

# Analysis of Robust Pole Clustering in a Good Ride Quality Region for Uncertain Matrices

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## Abstract

This paper presents a general analysis of robust pole clustering in a good ride quality region of aircraft, a specific non- $\Omega$ -transformable region, for uncertain matrices. The region is an intersection of a ring and a horizontal strip, located on the left half-plane. From experiments, it is known that the control system with poles located in this specific region provides a good ride quality for aircraft. The paper applies Rayleigh principle along the norm theory to analyze robust pole clustering within this good ride quality region since the generalized Lyapunov theory is not valid for non- $\Omega$ -transformable regions. The mainly concerned uncertainties are unstructured uncertainties. A simple extension of the results for structured uncertainties is also provided. Two examples illustrate the results for a perturbed closed-loop system matrix of F16 aircraft approximation model. The results are useful for robustness analysis and, especially, analysis of robust good ride quality of aircraft, shuttles, and space station via robust pole clustering in this region for their system matrices.

**Index terms:** Robust pole clustering, robust performance, uncertain systems, analysis, flight quality.

## 1. Introduction

Combining robust stability and robust performance is a fundamental control task and also a challenging task in control area. Due to the practical needs, robust control has been an important research area in control engineering. The main reason is that control systems are in face of a changing environment, model errors, approximations, perturbations, disturbances, and so on. A summary of robust control may be found in reprint volumes and some excellent books (such as Dorato [5-7], Barmish [3], Bhattacharyya et al. [4], Zhou et. al. [20], and others).

It is known by experiments that the flight control system with its poles staying in a specific approximate region bounded by two arcs and two lines in the left-half complex plane provides a good ride quality for flight [8, 10]. This region is called a Good Ride Quality Region (GRQR) in the literature [10]. It is well known that the system closed-loop poles determine the system stability and nature and dominate the system transient responses in the time domain. Therefore, if the aircraft control systems have their closed-loop poles robustly stayed within this specific region, then the aircraft will have a robust good ride quality performance, including robust stability. This is a motivation of this paper.

The control system may have state feedback, output feedback and observer to complete its control principles. The poles of the closed-loop system, which determine the system behavior, belong to the closed-loop system matrix. Thus, we need to consider robust pole clustering of an uncertain closed-loop system matrix. In view of few results about robust pole

clustering in the GRQR in the literature [14], we discuss the robust pole clustering in the GRQR for uncertain matrices with unstructured uncertainties here first. The discussion about the robust pole clustering in GRQR for control systems with state feedback, output feedback and observer will be followed in other papers.

Concerning robust pole clustering, the research can be carried out via the closed-loop system characteristic polynomial in the frequency domain, e.g., [2-4] and the closed-loop system matrix in the time domain [9, 11, 14-19, 21]. For example, there is research work [3, 4] via Karitonov's theorem [12] and edge theorem [2] in frequency domain. In time domain, most work is via generalized Lyapunov theorem [11, 16, 19], initially developed by Gutman and Jury [10] for the nominal matrix root clustering in  $\Omega$ -transformable regions. There is some new work via Rayleigh principle [14, 16, 17] and eigenvectors [15, 18] for  $\Omega$ -transformable and non- $\Omega$ -transformable regions in the time domain. Here, we focus on the time-domain approaches.

However, the concerned good ride quality region for aircraft belongs to a non- $\Omega$ -transformable region, where the General Lyapunov Theory is not valid, in the literature [10, 13]. Wang and Shieh [17] were the first to provide a general theory for analysis of robust pole clustering in general  $\Omega$ -regions, including both  $\Omega$ -transformable and non- $\Omega$ -transformable regions. Recently, Wang and Yedavalli [18] first discussed robust pole clustering in general  $\Omega$ -region via eigenvectors. This paper will specifically discuss robust pole clustering in the GRQR for uncertain matrices via Rayleigh principle approach developed first in [16, 17].

The considered region in the paper is an extended/generalized GRQR. It is an intersection of a ring with a center at  $c$  on the real axis and a horizontal strip on the left half-plane, taking the good ride quality region as a special case as  $c=0$ . It is noticed that there is much discussion on robust pole clustering in  $\Omega$ -transformable regions, but not for non- $\Omega$ -transformable regions, such as a GRQR. Therefore, comparison will focus on only the results of this paper, not the previous results in view of their non-validity for non- $\Omega$ -transformable regions. The paper mainly considers unstructured uncertainties. Then, these results are extended to structured uncertainties via their norms in a simple way.

