

Generalized Stability Preserving Maps

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Abstract

Matrix stability preserving maps can be used to provide a different characterization of the existence of a fixed order controller that robustly stabilizes a family of plants. In addition, stability preserving map tests can be used as the basis for robust controller synthesis/design methods. In this paper we develop additional stability preserving map tests and demonstrate their use for the robust stabilization of a physical system. Earlier work restricted the order of the controller to be one less than the order of the plant family. Here we allow controllers to have arbitrary order and show how this leads to the concept of a generalized stability preserving map. We conclude the paper with a discussion about tests for such maps.

KEYWORDS: Linear, Stability, Robustness, Synthesis

1 Introduction

Robust stabilization of families of linear time invariant plants has been a topic of investigation for several decades. Important classes of plant families that have been studied include those that consist of a finite number of systems and those that can be expressed in terms of a transfer function model with real parameter uncertainty. We know that the problem of simultaneous stabilization of a finite number of plants is a difficult one and in general elegant and computationally attractive solutions do not exist [2, 3, 9, 12, 13]. For systems with real parameters one may be able to express plant uncertainty using a special structure (e.g., multiplicative uncertainty) for which a solution can be given using H_∞ techniques [8]. However, such an approach can lead to conservative designs. Other approaches that have been suggested for systems with real parameter uncertainty include Parameter Space methods, [1], the QFT framework [10] and Finite Inclusions Theorem (FIT) design [7].

Recently, we introduced the notion of a matrix stability preserving map (SPM) [4, 5, 6] and showed how it can be used to formulate robust stabilization problems.

Not only does the concept provide new insight to the problem of robust stabilization but it leads to robust controller synthesis/design methods. We have demonstrated that the concept can be used for scalar as well as multivariable systems. In the literature one can find the notion of a *stability preserving mapping* defined in the context of Lyapunov based analysis of dynamical systems [11]. However, the matrix stability preserving maps discussed here appear to be different. Following the exposition in [11], stability preserving mappings map families of motions of a dynamical system to families of motions of another dynamical system (i.e., from one state space to another in the time domain). They are used to establish qualitative equivalences between dynamical systems. In contrast, matrix SPMs map polynomial coefficients to polynomial coefficients (i.e., frequency domain) and can be defined without any reference to dynamical systems as simply matrix properties. Furthermore, we use SPMs to develop methods for robust controller synthesis. Nevertheless, it would be worthwhile to investigate possible connections that may exist between the two concepts.

In our earlier work we began the development of the theory of stability preserving maps and presented a number of SPM tests. One of these, is the fact that an upper triangular matrix with “ones” on the main diagonal does generate a SPM. Another, is that a matrix with one stable row generates a SPM. In section 3 of this paper we continue the development of the theory by stating a new SPM test. We will confine our discussion to scalar maps but in view of [5], extensions to the multivariable case are immediate. In section 4 we apply the result to a physical system for the synthesis of a controller for robust stabilization. In section 5 we turn our attention to the robust stabilization problem where the order of the controller is not constrained. If higher order controllers are used we are led very naturally to the concept of a *generalized stability preserving map* (GSPM).

2 Stability Preserving Maps

First, let us recall the definition of a scalar stability preserving map [4]. With \mathbf{R} denoting the reals and \mathbf{R}^n the space of real n -vectors, let A be an $n \times m$ matrix

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with elements in \mathbf{R} , $\phi \in \mathbf{R}^n$, $\psi \in \mathbf{R}^m$ and $f_A : \mathbf{R}^n \rightarrow \mathbf{R}^m$ the function defined by:

$$f_A(\phi) = \phi * A = \psi \quad (1)$$

where “*” indicates vector-matrix multiplication. For notational simplicity the “*” will not be explicitly shown. The n -vector $\chi = [\chi_{n-1}, \chi_{n-2}, \dots, \chi_0]$ represents the coefficients of the degree $n - 1$ polynomial $\chi(s) = \chi_{n-1}s^{n-1} + \chi_{n-2}s^{n-2} + \dots + \chi_0$. The vector χ is called *stable* if the corresponding polynomial $\chi(s)$ has all its $n - 1$ roots in the left half complex plane (LHCP).

