Frequency Domain Curve Fitting: A Global Approach

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Abstract

We consider the problem of fitting fixed order rational SISO transfer functions to frequency response data using a least squares criterion. The main contribution is showing how this problem can be solved semi-globally by transforming it into a bilinear matrix inequality (BMI) problem. The BMI problem is in turn solved by a branch-and-bound approach. An application is given to 'linearization' of a complex dynamic multi-phase flow simulator.

1 Introduction

Fitting rational transfer functions to transfer function- or frequency response data has applications in several engineering fields, e.g. control engineering and signal processing. There are deterministic and stochastic approaches to this problem as surveyed in [1]. In the deterministic case one, essentially, does not explicitly characterize and take into account the (possible) noise associated with the data, while in the stochastic case one does. This paper presents a deterministic approach. We will consider a least squares criterion for measuring the fit and we will consider SISO systems only. There exists also work on using other criterions and/or dealing with MIMO problems, e.g. [2], [3], and [4].

Now, the problem of finding fixed order rational transfer functions fitting the data according to a least squares criterion on the magnitude of the errors, is generally nonconvex in all known parameterizations of the transfer function. Except from [5], have only non-global and local approaches been pursued for solving this problem, see [1] and [6]. (By non-global it is meant that the result, if any, may not even be a local minimum of the least squares criterion.) In [5], however, there is no global optimality guarantee. Herein, we provide a global approach with a semi-global optimality guarantee. The approach is to transform the problem into a bilinear matrix inequality (BMI) problem which is solved using as a basis a branch-and-bound algorithm given in [7].

The outline of the paper is as follows: Firstly, the problem is formally stated and we show how to transform it into a BMI problem. Then we develop the lower- and upper bounding problems associated with the branch-and-bound strategy, and discuss some extensions to and variations on the original problem. An application of the proposed algorithm to linearization of a complex dynamic multi-phase flow simulator is given before a concluding summary and conclusion section.

2 Problem statement

Given the transfer function data

$$\begin{aligned} h(j\omega_1) &= \operatorname{Re} h(j\omega_1) + j \operatorname{Im} h(j\omega_1) =: a_1 + jb_1 \\ &\vdots \\ h(j\omega_N) &= \operatorname{Re} h(j\omega_N) + j \operatorname{Im} h(j\omega_N) =: a_N + jb_N \end{aligned}$$

find, for given nonnegative n and d, the unknowns among $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_n, \hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d$, such that for

$$\hat{h}(s) := \frac{\hat{n}(s)}{\hat{d}(s)} := \frac{\hat{\alpha}_0 + \hat{\alpha}_1 s + \ldots + \hat{\alpha}_n s^n}{\hat{\beta}_0 + \hat{\beta}_1 s + \ldots + \hat{\beta}_d s^d}$$

the sum of squared errors

$$\mathrm{LS}_1(\hat{h}) := \sum_{i=1}^{N} \left| h(j\omega_i) - \hat{h}(j\omega_i) \right|^2$$

is minimized. This is generally a nonconvex nonlinear least squares problem. For a review on non-global and local approaches to this problem we refer to [1] and [6]. In the attempted global approach in [5], the problems of estimating the numerator and denominator are decoupled. The denominator is found first by solving a nonlinear programming problem locally, and then the numerator is found by solving a linear least squares problem that is parameterized by the denominator. The nonlinear programming problem associated with computing the denominator is, however, not proved to be convex and hence generally a non-global solution will be obtained. Next, we look into how this optimization problem can be approached globally using a branch-and-bound approach. For notational simplicity, we assume that all $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_n$, $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_d$ are unknown.

3 Transformation to a BMI problem

In the present context it is important to realize that any optimal solution to the least squares problem posed above is indeed equivalent to a solution that satisfies: given any $k \in \{0, \ldots, d\}$, then for all $(j, i) \in \{0, \ldots, n\} \times \{0, \ldots, d\} \setminus \{k\}$

$$\hat{\beta}_i, \hat{\alpha}_j \in [-1, 1], \hat{\beta}_k \in [0, 1] \tag{1}$$

Thus, we may without loss of generality assume that finite upper ($\overline{\hat{\beta}}_i$'s) and lower bounds ($\underline{\hat{\beta}}_i$'s) on the $\hat{\beta}_i$'s are known. Knowing the ranges of the time constants and/or resonance frequencies and relative damping coefficients, tighter such bounds than $[-1, 1]$ are easily found using simple interval arithmetic. Such ranges can be determined using e.g. a Bode plot of the given transfer function data (cf. Section 6). In addition to this non-conservative assumption of known finite bounds on the $\hat{\beta}_i$'s, we need to assume that the origin of R^d is not contained in the set of allowable denominator parameters, i.e. we assume that there exists $i \in \{0, \ldots, d\}$ such that $0 \notin [\underline{\hat{\beta}}_i, \overline{\hat{\beta}}_i]$. This is needed to avoid that the lower bounding problem developed in Section 4.1 degenerates. From (1) we understand that this assumption can be satisfied while introducing arbitrarily little conservatism; let $\hat{\beta}_k \in [\epsilon, 1]$, $0 < \epsilon < 1$. In this sense the obtained solution is semi-global. Let $\mathcal{H}$ denote the resulting associated set of transfer functions.