The paper is organized as follows. Section 2 presents a problem formulation, including the GRQR, the generalized GRQR and the system uncertainties. Section 3 provides the main results about robust pole clustering in that general GRQR for uncertain system matrices in presence of unstructured uncertainties. Then, in Section 4, the results are extended to structured uncertainties simply by the norm

theory. Two examples illustrate and compare the results in Section 5. Section 6 concludes the paper. Due to the page limit, all theorem proofs are omitted here.

## 2. Problem Formulation

It is known from experiments that the region of closed-loop system poles as shown in Fig. 1 provides a good ride quality for aircraft [8, 10]. This region is bounded by a pair of horizontal lines with frequencies  $\pm\omega_0$  and a ring with a center at the origin and radii  $r_1$  and  $r_2$  ( $r_1 > r_2$ ) in the left half plane, i.e.,  $\{(x, y) \mid -\omega_0^2 + y^2 < 0; x < 0; r_2^2 < x^2 + y^2 < r_1^2\}$ .

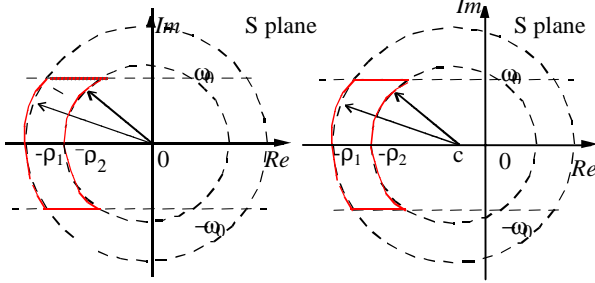


Fig.1. A Good Ride Quality Region

Fig.2. A Generalized Good Ride Quality Region

For generalization, we consider a general GRQR  $\Omega$ : an intersection of a left half-plane horizontal strip

$$\Omega_H = \{(x, y) \mid |y| < \omega_0, x < 0\} \quad (2.1)$$

and a ring symmetrical about real-axis

$$\Omega_R = \{(x, y) \mid r_2^2 < (x-c)^2 + y^2 < r_1^2\}. \quad (2.2)$$

The horizontal strip has a width  $2\omega_0$  and the ring has a center at  $c$  and radii  $r_1$  and  $r_2$  with  $r_1 > r_2$ . Then, the general GRQR  $\Omega$  is

$$\Omega = \Omega_H \cap \Omega_R. \quad (2.3)$$

Thus, the GRQR for aircraft is a specific region of  $\Omega$  as the ring center  $c = 0$ . It has a specification of the limit oscillation frequency  $\omega_0$  and the ring radii  $r_1$  and  $r_2$ . The values of  $r_1$  and  $r_2$  may be different based on different system models and their system powers.

Because a ring is not an  $\Omega$ -transformable region, the GRQR is a non- $\Omega$ -transformable region [10, 13]. An  $\Omega$ -transformable region and a non- $\Omega$ -transformable region are defined as follows.

**Definition 2.1.** ([10]) A region  $\Omega_v$  is  $\Omega$ -transformable if any two points  $\alpha, \beta \in \Omega_v$  implies  $\text{Re}[f(\alpha, \beta^*)] < 0$ , where function  $f(\lambda, \lambda^*) = f(x + jy, x - jy) = 0$  is the boundary function of the region  $\Omega_v$ .

In [13], a counter-example shows that generalized Lyapunov theorem is not valid for non- $\Omega$ -transformable region, just as stated in [10]. Thus, we will use Rayleigh principle approach for analysis of robust pole clustering in

the GRQR of aircraft.

Consider a closed-loop uncertain system

$$\dot{x} = [A + \Delta A(t, x(t))]x(t), \quad (2.4)$$

where  $t$  is the time variable,  $x(t)$  the state vector,  $A$  the nominal closed-loop system matrix,  $\Delta A$  the parameter uncertainty which is possibly time varying and/or state dependent,  $R$  is a real field over which the state and the uncertain parameters are defined. The concerned uncertainty belongs to a bounded set  $\Delta_R$  assumed to be a known compact subset of  $R^{n \times n}$ . In (2.4), the mapping  $\Delta A(\cdot): R \times R^n \rightarrow \Delta_R$  is assumed as Lebesgue measurable in the first argument and continuous in the second argument, called an admissible uncertainty. It is an admissible additive uncertainty. The uncertainties can be both time-invariant and time-variant as well as state-dependent and state-independent. However, only poles/eigenvalues of uncertain matrices  $A + \Delta A$  are considered here.