Definition 1 *The function f_A is called a Stability Preserving Map (SPM) if there exists some stable n -vector ϕ that is mapped to a stable m -vector $\psi = f_A(\phi) = \phi A$.*

We say that A generates the SPM f_A . In [4] we have shown that this concept plays an important role in robust stabilization. Consider the feedback system shown in Figure 1. The transfer function $P_i(s)$, $i \in \{1, 2, \dots, N\}$ is one of a finite number of order \bar{n} , strictly proper, single-input, single-output plants, $P_i(s) = \frac{n_{pi}(s)}{d_{pi}(s)}$

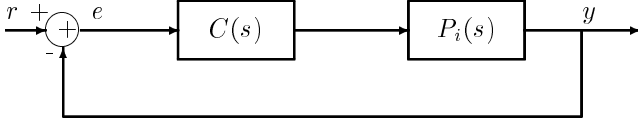


Figure 1: Unity Feedback Configuration

where $n_{pi}(s) = n_{i\bar{n}-1}s^{\bar{n}-1} + n_{i\bar{n}-2}s^{\bar{n}-2} + \dots + n_{i0}$ and $d_{pi}(s) = s^{\bar{n}} + d_{i\bar{n}-1}s^{\bar{n}-1} + \dots + d_{i0}$ are coprime for all values of i . The controller given by $C(s) = \frac{n_c(s)}{d_c(s)}$ is proper and degree $\bar{n} - 1$, with $n_c(s) = y_{\bar{n}-1}s^{\bar{n}-1} + y_{\bar{n}-2}s^{\bar{n}-2} + \dots + y_0$ and $d_c(s) = s^{\bar{n}-1} + x_{\bar{n}-2}s^{\bar{n}-2} + \dots + x_0$. The closed loop characteristic polynomials are of degree $2\bar{n} - 1$ and given by:

$$\begin{aligned} \phi_i(s) &= d_c(s)d_{pi}(s) + n_c(s)n_{pi}(s) \\ &= s^{2\bar{n}-1} + \phi_{i2\bar{n}-2}s^{2\bar{n}-2} + \dots + \phi_{i0} \end{aligned}$$

If we collect the controller coefficients in a single vector $x = [1 \ y_{\bar{n}-1} \ x_{\bar{n}-2} \ y_{\bar{n}-2} \ x_{\bar{n}-3} \ \dots \ x_0 \ y_0]$ and use \bar{n}^{th} order Sylvester Resultants we can write:

$$xS_{\bar{n}}(d_{pi}(s), n_{pi}(s)) = \phi_i, \quad i \in \{1, 2, \dots, N\}$$

For $N = 2$, if a single controller $C(s)$ (represented by x) exists that makes $\phi_1(s)$ and $\phi_2(s)$ stable then

$$\phi_1 \underbrace{S_{\bar{n}}(d_{p1}(s), n_{p1}(s))^{-1} S_{\bar{n}}(d_{p2}(s), n_{p2}(s))}_A = \phi_2 \quad (2)$$

or the function f_A is a SPM! The converse is also true, for if the matrix “ A ” defined in (2) generates a stability preserving map then the corresponding two plants are simultaneously stabilizable. Constructing a simultaneously stabilizing controller is simply done by computing $x = \phi_1 S_{\bar{n}}(d_{p1}(s), n_{p1}(s))^{-1}$. The concept of stability preserving maps thus leads very naturally to robust controller synthesis/design procedures. In the case of more than two plants a set of “ A ” matrices are generated along with the corresponding set of f_A maps. Specifically, for N plants the matrices $A_i = S_{\bar{n}}(d_{p1}(s), n_{p1}(s))^{-1} S_{\bar{n}}(d_{pi}(s), n_{pi}(s))$, $2 \leq i \leq N$. For simultaneous stabilization one needs to find a single stable polynomial that is mapped to stable polynomials by the corresponding maps. We then say that f_A is a *family* of SPM maps. The situation is similar if we have a plant transfer function which contains a (vector) parameter a . We form the set of matrices: $A_a = S_{\bar{n}}(d_{p0}(s), n_{p0}(s))^{-1} S_{\bar{n}}(d_p(s, a), n_p(s, a))$, $a \in \Omega_a$, where “0” indicates a nominal plant. For robust stabilization one needs to find a single stable polynomial that is mapped to stable polynomials by the corresponding maps (i.e., f_A is a *family* of SPM maps).