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Now, the problem is equivalent to

$$\min_{\substack{\hat{\alpha}_0,\hat{\alpha}_1,\dots,\hat{\alpha}_n,\hat{\beta}_0,\hat{\beta}_1,\dots,\hat{\beta}_d\\ e_1,\dots,e_N}} \sum_{i=1}^{N} e_i$$
$$\text{subject to}$$
$$\begin{aligned} \left| h(j\omega_i) - \hat{h}(j\omega_i) \right|^2 &\leq e_i, \quad i=1,\dots,N \\ \underline{\hat{\beta}}_i \leq \hat{\beta}_i &\leq \overline{\hat{\beta}}_i, \quad i=0,\dots,d \end{aligned}$$

This is equivalent to

$$\min_{\substack{\hat{\alpha}_0,\hat{\alpha}_1,\dots,\hat{\alpha}_n,\hat{\beta}_0,\hat{\beta}_1,\dots,\hat{\beta}_d\\ e_1,\dots,e_N}} \sum_{i=1}^{N} e_i$$
$$\text{subject to}$$
$$\begin{aligned} \left| \frac{h(j\omega_i)\hat{d}(j\omega_i) - \hat{n}(j\omega_i)}{\hat{d}(j\omega_i)} \right|^2 &\leq e_i, \quad i=1,\dots,N \\ \underline{\hat{\beta}}_i &\leq \hat{\beta}_i \leq \overline{\hat{\beta}}_i, \quad i=0,\dots,d \end{aligned}$$

which is equivalent to

$$\min_{\alpha,\beta,e} \sum_{i=1}^{N} e_i$$
$$\text{subject to}$$
$$\left| \frac{L^1_{i,\mathrm{Re}}(\alpha,\beta) + jL^1_{i,\mathrm{Im}}(\alpha,\beta)}{L^2_{i,\mathrm{Re}}(\beta) + jL^2_{i,\mathrm{Im}}(\beta)} \right|^2 \leq e_i, \quad i=1,\dots,N$$
$$0 \leq \beta_i \leq \overline{\hat{\beta}}_i - \underline{\hat{\beta}}_i =: \overline{\beta}_i, \quad i=0,\dots,d$$

where $\alpha := \hat{\alpha}$ and $\beta := \hat{\beta} - \underline{\hat{\beta}}$, and all the L's are *affine* functions

$$\begin{aligned} L^1_{i,\mathrm{Re}}(\alpha,\beta) &: = (a_i\omega^T_{i,e,d} - b_i\omega^T_{i,o,d})\left(\beta + \underline{\hat{\beta}}\right) - \omega^T_{i,e,n}\alpha \\ L^1_{i,\mathrm{Im}}(\alpha,\beta) &: = (b_i\omega^T_{i,e,d} + a_i\omega^T_{i,o,d})\left(\beta + \underline{\hat{\beta}}\right) - \omega^T_{i,o,n}\alpha \\ L^2_{i,\mathrm{Re}}(\beta) &: = \omega^T_{i,e,d}\left(\beta + \underline{\hat{\beta}}\right) \\ L^2_{i,\mathrm{Im}}(\beta) &: = \omega^T_{i,o,d}\left(\beta + \underline{\hat{\beta}}\right) \end{aligned}$$

and

$$\begin{aligned} &e := (e_1,\dots,e_N) \\ &\hat{\alpha}^T := (\hat{\alpha}_0,\dots,\hat{\alpha}_n), \hat{\beta}^T := (\hat{\beta}_0,\dots,\hat{\beta}_d) \\ &\omega^T_{i,e,l} := (1,0,-\omega_i^2,0,\omega_i^4,0,\dots,(-1)^{\frac{l-1}{2}}\omega_i^{l-1},0),\ l \text{ odd} \\ &\omega^T_{i,e,l} := (1,0,-\omega_i^2,0,\omega_i^4,0,\dots,(-1)^{\frac{l}{2}}\omega_i^{l}),\ l \text{ even} \\ &\omega^T_{i,o,l} := (0,\omega_i,0,-\omega_i^3,0,\omega_i^5,\dots,(-1)^{\frac{l-1}{2}}\omega_i^{l}),\ l \text{ odd} \\ &\omega^T_{i,o,l} := (0,\omega_i,0,-\omega_i^3,0,\omega_i^5,\dots,(-1)^{\frac{l}{2}-1}\omega_i^{l-1},0),\ l \text{ even} \end{aligned}$$

This follows since we have, for both $(c,\gamma,m) \in \left\{(\hat{n},\hat{\alpha},n),(\hat{d},\hat{\beta},d)\right\}$, that when m is odd

$$c(j\omega) = \sum_{k=0}^{\frac{m-1}{2}} (-1)^k\omega^{2k}\gamma_{2k} + j\sum_{k=0}^{\frac{m-1}{2}} (-1)^k\omega^{2k+1}\gamma_{2k+1}$$

while when m is even

$$c(j\omega) = \sum_{k=0}^{\frac{m}{2}} (-1)^k\omega^{2k}\gamma_{2k} + j\sum_{k=0}^{\frac{m}{2}-1} (-1)^k\omega^{2k+1}\gamma_{2k+1}$$