To account for possible uncertainties in system models, both parametric and unstructured uncertainty in plant models will be considered, though the paper mainly consider unstructured ones. For unstructured uncertainties, we assume that their spectral norms are bounded as

$$\|\Delta A\| \leq \delta, \quad \text{i.e.,} \quad \Delta_R := \{\Delta A \mid \|\Delta A\| \leq \delta, \Delta A \in R^{n \times n}\}. \quad (2.5)$$

The structured uncertainties (parametric uncertainties) are described as

$$\Delta A = \sum_{i=1}^k a_i A_i, \quad \text{i.e.,}$$

$$\Delta_R := \left\{ \Delta A \mid \Delta A = \sum_{i=1}^k a_i A_i, \underline{a}_i \leq a_i \leq \bar{a}_i, \Delta A \in R^{n \times n}, A_i \in R^{n \times n} \right\}, \quad (2.6)$$

where  $a_i$ ,  $i=1, \dots, k$ , are perturbation parameters and  $A_i$ ,  $i=1, \dots, k$ , are known constant matrices. Without loss of generality, we assume  $-1 \leq a_i \leq 1$ , i.e.,  $\underline{a}_i = -1$ ,  $\bar{a}_i = 1$ . This kind of structured uncertainties includes interval perturbation matrices as a special case, i.e., letting  $k$  be the number of the varying entries, and  $A_i$  be a matrix with only one nonzero entry. Here, we consider structured uncertainties as special "unstructured" uncertainties via their norms in Section 4. This simple way takes their norms in terms of their structures  $A_i$ ,  $i=1, \dots, k$ .

The objective is to discuss robust pole clustering in the generalized good ride quality region  $\Omega$  of aircraft in (2.3) for the uncertain system matrix  $A + \Delta A$  in (2.4) with unstructured uncertainties  $\Delta A$  in (2.5) or structured uncertainties  $\Delta A$  in (2.6).

## 3. Robust Pole Clustering in Good Ride Quality Region for Unstructured Uncertainties

This section, we consider robust pole clustering in the generalized GRQR  $\Omega$  for an uncertain system matrix  $A + \Delta A$ , where the perturbation matrix  $\Delta A$  is unstructured uncertainty. The objective is to find the conditions such that the poles of system matrix  $A + \Delta A$  robustly lie within the required GRQR  $\Omega$  for all admissible  $\Delta A$  in (2.5) via Rayleigh principle.

**Lemma 3.1.** (Rayleigh) For any Hermitian matrix  $H \in R^{n \times n}$  and non-zero vector  $v \in R^n$ , the following holds,

$$\underline{\lambda}(H) \leq \frac{v^* H v}{v^* v} \leq \bar{\lambda}(H) \quad (3.1)$$

where  $\underline{\lambda}(H)$  and  $\bar{\lambda}(H)$  are the minimum and the maximum eigenvalues of the Hermitian matrix  $H$  respectively.

First, we consider robust pole clustering in the horizontal strip  $\Omega_H$  in (2.1), then in the ring  $\Omega_R$  in (2.2), and finally in the GRQR  $\Omega$  in (2.3).

Notice that robust pole clustering in the whole horizontal strip for real uncertain matrices  $A + \Delta A$  is equivalent to robust pole clustering below the line  $y = \omega_0$ , i.e.,  $y < \omega_0$ , since the real uncertain matrices have their poles located symmetrically about the real axis.

**Theorem 3.1.** The poles of uncertain matrices  $A + \Delta A$  with unstructured uncertainties in (2.5) are robustly clustering in the left half horizontal strip  $\Omega_H$  in (2.1) if

$$\delta < \min \left\{ \omega_0 - \frac{1}{2} \bar{\lambda}(jA^T - jA), \frac{1}{\bar{\sigma}(P)} \right\} \quad \text{or} \quad (3.2)$$

$$\delta < \min \left\{ \omega_0 - \frac{1}{2} \bar{\sigma}(A - A^T), \frac{1}{\bar{\sigma}(P)} \right\} \quad (3.3)$$

where  $\omega_0$  is the boundary frequency of the horizontal strip  $\Omega_H$ ,  $\delta$  the norm bound of the uncertainties  $\Delta A$ ,  $\bar{\sigma}(\cdot)$  the maximum singular value of the matrix, i.e., the spectrum norm, matrix  $P$  the solution matrix from the following Lyapunov equation

$$A^T P + PA = -2I. \quad (3.4)$$

Further notice that the robust pole clustering in  $\Omega_H$  of (2.1) for a real  $A + \Delta A$  is equivalent to the following:

- a)  $A + \Delta A$  is robustly stable; and
- b)  $-j(A + \Delta A) - \omega_0 I$  is robustly stable.