3 Another SPM Test

In [4] we proved that if A is an $n \times n$ which has some row that is a stable (degree $n - 1$) polynomial, the f_A it generates is a SPM. In [6] we showed that if several combinations of sums of rows generate a stable (degree $n - 1$) polynomial the f_A it generates is a SPM. Here we will focus on the sum of four consecutive rows.

Lemma 1 *Let A be $n \times n$ with four consecutive rows whose sum is a degree $n - 1$ stable polynomial. The corresponding f_A is a SPM.*

Proof: We consider two cases. In Case 1, the four rows in question are the first four rows of the matrix and in Case 2, any other four consecutive rows.

Case 1: Suppose that the sum of rows 1, 2, 3 and 4 of matrix A forms a degree $n - 1$ stable polynomial ψ . We need to construct a stable degree $n - 1$ polynomial that gets mapped by f_A to some stable degree $n - 1$ polynomial. Consider the polynomial:

$\phi(s) = (s + \frac{1}{q})^{n-5}(qs^4 + qs^3 + (q+2)s^2 + qs + 1)$ where q is positive, real and “large.” It is the product of stable polynomials and thus is stable. After expanding the product it can be seen that the coefficients of the terms s^{n-1} , s^{n-2} , s^{n-3} and s^{n-4} are q , $q + n - 5$, $q + 2 + n - 5 + \frac{(n-5)(n-6)}{2q}$ and $q + \frac{(n-5)(q+2)}{q} + \frac{(n-5)(n-6)}{2q} + \frac{(n-5)(n-6)(n-7)}{6q^2}$ respectively. Lower powers of s have coefficients that include terms $(1/q)^j$ where $j \geq 0$. When ϕA is formed it generates a polynomial which can be expressed as: $q\psi + \chi$ where χ has entries that

include terms $(1/q)^j$ where $j \geq 0$. By making q “large” the roots of ϕA can be made to lie arbitrarily close to the roots of ψ which is stable.

Case 2: Suppose that the sum of rows $i - 3, i - 2, i - 1$ and i ($5 \leq i \leq n$) of matrix A forms a degree $n - 1$ stable polynomial ψ . We need to construct a stable degree $n - 1$ polynomial that gets mapped by f_A to some stable degree $n - 1$ polynomial. Consider the polynomial:

$\phi(s) = (s + q)^{j-1} (s + \frac{1}{q})^{n-3-j} (s^4 + qs^3 + (q+2)s^2 + qs + q)$ where q is positive, real and “large,” and $1 \leq j \leq n - 4, i = j + 4$. It is a product of stable polynomials and thus is stable. Consider first the product $g(s) = (s + q)^{j-1} (s + \frac{1}{q})^{n-4-j}$ and focus on the coefficient of s^{n-4-j} . For large enough q it will be dominated by the term q^{j-1} . Furthermore, all other coefficients of $g(s)$ in this product will contain terms which are lower powers of q (and can be negative). When $g(s)$ is multiplied by $s^4 + qs^3 + (q+2)s^2 + qs + q$ to form $\phi(s)$, the coefficients of the terms $s^{n-1-j}, s^{n-2-j}, s^{n-3-j}$ and s^{n-4-j} will include the same q^j term (call it $\alpha(q)$), and all other coefficients will have terms with lower powers of q . When ϕA is formed it generates a polynomial which can be expressed as: $\alpha(q)\psi + \chi$ where χ has entries that include terms with lower powers of q . By making q “large” the roots of ϕA can be made to lie arbitrarily close to the roots of ψ which is stable. \square

It should be clear that many more results can be formulated that make use of the ideas expressed in the above proof. In conjunction with these results one can also make use of the following observation:

Observation Let $\phi(s) = s^n + qf(s) + g(s)$ be a degree n polynomial, $f(s)$ a degree $n - 1$ stable polynomial, $g(s)$ a polynomial of degree at most $n - 1$ and q a positive constant. For large enough q the polynomial $\phi(s)$ is stable.