Thus, the problem is equivalent to

$$\min_{\alpha,\beta,e} \sum_{i=1}^{N} e_i$$
$$\text{subject to}$$
$$\begin{aligned} L^1_{i,\mathrm{Re}}(\alpha,\beta)^2 + L^1_{i,\mathrm{Im}}(\alpha,\beta)^2 &\leq e_iL_i(\beta),\ i \in \{1,\dots,N\}, \\ 0 &\leq \beta_j \leq \overline{\beta}_j,\ j \in \{0,\dots,d\} \end{aligned}$$

where $L_i(\beta) := L^2_{i,\mathrm{Re}}(\beta)^2 + L^2_{i,\mathrm{Im}}(\beta)^2$. Using that a matrix is positive semi-definite if and only if all principal submatrices have nonnegative determinants, we get that this is equivalent to a problem of minimizing a linear cost function subject to some *bilinear* matrix inequalities (BMIs), more precisely

$$\min_{\alpha,\beta,e} \sum_{i=1}^{N} e_i \tag{2}$$
$$\text{subject to } \forall i \in \{1,\dots,N\}$$
$$\begin{bmatrix} e_i & L^1_{i,\mathrm{Re}}(\alpha,\beta) & 0 \\ L^1_{i,\mathrm{Re}}(\alpha,\beta) & L_i(\beta) & L^1_{i,\mathrm{Im}}(\alpha,\beta) \\ 0 & L^1_{i,\mathrm{Im}}(\alpha,\beta) & e_i \end{bmatrix} \geq 0$$
$$0 \leq \beta_j \leq \overline{\beta}_j,\ j \in \{0,\dots,d\}$$

As a summary we have now transformed the least squares fitting problem into a BMI problem. Many control engineering problems can be formulated as linear matrix inequality (LMI) problems, see e.g. [8], [9], and [10], or BMI problems, see e.g. [10].

4 Solving the BMI problem

Attention is now given to solving (2) to global optimality using a branch-and-bound based algorithm developed in [7] as a basis. To this end, we need to lower- and upper bound this problem over hyper rectangles contained in $[0,\overline{\beta}]$. Next, we will show that this can be done by concentrating on the $(2,2)$-element in the matrix in (2). First, we consider the lower bounding problem.

4.1 LMI lower bounding

The $(2,2)$ element consists of quadratic and bilinear terms.

$$L_i(\beta) = \left(\beta + \underline{\hat{\beta}}\right)^T D_i \left(\beta + \underline{\hat{\beta}}\right)$$

where $D_i := \omega_{i,e,d}\omega^T_{i,e,d} + \omega_{i,o,d}\omega^T_{i,o,d}$. Let $D(i,j)$ denote the element located at the ith row and jth column of a matrix D. Thus

$$\begin{aligned} L_i(\beta) &= 2\beta^T D_i\underline{\hat{\beta}} + \underline{\hat{\beta}}^T D_i\underline{\hat{\beta}} \\ &+ \sum_{k=0}^{d} \beta_k^2 D_i(k+1,k+1) + 2\sum_{(l,m)=(0,1),l<m}^{(d-1,d)} \beta_l\beta_m D_i(l+1,m+1) \end{aligned}$$

Now, define the affine function

$$\begin{aligned} L_i^z(z,\beta) &:= 2\beta^T D_i\underline{\hat{\beta}} + \underline{\hat{\beta}}^T D_i\underline{\hat{\beta}} + \sum_{k=0}^{d} z_k D_i(k+1,k+1) \\ &+ 2\sum_{k=d+1}^{N_{dp}-1} z_k D_i(\mathrm{EN}_d^{-1}(k)_1 + 1, \mathrm{EN}_d^{-1}(k)_2 + 1) \end{aligned}$$

where $N_{dp} := \frac{(d+1)^2+(d+1)}{2}$ is the number of different products of variables appearing in $L_i(\beta)$. The inverse element number mapping EN_d^{-1} (or, `ENinv`) for a symmetric matrix is defined by the following pseudo code:

```
for k=0:d
  ENinv(k)=(k,k);
end
k=d+1;
for i=0:d-1, % row
  for j=i+1:d, % column
    ENinv(k)=(i,j);
    k=k+1;
  end
end
```

Clearly, problem (2) is equivalent to

$$\begin{aligned}
&\min_{\alpha,\beta,e}\sum_{i=1}^{N} e_i \qquad (3)\\
&\text{subject to } \forall i \in \{1,\dots,N\}\\
&\begin{bmatrix} e_i & L^1_{i,\mathrm{Re}}(\alpha,\beta) & 0\\ L^1_{i,\mathrm{Re}}(\alpha,\beta) & L^z_i(z,\beta) & L^1_{i,\mathrm{Im}}(\alpha,\beta)\\ 0 & L^1_{i,\mathrm{Im}}(\alpha,\beta) & e_i \end{bmatrix} \geq 0\\
&0 \leq \beta_j \leq \overline{\beta}_j,\ j \in \{0,\dots,d\}\\
&\beta_j^2 = z_j,\ j \in \{0,\dots,d\}\\
&\beta_{\mathrm{EN}_d^{-1}(k)_1}\beta_{\mathrm{EN}_d^{-1}(k)_2} = z_k,\ k \in \{d+1,\dots,N_{dp}\}
\end{aligned}$$

Notice that this is a nonconvex problem involving LMIs, a box constraint, and quadratic- and bilinear equality constraints.