The above statement b) is equivalent to robust pole clustering in the whole horizontal strip  $|y| < \omega_0$  by  $y < \omega_0$ . Now, we have the following theorem.

**Theorem 3.2.** The poles of uncertain matrices  $A + \Delta A$  with unstructured uncertainties in (2.5) are robustly clustering in the half horizontal strip  $\Omega_H$  in (2.1) if

$$\delta < \min \left\{ \frac{1}{\bar{\sigma}(P_1)}, \frac{1}{\bar{\sigma}(P_2)} \right\}, \quad (3.5)$$

where matrices  $P_1$  and  $P_2$  are the following Lyapunov matrices respectively,

$$A^T P_1 + P_1 A = -2I, \quad (3.6)$$

$$(-jA - \omega_0 I)^* P_2 + P_2 (-jA - \omega_0 I) = -2I. \quad (3.7)$$

Now, we consider robust pole clustering in the ring  $\Omega_R$  in (2.2).

**Theorem 3.3.** The poles of uncertain matrices  $A + \Delta A$  with unstructured uncertainties in (2.5) are robustly clustering in

the ring  $\Omega_R$  in (2.2) if

$$\delta < \min \left\{ r_1 - \bar{\sigma}(A - cI), \frac{\sigma^2(A - cI) - r_2^2}{2\bar{\sigma}(A - cI)} \right\} \quad (3.8)$$

It is well known that any similarity transformation will not change eigenvalues/poles of a matrix. However, a similarity transformation may reduce conservative of sufficient conditions of robust stability and robust pole clustering. Therefore, we may apply the above results to the uncertain matrices  $T^{-1}(A + \Delta A)T$  instead of  $A + \Delta A$ . Usually,  $T$  is selected such that matrix  $A_T = T^{-1}AT$  be diagonal if matrix  $A$  can be diagonalized. However, similarity transformation matrix  $T$  may be any non-singular matrix. We give these results as follows.

**Theorem 3.4.** The poles of uncertain matrices  $A + \Delta A$  with unstructured uncertainties in (2.5) are robustly clustering in the left half horizontal strip  $\Omega_H$  in (2.1) if

$$\delta < \min \{ \omega_0 - 0.5\bar{\sigma}(A_T - A_T^T), 1/\bar{\sigma}(P) \} / \kappa(T) \quad (3.9)$$

where  $\omega_0$  is the boundary frequency of the horizontal strip  $\Omega_H$ ,  $\delta$  is the norm bound of the uncertainties  $\Delta A$ ,  $A_T = T^{-1}AT$ ,  $\kappa(T) = \bar{\sigma}(T)/\underline{\sigma}(T)$ , matrix  $T$  is any non-singular matrix, matrix  $P$  is the solution matrix from the Lyapunov equation

$$A_T^T P + PA_T = -2I. \quad (3.10)$$

**Theorem 3.5.** The poles of uncertain matrices  $A + \Delta A$  with unstructured uncertainties in (2.5) are robustly clustering in the left half horizontal strip  $\Omega_H$  in (2.1) if

$$\delta < \min \{ 1/\bar{\sigma}(P_1), 1/\bar{\sigma}(P_2) \} / \kappa(T), \quad (3.11)$$

where  $T$  is any non-singular matrix, and matrices  $P_1$  and  $P_2$  are the following Lyapunov matrices respectively,

$$A_T^T P_1 + P_1 A_T = -2I, \quad (3.12)$$

$$(-jA_T - \omega_0 I)^* P_2 + P_2 (-jA_T - \omega_0 I) = -2I. \quad (3.13)$$

**Theorem 3.6.** The poles of uncertain matrices  $A + \Delta A$  with unstructured uncertainties in (2.5) are robustly clustering in the ring  $\Omega_R$  in (2.2) if

$$\delta < \min \left\{ r_1 - \bar{\sigma}(A_T - cI), \frac{\sigma^2(A_T - cI) - r_2^2}{2\bar{\sigma}(A_T - cI)} \right\} / \kappa(T). \quad (3.14)$$

Furthermore, if  $A_T$  is diagonal, then condition (3.14) is reduced to

$$\delta < \min \left\{ r_1 - \max |\lambda(A) - c|, \frac{\min |\lambda(A) - c|^2 - r_2^2}{2 \max |\lambda(A) - c|} \right\} / \kappa(T). \quad (3.15)$$

Now, we are ready to derive the main results of the paper by combining the above results as follows.

