters are not known apriori the associated perspective group describes orbits of "large" dimension. Hence the number of identifiable parameters are quite small (7, 5 and 9 in our examples) compared to the total number of parameters. In the latter case, when the calibration parameters are known, 11 of the possible 16 parameters under perspective projection are identifiable and has been identified in our example. We parameterize orbits of the perspective dynamical systems, and show via simulation that the parameters can be identified using an EKF. In this way, we provide a geometric framework to study the problem of parameter identification of a linear dynamic system with perspective and orthographic observations.

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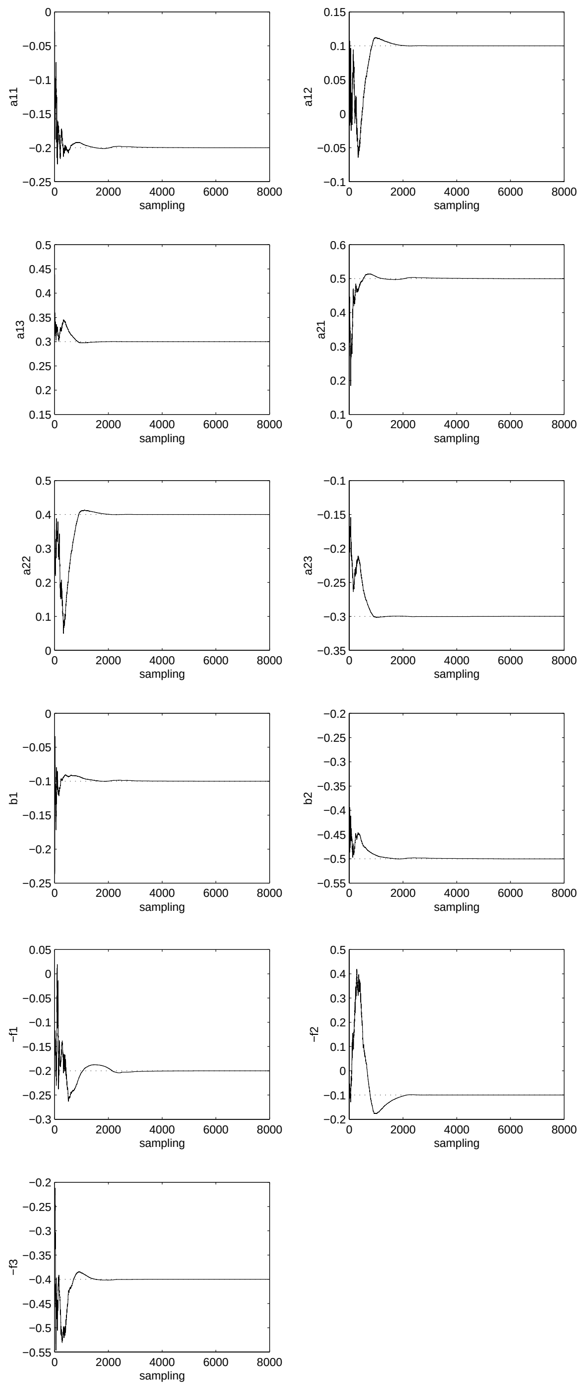

Figure 2: Estimation of parameters in (4.4)

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