We utilize the following results for lower bounding this problem over hyperrectangles contained in $[0,\overline{\beta}]$.

Lemma 1 ([7]) *Let the reals $p, q, c,$ and d satisfy $0 \leq p < q$ and $0 \leq c < d$, then*

$$p \leq x \leq q, c \leq y \leq d$$

if and only if there exists a real z satisfying

$$\begin{aligned} z &\geq \max\{cx + py - pc, dx + qy - qd\} \qquad (4)\\ z &\leq \min\{cx + qy - qc, dx + py - pd\} \qquad (5)\end{aligned}$$

Moreover, given x and y satisfying $p \leq x \leq q$ and $c \leq y \leq d$, then $z = xy$ will satisfy the inequalities (4)-(5).

Lemma 2 *Let the reals $p, q, c,$ and d satisfy $0 \leq p < q$ and $0 \leq c < d$, then the set*

$$\{(x,y,z)|(4)-(5) \text{ is satsifed}\} \qquad (6)$$

is the convex hull of the set

$$\{(x,y,z)|z = xy, p \leq x \leq q, c \leq y \leq d\} \qquad (7)$$

Proof: *The points (p,c,pc), (q,c,qc), (p,d,pd), and (q,d,qd) must be contained in the convex hull of (7). It is straightforward to see that (6) is the convex hull of these points. In addition, it follows from Lemma 1 that (7) is contained in (6). Thus, the result follows.* ■

Fact 3 *Let p and q be any reals satisfying $p < q$, then the set*

$$\{(x,z)| \begin{bmatrix} z & x\\ x & 1\end{bmatrix} \geq 0, (p+q)x - pq \geq z\}$$

is the convex hull of the set $\{(x,z)|z = x^2, p \leq x \leq q\}$.

From the above we understand that the tightest convex relaxation of problem (2) over a box $[\beta^l,\beta^u] \subseteq [0,\overline{\beta}]$ is given by the LMI problem

$$\begin{aligned}
&(\tilde{\alpha},\tilde{\beta},\tilde{e},\tilde{z}) := \arg\min_{\alpha,\beta,e,z}\sum_{i=1}^{N} e_i\\
&\text{subject to } \forall j \in \{1,\dots,N\}\\
&\begin{bmatrix} e_j & L^1_{j,\mathrm{Re}}(\alpha,\beta) & 0\\ L^1_{j,\mathrm{Re}}(\alpha,\beta) & L^z_j(z,\beta) & L^1_{j,\mathrm{Im}}(\alpha,\beta)\\ 0 & L^1_{j,\mathrm{Im}}(\alpha,\beta) & e_j \end{bmatrix} \geq 0\\
&\begin{bmatrix} z_k & \beta_k\\ \beta_k & 1\end{bmatrix} \geq 0, \quad k \in \{0,\dots,d\}\\
&\beta^l_k\beta_k + \beta^u_k\beta_k - \beta^l_k\beta^u_k \geq z_k, \quad k \in \{0,\dots,d\}\\
&\max\{g^1_{i,\max}, g^2_{i,\max}\} \leq z_i, \quad i = \{d+1,\dots,N_{dp}-1\}\\
&\min\{g^1_{i,\min}, g^2_{i,\min}\} \geq z_i, \quad i = \{d+1,\dots,N_{dp}-1\}\\
&g^1_{i,\max} =\\
&\beta^l_{\mathrm{EN}_d^{-1}(i)_2}\beta_{\mathrm{EN}_d^{-1}(i)_1} + \beta^l_{\mathrm{EN}_d^{-1}(i)_1}\beta_{\mathrm{EN}_d^{-1}(i)_2} - \beta^l_{\mathrm{EN}_d^{-1}(i)_2}\beta^l_{\mathrm{EN}_d^{-1}(i)_1}\\
&g^2_{i,\max} =\\
&\beta^u_{\mathrm{EN}_d^{-1}(i)_2}\beta_{\mathrm{EN}_d^{-1}(i)_1} + \beta^u_{\mathrm{EN}_d^{-1}(i)_1}\beta_{\mathrm{EN}_d^{-1}(i)_2} - \beta^u_{\mathrm{EN}_d^{-1}(i)_2}\beta^u_{\mathrm{EN}_d^{-1}(i)_1}\\
&g^1_{i,\min} =\\
&\beta^l_{\mathrm{EN}_d^{-1}(i)_2}\beta_{\mathrm{EN}_d^{-1}(i)_1} + \beta^u_{\mathrm{EN}_d^{-1}(i)_1}\beta_{\mathrm{EN}_d^{-1}(i)_2} - \beta^l_{\mathrm{EN}_d^{-1}(i)_2}\beta^u_{\mathrm{EN}_d^{-1}(i)_1}\\
&g^2_{i,\min} =\\
&\beta^u_{\mathrm{EN}_d^{-1}(i)_2}\beta_{\mathrm{EN}_d^{-1}(i)_1} + \beta^l_{\mathrm{EN}_d^{-1}(i)_1}\beta_{\mathrm{EN}_d^{-1}(i)_2} - \beta^u_{\mathrm{EN}_d^{-1}(i)_2}\beta^l_{\mathrm{EN}_d^{-1}(i)_1}
\end{aligned}$$