**Theorem 3.7.** The poles of uncertain matrices  $A + \Delta A$  with unstructured uncertainties in (2.5) are robustly clustering in the general good ride quality region (GRQR)  $\Omega$  in (2.3) if any one condition of Theorems 3.1, 3.2, 3.4 or 3.5 holds and any one of Theorems 3.3 or 3.6 holds.

#### 4. Robust Pole Clustering in Good Ride Quality Region for Structured Uncertainties

In this section, the structured uncertainty (2.6) is considered. Similarly, we consider robust pole clustering first in the left horizontal region  $\Omega_H$ , then in the ring  $\Omega_R$ , and finally in the general GRQR  $\Omega$  in (2.3). We first check the norm bound of the structured uncertainties in term of their structures  $A_i$ ,  $i=1, \dots, k$ , with the normalized perturbation parameters. Then, we apply the above section results to structured uncertainties. It may have some conservatism, but it is an easy and good way to apply the results in Section 3 to structured uncertainties. (Discussions on structured uncertainties via a direct way is in details in [14].)

Based on the Theorem in [21] (Zhou and Khargonekar), we can represent the norm bound of structured uncertainties in (2.6) as the following Lemma 4.1, in view of the normalized  $a_i$ ,  $i=1, \dots, k$ .

**Lemma 4.1.** The normalized structured uncertainties for (2.6)

$$\begin{aligned} \Delta A &= \sum_{i=1}^k a_i A_i, \\ \Delta_R &:= \left\{ \Delta A \mid \Delta A = \sum_{i=1}^k a_i A_i, -1 \leq a_i \leq 1, \Delta A \in R^{n \times n}, A_i \in R^{n \times n} \right\}, \end{aligned} \quad (4.1)$$

has its norm-bound as follows:

$$\begin{aligned} \text{(i)} \quad \|\Delta A\| &\leq \sqrt{k} \bar{\sigma}(A_e), \quad \text{(ii)} \quad \|\Delta A\| \leq \sum_{i=1}^k \bar{\sigma}(A_i), \\ \text{(iii)} \quad \|\Delta A\| &\leq \bar{\sigma} \left( \sum_{i=1}^k |A_i| \right), \end{aligned} \quad (4.2)$$

where  $A_e = [A_1 \ A_2 \ \dots \ A_k]$ , and  $|A_i|$  is a matrix taking the absolute value of its each entry.

Now, we extend the results in Section 3 to structured uncertainties in (4.1) by Lemma 4.1.

**Theorem 4.1.** The poles of uncertain matrices  $A + \Delta A$  with structured uncertainties in (4.1) are robustly clustering in the left half horizontal strip  $\Omega_H$  in (2.1) if one of the following

$$\text{(i)} \quad \sqrt{k} \bar{\sigma}(A_e), \quad \text{(ii)} \quad \sum_{i=1}^k \bar{\sigma}(A_i), \quad \text{(iii)} \quad \bar{\sigma} \left( \sum_{i=1}^k |A_i| \right), \quad (4.3)$$

is less than

$$\begin{aligned} \text{(a)} \quad \min &\left\{ \omega_0 - \frac{1}{2} \bar{\sigma}(A - A^T), \frac{1}{\bar{\sigma}(P_1)} \right\}, \quad \text{or} \\ \text{(b)} \quad \min &\left\{ \frac{1}{\bar{\sigma}(P_1)}, \frac{1}{\bar{\sigma}(P_2)} \right\}, \end{aligned} \quad (4.4)$$

where matrices  $P$ ,  $P_1$ , and  $P_2$  are Lyapunov matrices from equations (3.4), (3.6) and (3.7), respectively.

**Theorem 4.2.** The poles of uncertain matrices  $A + \Delta A$  with structured uncertainties in (4.1) are robustly clustering in the ring  $\Omega_R$  in (2.2) if any one in (4.3) is less than

$$\min \left\{ r_1 - \bar{\sigma}(A - cI), \frac{\underline{\sigma}^2(A - cI) - r_2^2}{2\bar{\sigma}(A - cI)} \right\}. \quad (4.5)$$

Similarly to the above section, we can use similarity transformation to get the following results.