This can be immediately seen from root locus arguments. For large enough q , $n - 1$ roots of $\phi(s)$ will get arbitrarily close to the roots of $f(s)$ which is stable and the n^{th} root will tend to $-\infty$. \square

4 Application to a Physical Example

In the previous section we developed a test for stability preserving maps. We also have pointed out that SPM tests lead to robust stabilization methods. We demonstrate this fact by considering a physical example. Consider the “crane” example discussed in [1]. The state space (linearized) description of the system is given by:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_\ell g}{m_c} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(m_c + m_\ell)g}{m_c \ell} & 0 \end{bmatrix} z + 1/m_c \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/\ell \end{bmatrix} u$$

The system states are: z_1 , crab position, z_2 , crab velocity, z_3 , rope angle and z_4 , rope angular velocity. The input u represents the force accelerating the crab. In this example we let the crab mass m_c be equal to 10 and the acceleration due to gravity g be equal to 10. The rope length ℓ and the load mass m_ℓ are considered to be real parameters that take values in some given region. With $a_1 = 1/\ell$ and $a_2 = m_\ell$ the parameter range Ω_a is given by $a_1 \in [1/50, 1/2], a_2 \in [1, 10]$. We assume that all states are available for feedback and consider the following synthesis problem: Does there exist a constant gain controller that robustly stabilizes the system? If one exists construct it.

First, we obtain a transfer function model for the plant family and a right matrix fraction description is given by:

$$P(s, a_1, a_2) = \underbrace{\begin{bmatrix} \frac{s^2}{10} + a_1 \\ \frac{s}{10} + a_1 s \\ -\frac{a_1 s^2}{10} \\ -\frac{a_1 s^3}{10} \end{bmatrix}}_{N(s, a_1, a_2)} \underbrace{(s^4 + a_1(a_2 + 10)s^2)^{-1}}_{d(s, a_1, a_2)} \quad (3)$$

With the constant gain controller $x = [y_1 \ y_2 \ y_3 \ y_4]$ the closed loop characteristic polynomial family is:

$$\phi(s, a_1, a_2) = xN(s, a_1, a_2) + d(s, a_1, a_2) \quad (4)$$

The synthesis question can now be rephrased: Does there exist some x that robustly stabilizes $\phi(s, a_1, a_2)$ for all parameters in Ω_a ? The family of zero order Sylvester Resultants is:

$$S(d(s, a_1, a_2), N(s, a_1, a_2)) = \begin{bmatrix} 1 & 0 & a_1(a_2 + 10) & 0 & 0 \\ 0 & 0 & 1/10 & 0 & a_1 \\ 0 & 1/10 & 0 & a_1 & 0 \\ 0 & 0 & -a_1/10 & 0 & 0 \\ 0 & -a_1/10 & 0 & 0 & 0 \end{bmatrix}$$

We are asking whether a fixed x exists such that:

$$xS(d(s, a_1, a_2), N(s, a_1, a_2)) = \psi(a_1, a_2) \quad (5)$$

the polynomial ψ is stable for all parameter values. Let $a_{10} = 1/10$ and $a_{20} = 1$ be considered the “nominal”

values, S_0 be the corresponding Resultant and let $A = S_0^{-1}S(d(s, a_1, a_2), N(s, a_1, a_2))$:

$$A = \begin{bmatrix} 1 & 0 & a_1 a_2 - a_1 & 0 & 0 \\ 0 & 10a_1 & 0 & 0 & 0 \\ 0 & 0 & 10a_1 & 0 & 0 \\ 0 & 1 - 10a_1 & 0 & 10a_1 & 0 \\ 0 & 0 & 1 - 10a_1 & 0 & 10a_1 \end{bmatrix} \quad (6)$$