(The g_is are used for notational convenience only. They are substituted for in the min and max operations, and are not considered variables in the optimization.)

Observe that each of the inequalities involving the min and max operators is equivalent to two linear inequalities. Also, note that if the origin were not excluded from the allowable denominator vectors, $\hat{\beta}$, the optimal value of the lower bounding problem would degenerate to 0 independently of the problem data, and hence be useless.

4.2 LMI Upper bounding

When it comes to upper bounding the problem (2) over a box $[\beta^l,\beta^u]$ we use the fact that element $(2,2)$ is differentiable and convex. From this observation we immediately get that the following LMI problem provides an upper bound over $[\beta^l,\beta^u]$

$$\begin{aligned}
&(\alpha^*,\beta^*,e^*) := \arg\min_{\alpha,\beta,e}\sum_{i=1}^{N} e_i\\
&\text{subject to } \forall i \in \{1,\dots,N\}\\
&\begin{bmatrix} e_i & L^1_{i,\mathrm{Re}}(\alpha,\beta) & 0\\ L^1_{i,\mathrm{Re}}(\alpha,\beta) & \tilde{L}_i(\beta) & L^1_{i,\mathrm{Im}}(\alpha,\beta)\\ 0 & L^1_{i,\mathrm{Im}}(\alpha,\beta) & e_i \end{bmatrix} \geq 0\\
&\beta^l_j \leq \beta_j \leq \beta^u_j,\ j \in \{0,\dots,d\}
\end{aligned}$$

where $\tilde{L}_i(\beta) := L_i\left(\tilde{\beta}\right) + \nabla L_i\left(\tilde{\beta}\right)(\beta - \tilde{\beta})$ and $\tilde{\beta}$ is given by the solution to the lower bounding problem. This procedure can be iterated (let $\tilde{\beta} \leftarrow \beta^*$) to convergence, or halted at any iteration providing an upper bound. We refer to [7] for a more conservative but general and less computational expensive approach to upper bounding BMI problems. Note that instead of restricting the upper bounding to the box $[\beta^l,\beta^u]$ one can and should perform the upper bounding described above by only requiring $\beta \in [0,\overline{\beta}]$ (or, even without any bound on β). The point is that the resulting upper bound will guaranteed be lower than what would be obtained by merely substituting the result from the lower bounding problem and computing the fit, which also would provide an upper bounding strategy that provides global optimality guarantees.

4.3 Branching rule

Having presented the bounding part of our branch-and-bound approach, we now proceed to the branching part. The branching rule presented next is taken from [7]. Pick the box $[\beta^l, \beta^u]$ with the lowest lower bound, and branch (or, split this box) by first evaluating

$$i^* := \arg\max_i |\tilde{z}_i - \tilde{\beta}_{\mathrm{EN}_d^{-1}(i)_1} \tilde{\beta}_{\mathrm{EN}_d^{-1}(i)_2}|$$

and then create two new boxes by splitting this box by the plane

$$\beta_{\mathrm{EN}_d^{-1}(i^*)_1} = \frac{1}{2} \left(\tilde{\beta}_{\mathrm{EN}_d^{-1}(i^*)_1} + \frac{\tilde{z}_{i^*}}{\tilde{\beta}_{\mathrm{EN}_d^{-1}(i^*)_2}} \right)$$

The rationale behind choosing this branching rule, is that one is then forcing

$$\tilde{z}_i - \tilde{\beta}_{\mathrm{EN}_d^{-1}(i)_1} \tilde{\beta}_{\mathrm{EN}_d^{-1}(i)_2} \to 0$$

as the number of iterations in the branch-and-bound algorithm tends to infinity, which is what is required at the solution, cf. (3).

4.4 Branch-and-bound and complicating variables

Using the developed upper- and lower bounding LMI problems and branching rule, one can device a convergent branch-and-bound algorithm [7] providing as a result an ε-optimal solution to the fitting problem. By ε-optimal it is, herein, meant that when the algorithm terminates with a solution $\hat{h}^*$ it holds that

$$\frac{\mathrm{LS}_1(\hat{h}^*) - \min_{\hat{h} \in \mathcal{H}} \mathrm{LS}_1(\hat{h})}{\mathrm{LS}_1(\hat{h}^*)} \leq \varepsilon. \quad (8)$$

As for the details of the algorithm we refer to the LMI based BMI feasibility problem algorithm LMIBBA given in [7], a proof of finite ε-optimal termination is also provided therein.