**Theorem 4.3.** The poles of uncertain matrices  $A + \Delta A$  with structured uncertainties in (4.1) are robustly clustering in the left half horizontal strip  $\Omega_H$  in (2.1) if any one in (4.3) is less than

$$\begin{aligned} \text{(a)} \quad \min &\left\{ \omega_0 - \frac{1}{2} \bar{\sigma}(A_T - A_T^*), \frac{1}{\bar{\sigma}(P_1)} \right\} / \kappa(T), \quad \text{or} \\ \text{(b)} \quad \min &\left\{ \frac{1}{\bar{\sigma}(P_1)}, \frac{1}{\bar{\sigma}(P_2)} \right\} / \kappa(T), \end{aligned} \quad (4.6)$$

where  $T$  is any non-singular matrix,  $A_T = T^{-1}AT$ ,  $\kappa(T) = \bar{\sigma}(T) / \underline{\sigma}(T)$  is the condition number of matrix  $T$ , and matrices  $P_1$  and  $P_2$  are the Lyapunov matrices from (3.12) and (3.13), respectively.

Furthermore, if

$$\text{(i)} \quad \sqrt{k} \bar{\sigma}(A_{eT}), \quad \text{(ii)} \quad \sum_{i=1}^k \bar{\sigma}(A_{iT}), \quad \text{(iii)} \quad \bar{\sigma} \left( \sum_{i=1}^k |A_{iT}| \right), \quad (4.7)$$

is less than

$$\begin{aligned} \text{(a)} \quad \min &\left\{ \omega_0 - \frac{1}{2} \bar{\sigma}(A_T - A_T^*), \frac{1}{\bar{\sigma}(P_1)} \right\}, \quad \text{or} \\ \text{(b)} \quad \min &\left\{ \frac{1}{\bar{\sigma}(P_1)}, \frac{1}{\bar{\sigma}(P_2)} \right\}, \end{aligned} \quad (4.8)$$

where  $A_{iT} = T^{-1}A_iT$  and  $A_{eT} = T^{-1}A_eT$ .

**Theorem 4.4.** The poles of uncertain matrices  $A + \Delta A$  with structured uncertainties in (4.1) are robustly clustering in the ring  $\Omega_R$  in (2.2) if any one in (4.3) is less than

$$\min \left\{ r_1 - \bar{\sigma}(A_T - cI), \frac{\underline{\sigma}^2(A_T - cI) - r_2^2}{2\bar{\sigma}(A_T - cI)} \right\} / \kappa(T), \quad (4.9)$$

or any one in (4.7) is less than

$$\min \left\{ r_1 - \bar{\sigma}(A_T - cI), \frac{\underline{\sigma}^2(A_T - cI) - r_2^2}{2\bar{\sigma}(A_T - cI)} \right\}. \quad (4.10)$$

Furthermore, if  $A_T$  is diagonal, then condition (4.10) is reduced to

$$\min \left\{ r_1 - \max |\lambda(A) - c|, \frac{\min |\lambda(A) - c|^2 - r_2^2}{2 \max |\lambda(A) - c|} \right\}. \quad (4.11)$$

Now, we combine the above results for robust pole clustering in the generalized GRQR for structured uncertain matrices.

**Theorem 4.5.** The poles of uncertain matrices  $A + \Delta A$  with structured uncertainties in (4.1) are robustly clustering in the general good ride quality region  $\Omega$  in (2.3) if any one condition of Theorems 4.1 or 4.3 holds and any one of Theorems 4.2 or 4.4 holds.

#### 5. Examples

In this section, we discuss two examples. Both consider a flight system (Al-Summi *et al.* [1]). However, one considers bounded unstructured uncertainties and the other considers bounded structured uncertainties. The system is the short-period approximation to the F-16 dynamics linearized about the nominal flight condition: speed 502 ft/s, 0 altitude, level flight, 300 psf dynamic pressure, with C.G. at 0.4. The basic

aircraft states of interest are the pitch  $q$  and the angle of attack  $\alpha$ . The control input is the elevator deflection. The nominal open-loop system has

$$A = \begin{bmatrix} -1.0188 & 0.905284 \\ 4.0639 & -0.77013 \end{bmatrix}, \quad B = \begin{bmatrix} -0.00212 \\ -0.16919 \end{bmatrix}.$$