Equivalently, we are asking whether the matrix A generates the family of stability preserving maps f_A . In particular, whether there exists a *single* degree four polynomial ϕ such that ϕA is stable for all values of the parameters. We approach this synthesis problem using the tools developed in the previous section. Let r_i be the i^{th} row of A . One can verify that the degree three polynomial formed by $r_2 + r_3 + r_4 + (1/10)r_5$ is equal to:

$$f(s, a_1) = s^3 + (9a_1 + 1/10)s^2 + 10a_1 s + a_1 \quad (7)$$

and is stable (via root locus arguments) for all for all values of the uncertain parameters (a_2 is absent). Now consider the polynomial $[1 \ q \ q+2 \ q \ 1/10q]$ and its image under f_A (i.e., $[1 \ q \ q+2 \ q \ 1/10q]A$). The polynomial $[1 \ q \ q+2 \ q \ 1/10q]$ is degree four and stable for all positive q . Its image has the form: $\psi(s, a_1, a_2) = s^4 + qf(s, a_1) + g(s, a_1, a_2)$, for some $g(s, a_1, a_2)$ of degree at most three. Recall from the Observation that such a polynomial is robustly stable for “large enough” q . When $q = 1000$ the image is $s^4 + 1000s^3 + (a_1 a_2 + 9019a_1 + 100)s^2 + 10000a_1 s + 1000a_1$. Using value set ideas and the special structure of the uncertainty we see that at frequencies, $\omega_1 = 1/10, \omega_2 = 1/3, \omega_3 = 5, \omega_4 = 100$, the value set is in the $1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}$ and 4^{th} quadrants respectively. The Finite Inclusions Theorem [7] then guarantees that the image family is robustly stable. Therefore, given the plant family, we have identified a specific polynomial that is mapped by A to stable polynomials making f_A a SPM family. This implies that a constant state feedback controller exists that robustly stabilizes the plant family. The controller is constructed as:

$$\begin{aligned} x &= \psi(a_{10}, a_{20})S_0^{-1} \\ &= [1 \ 1000 \ 10000 \ -90090 \ 0] \end{aligned} \quad (8)$$

We summarize the three-step synthesis procedure:

Robust Controller Synthesis

- Form the matrix family A from plant information and show that it generates a stability preserving map. In this example this is accomplished by recognizing that an appropriate weighted sum of its rows forms a robustly stable polynomial.

- Use the “weights” (parameterized by q) to construct a stable polynomial which for large q is guaranteed to map via A to stable polynomials. Since f_A is a SPM this implies that a controller exists that robustly stabilizes the family.
- Construct the robustly stabilizing controller using matrix operations.

5 Generalizations of the SPM concept

In section 2 we pointed out how the robust stabilization problem can be approached using the SPM concept. For plant families of order \bar{n} a controller of order $\bar{n} - 1$ is employed. Well, we would like to have the ability to employ the SPM formulation for both lower order and higher order controllers. In the first case, even though we have fewer “degrees of freedom” we may still be able to robustly stabilize. When we are able to use higher order controllers we would like to be able to “capitalize” on the additional degrees of freedom. We will address both cases and begin with controllers of order q lower than $\bar{n} - 1$. The Sylvester Resultant will be order $q + 1$, nonsquare, having fewer rows than columns and being full row rank. The characteristic polynomial relationship in coefficient space becomes:

$$xS_{q+1}(d_{pi}(s), n_{pi}(s)) = \phi_i, \quad i \in \{1, 2, \dots, N\}$$

For $i = 1$ if we multiply both sides on the right by $S_{q+1}(d_{p1}(s), n_{p1}(s))^T$ we have:

$$\begin{aligned} xS_{q+1}(d_{p1}(s), n_{p1}(s))S_{q+1}(d_{p1}(s), n_{p1}(s))^T &= \\ \phi_1 S_{q+1}(d_{p1}(s), n_{p1}(s))^T & \end{aligned}$$

The matrix multiplying x is now square and invertible. One can “solve” for x and write:

$$\begin{aligned} x &= \phi_1 S_{q+1}(d_{p1}(s), n_{p1}(s))^T \\ & \quad (S_{q+1}(d_{p1}(s), n_{p1}(s))S_{q+1}(d_{p1}(s), n_{p1}(s))^T)^{-1} \end{aligned}$$

This controller will simultaneously stabilize two plants if

$$\underbrace{S_{q+1}(d_{p1}, n_{p1})^T (S_{q+1}(d_{p1}, n_{p1})S_{q+1}(d_{p1}, n_{p1})^T)^{-1} S_{q+1}(d_{p2}, n_{p2})}_{A}$$

A generates a SPM! We again see the emergence of the concept of a stability preserving map which leads to the result:

Theorem 1 *Two strictly proper, degree \bar{n} plants $P_1(s) = \frac{n_{p1}(s)}{d_{p1}(s)}$ and $P_2(s) = \frac{n_{p2}(s)}{d_{p2}(s)}$ can be simultaneously stabilized by a proper degree q controller ($q < \bar{n} - 1$) if f_A is a stability preserving map where A is the matrix $S_{q+1}(d_{p1}, n_{p1})^T (S_{q+1}(d_{p1}, n_{p1}) S_{q+1}(d_{p0}, n_{p0})^T)^{-1} S_{q+1}(d_{p2}, n_{p2})$*

Notice that this is only a sufficient condition. In the case when we have more than two plants or the plant family involves parametric uncertainty we would again need to employ the concept of SPM families. Specifically, consider the strictly proper plant family $P(s, a)$ where a is a vector parameter that takes values in some set Ω_a . Let “0” indicate a nominal plant. Consider the family of maps $f_{A(a)}$ where $A(a) =$

$S_{q+1}(d_{p0}, n_{p0})^T (S_{q+1}(d_{p0}, n_{p0}) S_{q+1}(d_{p0}, n_{p0})^T)^{-1} S_{q+1}(d_{p(s), a}, n_{p(s), a})$. The plant family will be robustly stabilized by a degree q controller if $f_{A(a)}$ is a SPM family (i.e., there is one stable polynomial ϕ that is mapped to stable polynomials by $f_{A(a)}$).

Let us now focus our attention to the case when controllers of higher order than $\bar{n} - 1$ are used. The Sylvester Resultant will be nonsquare, have more rows than columns and full column rank. Here more than one controller can assign the same characteristic polynomial. With perhaps some reordering of rows the characteristic polynomial relationship for the first plant can now be written as:

$$x S_{q+1}(d_{p1}(s), n_{p1}(s)) = [x_t \ x_b] \begin{bmatrix} S_{1t} \\ S_{1b} \end{bmatrix} = \phi_1$$

where S_{1t} is invertible. We can then write:

$$[x_t \ x_b] \begin{bmatrix} I \\ S_{1b} S_{1t}^{-1} \end{bmatrix} = \phi_1 S_{1t}^{-1}$$

where $x_t = \phi_1 S_{1t}^{-1} - x_b S_{1b} S_{1t}^{-1}$. If we now consider the characteristic polynomial relationship for the second plant we have:

$$[x_t \ x_b] \begin{bmatrix} S_{2t} \\ S_{2b} \end{bmatrix} = \phi_2$$

which can be expressed as:

$$\underbrace{\phi_1 S_{1t}^{-1} S_{2t}}_A + x_b \underbrace{[S_{2b} - S_{1b} S_{1t}^{-1} S_{2t}]}_B = \phi_2$$

$$\phi_1 A + x_b B = \phi_2 \quad (9)$$

Therefore, there exists a matrix $\begin{bmatrix} A \\ B \end{bmatrix}$, that maps the vector $[\phi_1 \ x_b]$ (ϕ_1 stable and x_b arbitrary) to a stable vector ϕ_2 . In other words, for the simultaneous stabilization of the two plants the map $f_{(A,B)} : \mathbf{R}^n \times \mathbf{R}^l \rightarrow \mathbf{R}^n$ defined by:

$$f_{(A,B)}(\phi, x) = [\phi \ x] * \begin{bmatrix} A \\ B \end{bmatrix} = \psi \quad (10)$$

is a *Generalized Stability Preserving Map* (GSPM). In order to capture all the degrees of freedom afforded by the use of higher order controllers we are “forced” to generalize the notion of a stability preserving map. The result is quite intuitive as we see how the additional degrees of freedom in the controller translate into the presence of “free” parameters (e.g., the vector x) which can potentially be used to generate a stable polynomial ψ . One can show that this is a necessary condition for simultaneous stabilization as well. Suppose that the map in (9) is a GSPM. Then there exist ϕ, x, ψ (ϕ stable degree n , x arbitrary and ψ stable degree n) such that:

$$\phi S_{1t}^{-1} S_{2t} + x [S_{2b} - S_{1b} S_{1t}^{-1} S_{2t}] = \psi$$

Equivalently,

$$\underbrace{[(\phi - x S_{1b}) S_{1t}^{-1} \ x]}_C \begin{bmatrix} S_{2t} \\ S_{2b} \end{bmatrix} = \psi$$