A very important feature of LMIBBA is that the branching is done on a set of dimension equal to the number of unknowns in the denominator polynomial. This number is the so-called number of *complicating variables*, since the problem is an LMI problem if these are fixed. Having made possible, using BMIs, such a separation of the variables is important when it comes to devising branch-and-bound algorithms for solving the associated problem globally, since the curse of dimensionality generally renders a branch-and-bound approach nonviable in practice if the dimension of the set one is branching on is larger than, say, $5-10$. Thus, if one applies a branch-and-bound algorithm that branches on a set with dimension equal to $\dim(\hat{\alpha}) + \dim(\hat{\beta})$ only very low complexity transfer functions can generally be used as fitting candidates in practice.

5 Extensions and an 'inverse' criterion

5.1 Scaling and weighting

It is trivial to modify the above to the more general criterion

$$\mathrm{LS}_1(\hat{h}, W_1) := \sum_{i=1}^{N} \left| h(j\omega_i) - \hat{h}(j\omega_i) \right|^2 W_1(\omega_i)$$

where W_1 is some scalar weighting function. One could for example choose

$$W_1(\omega_i) = \frac{1}{|h(j\omega_i)|^2} \tilde{W}(\omega_i)$$

where the fraction scales the error so as to make all the errors contribute relatively equal to the sum of squares, the weighting function $\tilde{W}$ may then be used to implement knowledge of the relative goodness of the data.

5.2 Stable estimate

Often one would like the estimated $\hat{h}$ to be stable. If the estimate obtained using the approach described above does not render the estimated transfer function stable, this can be enforced by adding the following well known Liapunov conditions for stability of descriptor systems

$$A^T P + P^T A < 0, \; P^T E = E^T P \geq 0$$

where the matrices $A, E \in R^{d \times d}$ are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ & 0 & \ddots & \ddots & \\ & & \ddots & 1 & 0 \\ & & & 0 & 1 \\ -\hat{\beta}_0 & -\hat{\beta}_1 & \cdots & -\hat{\beta}_{d-2} & -\hat{\beta}_{d-1} \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & & \\ 0 & \ddots & \ddots & \\ & \ddots & 1 & 0 \\ & & 0 & \hat{\beta}_d \end{bmatrix}$$

A and E are affine in β, thus the Liapunov conditions constitutes another BMI (notice that the P matrix is unknown). The main point to notice here is that the number of complicating variables is the same as before, thus the computational time is not expected to be significantly increased by adding this stability constraint. Bounds on the elements of the Liapunov matrix P are required to obtain the associated LMI lower bounding problem. However, all these bounds can, in the present context, be taken to be the intervals $[-1, 1]$ (off diagonal elements) and $[0, 1]$ (diagonal elements) without loss of generality.

As for the lower bounding, one must include a relaxation of the Liapunov conditions analogous to the relaxation of the other BMIs. Note that in this case, the lower bounding problem may become infeasible, meaning that there are no stable estimates in the current box.

A different upper bounding strategy is also required. If the Liapunov conditions are satisfied for the $\tilde{A}, \tilde{E}$, and $\tilde{P}$ obtained from the lower bounding problem, one fixes the Liapunov matrix and goes ahead as in Section 4.2. If the Liapunov conditions are not satisfied there are two possibilities: *(i)* the estimate is unstable. In this case one may reflect the unstable poles about the imaginary axis, and find a Liapunov matrix for the new estimate and proceed as when the Liapunov conditions are satisfied; *(ii)* $\tilde{P}$ is not a Liapunov matrix. In this case one finds a Liapunov matrix and proceeds as when the Liapunov conditions are satisfied.

5.3 'Inverse' criterion

Motivated by the fact that the fewer complicating variables the better, we propose the following 'inverse' criterion

$$\mathrm{LS}_2(\hat{h}, W_2) := \sum_{i=1}^{N} \left| \frac{1}{h(j\omega_i)} - \frac{1}{\hat{h}(j\omega_i)} \right|^2 W_2(\omega_i)$$

having $\mathbf{dim}(\hat{\alpha})$ (or, order of the numerator) complicating variables. The rationale behind this is that the numerator order is typically less than the order of the denominator. The minimization of this 'inverse' criterion is done in exactly the same manner as for the standard criterion LS_1, it is essentially just to exchange $\hat{n}$ and $\hat{d}$ in the derivations above. Of course, one must also change LS_1 to LS_2 in the termination criterion (8). Note that the stability constraint should not be combined with this criterion, since it would lead to $\mathbf{dim}(\hat{\alpha}) + \mathbf{dim}(\hat{\beta})$ complicating variables. As for the relationship between the two criteria, we note that

$$\mathrm{LS}_2(\hat{h}, W_2) = \mathrm{LS}_1(\hat{h}, W_1)$$

if

$$W_2(\omega_i) = W(\omega_i) \text{ and } W_1(\omega_i) = \frac{W(\omega_i)}{\left|h(j\omega_i)\hat{h}(j\omega_i)\right|^2}$$