An indefinite  $Q$  is used in this case, and the elevator actuator is neglected,

$$Q = \begin{bmatrix} 1725.4 & 318.7 \\ 318.7 & -295 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From the Riccati equation, the solution matrix  $P$  is indefinite as follows due to indefinite  $Q$ ,

$$P = \begin{bmatrix} 673.7292 & 306.9523 \\ 306.9523 & 69.4504 \end{bmatrix}.$$

The nominal closed-loop system matrix is

$$A_{c0} = A - BR^{-1}B^T P = \begin{bmatrix} -1.1319 & 0.8790 \\ -4.9643 & -2.8683 \end{bmatrix}.$$

The eigenvalues of the nominal closed-loop system are at the desired location:  $-2 \pm j1.9$  which are suitable in terms of flying qualities. This is the reason why the matrix  $Q$  is selected as an indefinite matrix and the solution matrix  $P$  is indefinite [1]. The closed-loop system poles are within the good ride quality region. The nominal poles have a frequency  $1.9 < \omega_0$  and a radius  $r = 2.7586$  from the origin, which are within the region.

Here, we take a tight region of a good ride quality region as  $\omega_0 = 2.15$ ,  $r_1 = 3.05$ , and  $r_2 = 2.47$ . However, there are perturbations due to the model approximation, changes of the environment parameters, and so on. Let the total equivalent perturbation be  $\Delta A$ . The following examples are to check the allowed bounds of both unstructured and structured uncertainties  $\Delta A$  for robust pole clustering in this GRQR. Example 1 concerns an unstructured perturbation  $\Delta A$  in (2.5), while Example 2 concerns a structured perturbation  $\Delta A$  in (4.1).

**Example 1.** Consider the above-mentioned F16 approximation model with a norm-bounded unstructured perturbation  $\Delta A$  in (2.5). We need to find the allowed uncertainty bound for robust pole clustering in the above GRQR with  $\omega_0 = 2.15$ ,  $r_1 = 3.05$  and  $r_2 = 2.47$ . The results from Theorem 3.1-3.7 are listed in Table 1.

**Table 1. The allowed norm-bound of unstructured uncertainties for robust pole clustering in GRQR**

	Th. 3.1	Th. 3.2	Th. 3.3	Th. 3.4	Th. 3.5	Th. 3.6
	$\Omega_H$	$\Omega_H$	$\Omega_R$	$\Omega_H$	$\Omega_H$	$\Omega_R$
$\ \Delta A\  \leq \delta <$	not available	0.1172	not available	0.0924	0.0924	0.1011

From Table 1, we know that the allowed norm bound of unstructured uncertainties is 0.1011 for robust pole clustering in the selected GRQR. Theorem 3.2 is powerful for the left half horizontal strip  $\Omega_H$ . From this example, we notice that similarity transformation is useful and see Theorem 3.6 good

for the ring  $\Omega_R$ . The result of Theorem 3.7 is from Theorems 3.2 and 3.6.

**Example 2.** Consider the above-mentioned F16 approximation model but with a norm-bounded structured perturbation  $\Delta A$  in (4.1). We consider two structured uncertainties and check if these uncertainties are allowed for robust pole clustering in the above good ride quality region with  $\omega_0 = 2.15$ ,  $r_1 = 3.05$ , and  $r_2 = 2.47$ .

(a). Consider  $\Delta A = a_1 A_1 + a_2 A_2$ , where  $A_1 = \begin{bmatrix} 0 & 0.020 \\ 0 & 0.016 \end{bmatrix}$ ,

$A_2 = \begin{bmatrix} 0 & 0 \\ 0.058 & 0.035 \end{bmatrix}$ ,  $-1 \leq a_i \leq 1, i = 1, 2$ . From Lemma 4.1,

we have  $\|\Delta A\| \leq \sqrt{2} \bar{\sigma}([A_1 \ A_2]) = 0.0987$ ,

$\|\Delta A\| \leq \sum_{i=1}^2 \bar{\sigma}(A_i) = 0.0934$ , and  $\|\Delta A\| \leq \bar{\sigma}(\sum_{i=1}^2 |A_i|) = 0.0784$  listed

in Table 2. They are all less than 0.1011. Therefore, the system poles of  $A_{c0} + \Delta A$  are robustly clustering in the selected good ride quality region by Theorem 4.5.