The controller C stabilizes system 2. It also stabilizes system 1 since:

$$[(\phi - x S_{1b}) S_{1t}^{-1} \ x] \begin{bmatrix} S_{1t} \\ S_{1b} \end{bmatrix} = \phi$$

We have thus proved:

Theorem 2 *Two strictly proper, degree \bar{n} plants $P_1(s) = \frac{n_{p1}(s)}{d_{p1}(s)}$ and $P_2(s) = \frac{n_{p2}(s)}{d_{p2}(s)}$ can be simultaneously stabilized by a proper degree q controller ($q > \bar{n} - 1$) if and only if $f_{A,B}$, $A = S_{1t}^{-1} S_{2t}$, $B = S_{2b} - S_{1b} S_{1t}^{-1} S_{2t}$, is a generalized stability preserving map ($S_{q+1}(d_{pi}(s), n_{pi}(s)) = \begin{bmatrix} S_{it} \\ S_{ib} \end{bmatrix}$, $i \in \{1, 2\}$).*

For a strictly proper plant family $P(s, a)$ where a is a vector parameter that takes values in some set Ω_a we proceed using GSPM families. In particular, the plant family will be robustly stabilized if $f_{A(a), B(a)}$, $A(a) = S_{0t}^{-1} S_{at}$, $B = S_{ab} - S_{0b} S_{0t}^{-1} S_{at}$ is family of GSPMs (i.e., a stable ϕ and a specific x are mapped by $f_{A(a), B(a)}$ to stable polynomials).

These results extend the formulation for robust stabilization to the case when controllers have arbitrary order. In order to use these results as the basis for a robust synthesis procedure we need to develop GSPM tests. Proofs of results presented earlier can be modified to yield GSPM tests. Here we just give one example.

Lemma 2 Let A, B be $n \times n$ and $\ell \times n$ matrices respectively. Let the sum of the i^{th} row of A and the j^{th} row of B be a degree $n - 1$ stable polynomial. The corresponding $f_{A,B}$ is a GSPM.

Proof: Suppose that the sum of the polynomial which corresponds to the i^{th} row, $i \in \{1, 2, \dots, n\}$, of matrix A and the j^{th} row, $j \in \{1, 2, \dots, \ell\}$, of matrix B is the stable degree $n - 1$ polynomial ψ . We will construct a stable polynomial ϕ and an ℓ -vector x that get mapped by $f_{A,B}$ to a stable polynomial. Let $x = [0 \dots 0 q^{i-1} 0 \dots 0]$ be the vector with q^{i-1} in the j^{th} component. Let $\phi(s) = (s + q)^{i-1} (s + \frac{1}{q})^{n-i}$ where q is positive, real and “large.” This means that:

$$\begin{aligned} \phi(s) &= (s + q)^{i-1} (s + \frac{1}{q})^{n-i} \\ &= (s^{i-1} + (i-1)qs^{i-2} + \dots + q^{i-1}) \\ &\quad (s^{n-i} + \dots + (\frac{1}{q})^{n-i}) \end{aligned}$$

Consider the coefficient of s^{n-i} in this polynomial. It will be dominated by the term q^{i-1} for q large enough. Furthermore, all other coefficients will contain lower powers of q (they can be negative). When the expression $\phi A + xB$ is formed it can be expressed as: $q^{i-1}\psi + \chi$ where χ has entries with lower powers of q than $i - 1$ (they can be negative). By making q larger the roots of $\phi A + xB$ can be made to lie arbitrarily close to the roots of ψ which is stable. \square

6 Conclusions

In this paper we presented more tests for checking whether some matrix A generates a stability preserving map and then used them for the robust stabilization of a physical system. We extended the formulation to cover the case of controllers of arbitrary order and showed how the notion of stability preserving maps can be generalized. We also briefly discussed GSPM tests. The work can be extended to stability regions other than the left half complex plane so one can talk about \mathcal{D} -stability preserving maps. By allowing matrices to have complex entries the analysis can be carried out over the field of complex numbers which has ramifications in robust performance synthesis/design.

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