or if

$$W_2(\omega_i) = |h(j\omega_i)|^2 \, W(\omega_i) \text{ and } W_1(\omega_i) = \frac{W(\omega_i)}{\left|\hat{h}(j\omega_i)\right|^2}$$

for some scalar weighting function W. Considering the first case, this non-scaled weighted LS_2 criterion is equivalent to the scaled and weighted LS_1 criterion but with weight $\tilde{W} = \frac{W}{\left|\hat{h}(j\omega_i)\right|^2}$. Regarding the latter case we see that this scaled and weighted LS_2 criterion is equivalent—assuming the same weighting function W—to the scaled and weighted LS_1 criterion using $\frac{1}{\left|\hat{h}(j\omega_i)\right|^2}$ as scaling instead of $\frac{1}{|h(j\omega_i)|^2}$.

6 Example: 'Linearizing' a complex simulator

The developed fitting method have been applied to identifying a linear model of a simulated unstable gas-lifted oil well around an operating point [11]. The oil-well has been simulated using OLGA[1] which is the state-of-the-art dynamic multi-phase flow simulator. There are, however, approximately 800 states associated with the numerical solution of the associated PDEs, and thus it is hard to get a low order model for control design based on computing the associated Jacobian at an operating point. Instead we have by trial-and-error found a stabilizing controller for the chosen operating point, and thereafter performed a frequency response test in closed-loop from which we can compute the (noise-free) transfer function data of interest. (The reason we use the phrase 'transfer function data' rather than 'frequency response data' is that in this case the transfer function of interest is in fact unstable and thus the phrase 'frequency response data' would not make sense.) The transfer function was chosen to be the one from the production choke to the casing head pressure [11], but this is not really of interest herein. The main point here is to investigate the performance of the proposed algorithm on these data using the two different criteria LS_1 and LS_2 with and without scaling.

The estimator was chosen to be of 6th order with 4 zeros, or

$$\hat{h}(s) = \frac{\hat{\alpha}_0 + \hat{\alpha}_1 s + \hat{\alpha}_2 s^2 + \hat{\alpha}_3 s^3 + \hat{\alpha}_4 s^4}{\hat{\beta}_0 + \hat{\beta}_1 s + \hat{\beta}_2 s^2 + \hat{\beta}_3 s^3 + \hat{\beta}_4 s^4 + \hat{\beta}_5 s^5 + \hat{\beta}_6 s^6}$$

and ε was chosen to be 0.01. The data of the test cases are given in Table 1. The associated weights were identically equal to 1 in all four cases.

[1]OLGA is an abbreviation for OiL and GAs and is a product of Scandpower, Kjeller, Norway.

Case	Crit.	Scal.	#It	Comp.time
1	LS_1	no	6347	50 h
2	LS_1	yes	110	1 h
3	LS_2	yes	74	0.5 h
4	LS_2	no	6	3 min

Table 1: The data for the test cases

When criterion LS_1 was used the following bounds were imposed on the $\hat{\beta}_i$'s

$$\frac{1}{3.0864e5}\begin{bmatrix} 1 \\ -0.0078e3 \\ -0.1438e3 \\ -1.5320e3 \\ -5.8833e3 \\ -9.9700e3 \\ 0.100e3 \end{bmatrix} \leq \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \\ \hat{\beta}_6 \end{bmatrix} \leq \frac{1}{3.0864e5}\begin{bmatrix} 1 \\ 0.0002e5 \\ 0.0049e5 \\ 0.0778e5 \\ 0.6159e5 \\ 2.0987e5 \\ 3.0864e5 \end{bmatrix}$$

while when LS_2 was used the $\hat{\alpha}_i$'s were bounded as follows

$$\frac{1}{4.0000e3}\begin{bmatrix} 1 \\ 9.5000 \\ 28.5000 \\ 32.0000 \\ 10.0000 \end{bmatrix} \leq \begin{bmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \\ \hat{\alpha}_4 \end{bmatrix} \leq \frac{1}{4.0000e3}\begin{bmatrix} 1 \\ 0.0420e3 \\ 0.5800e3 \\ 3.0000e3 \\ 4.0000e3 \end{bmatrix}$$

These bounds were found by: *(i)* regarding $\hat{h}(s)$ as

$$K\frac{(1+T_1 s)(1+T_2 s)(1+T_3 s)(1+T_4 s)}{(1+T_5 s)(1+T_6 s)(\frac{1}{\omega_1^2}s^2 + 2\xi_1\frac{1}{\omega_1}s + 1)(\frac{1}{\omega_2^2}s^2 + 2\xi_2\frac{1}{\omega_2}s + 1)}$$

(ii) using the Bode plot to decide that

$$\begin{aligned} T_1, T_2 &\in [\frac{1}{0.5}, \frac{1}{0.1}], T_3 \in [\frac{1}{0.2}, \frac{1}{0.05}] \\ T_4 &\in [\frac{1}{2}, \frac{1}{0.5}], T_5, T_6 \in [\frac{1}{2}, \frac{1}{0.1}] \\ \omega_1 &\in [0.06, 0.1], \omega_2 \in [0.3, 0.5] \\ \xi_1 &\in [-0.3, -0.05], \xi_2 \in [0.3, 0.8] \end{aligned}$$

and *(iii)* interval arithmetic. When performing the interval arithmetic we utilized, by adding before multiplying, that $x(y+z) \subset xy + xz$ (x, y and z are intervals).

In all the test cases the upper bounding was done without iterating (cf Section 4.2). The branch-and-bound algorithm was implemented in MATLAB with the LMI-Lab in [12] as the core component on a 450 MHz PC.

Figure 1 shows the fit corresponding to case 1, we also let it serve—due to space limitations—to show the fits associated with cases 2 and 3, since those fits cannot be visually differed from the fit of case 1. The fit is slightly different for case 4, see Figure 2 (notice that the data have been scaled for numerical reasons in this case). From a computational time point of view the non-scaled LS_2 criterion (case 4) is superior in this example. In cases 1 to 3 the computational times are fairly long, in particular in case 1. Although the computational time of test case 1 is huge, it must be noted that in this case and other cases as well, the final and best fit typically does not differ very much from the temporarily best fit obtained during the first few iterations. The remaining iterations of the branch-and-bound algorithm just increases the lower bound until ε-global optimality is certified. In particular, the fit is 0.31-optimal after the first

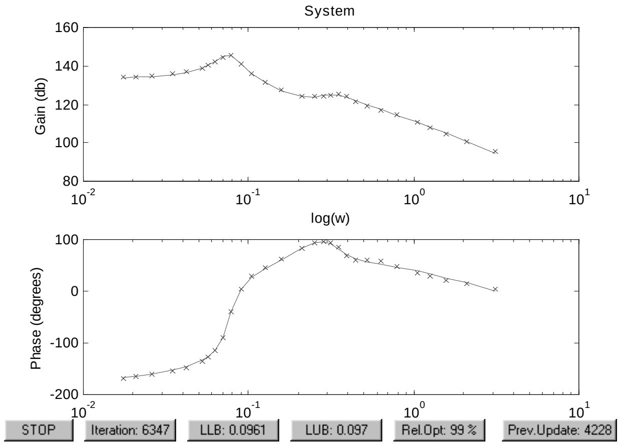

Figure 1: Test case # 1. See Table 1 for test case data.

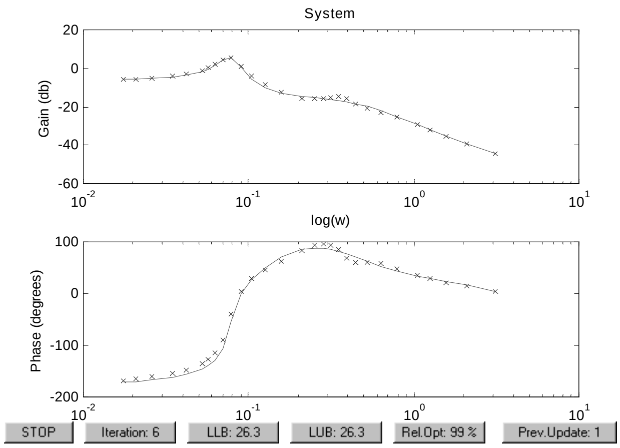

Figure 2: Test case # 4. See Table 1 for test case data.

iteration of test case 1 and can (almost) not be visually differed from the 0.01-optimal fit in Figure 1.Test case # 4. See Table 1 for test case data.

7 Summary and conclusion

In this paper we have presented a new and *global* approach to the problem of fitting fixed order SISO rational transfer functions to transfer function data. By this we mean that we can guarantee to find the best possible fit, according to the standard least squares criterion, within the model class we are considering over the considered (denominator) parameter range. To the best of the author's knowledge previous approaches in the literature can not guarantee this. Any parameter range is allowed provided the origin is not an element of it. Thus, the problem is solved in a semi-global sense.

We have shown how one can force the estimate to be stable and discussed other extensions and variations as well. Furthermore, is it illustrated how prior knowledge such as ranges of time constants and/or resonance frequencies and relative damping coefficients can be included by using simple interval arithmetic. Such ranges can often be found by considering e.g. a Bode plot of the data.

When it comes to the practical usefulness of the algorithm, more experience must be gained to be conclusive. However, it seems to work well—in the sense that convergence to an ε-optimal solution (with ε less than, say, 0.05) takes place in a reasonable amount of time—for problems having either few unknown poles (using LS_1) or few unknown zeros (using LS_2). Also, it is important to notice that fits obtained after a few iterations often are very good. This has also been the case for more complex examples.

The computational time might be reduced by changing the lower bounding problems from LMI problems to linear programming (LP) problems as described in [7]. It may also be that the problem structure lends itself to the design of much more efficient tailor made LMI/LP solvers.

Finally, it should be mentioned that it seems that the same approach can be used to solve the more general problem of finding the maximum likelihood solution, but in this case both the numerator and denominator parameters are seemingly going to be so-called complicating variables.

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