(b). Consider  $\Delta A = \sum_{i=1}^4 a_i A_i$ , where  $A_1 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0 \end{bmatrix}$ ,

$A_2 = \begin{bmatrix} 0 & 0.09 \\ 0 & 0 \end{bmatrix}$ ,  $A_3 = \begin{bmatrix} 0 & 0 \\ 0.04 & 0 \end{bmatrix}$ ,  $A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0.03 \end{bmatrix}$ ,

$-1 \leq a_i \leq 1, i = 1, \dots, 4$ . This structured uncertainty is just an

interval matrix  $A_I = [-A_b \ A_b]$ , where  $A_b = \begin{bmatrix} 0.02 & 0.09 \\ 0.04 & 0.03 \end{bmatrix}$ .

We have  $\bar{\sigma}(\sum_{i=1}^4 |A_i|) = \bar{\sigma} \left( \begin{bmatrix} 0.02 & 0.09 \\ 0.04 & 0.03 \end{bmatrix} \right) = 0.1005 < 0.1011$ .

Then, by Theorem 4.5, the system poles of  $A_{c0} + \Delta A$  are robustly clustering in the selected good ride quality region.

Detail comparison of results is listed in the following two tables, where matrix  $T$  is selected to diagonalize matrix  $A$ .

**Table 2. The norm-bound of structured uncertainties in Example 2**

	$\sqrt{k} \bar{\sigma}(A_e)$	$\sum_{i=1}^k \bar{\sigma}(A_i)$	$\bar{\sigma}(\sum_{i=1}^k  A_i )$	$\sqrt{k} \bar{\sigma}(A_e T)$	$\sum_{i=1}^k \bar{\sigma}(A_i T)$	$\bar{\sigma}(\sum_{i=1}^k  A_i T )$
Example 2 (a)	0.0987	0.0934	0.0784	0.0945	0.0983	0.0983
Example 2 (b)	0.1844	0.18	0.1005	0.4734	0.3086	0.3086

**Table 3. Allowed norm-bound for robust pole clustering in GRQR**

	Th. 4.1 (a) $\Omega_H$	Th. 4.1 (b) $\Omega_H$	Th. 4.2 $\Omega_R$	Th. 4.3 (4.6) (a) $\Omega_H$	Th. 4.3 (4.6) (b) $\Omega_H$
$\ \Delta A\  \leq \delta <$	not available	0.1172	not available	0.0924	0.0924
$\ \Delta A_T\  \leq \delta <$					

	Th. 4.3 (4.8) (a) $\Omega_H$	Th. 4.3 (4.8) (b) $\Omega_H$	Th. 4.4 (4.9) $\Omega_R$	Th. 4.4 (4.10) $\Omega_R$
$\ \Delta A\  \leq \delta <$			0.1011	
$\ \Delta A_T\  \leq \delta <$	0.25	0.25		0.2736

From the results, we see that a diagonalization similarity transformation is useful for testing region  $\Omega_R$ , but not for region  $\Omega_H$  in this example. It is good for the part (a) of this example, but not for the part (b) of this example. This means it may be useful for some cases, but not always. However, in theoretical viewpoint, general similarity transformation is always useful because similarity transformation matrix  $T$  can be any non-singular matrix which includes an identity matrix making the original non-transformation result as a special case.

## 6. Conclusion

In this paper, we have discussed the analysis of robust pole clustering in a good ride quality region of aircraft for uncertain matrices. The good ride quality region is as shown in Fig.1. It is well known from experiments that this region provides good ride quality for aircraft. The paper here considers a generalized good ride quality region with a ring center at  $c$  as shown in Fig.2, making the GRQR as a special case with  $c = 0$ . However, both are non- $\Omega$ -transformable regions. The difficulty is that the generalized Lyapunov theory is not valid for non- $\Omega$ -transformable regions. An approach based on Rayleigh principle is developed for analysis of robust pole clustering in the general GRQR, a specific non- $\Omega$ -transformable region. The considered uncertainties include both unstructured and structured uncertainties. The main results are for unstructured uncertainties about their allowed norm bound such that the uncertain system matrix eigenvalues/poles are robustly clustering in the generalized GRQR. Then, they are extended to structured uncertainties via their norms in terms of their structures. The results are sufficient conditions. Similarity transformation is also introduced to reduce conservative. The discussion also includes analysis of robust pole clustering in horizontal strips, left half-horizontal strips, and rings. Two examples illustrate the results about analysis of robust pole clustering in the selected good ride quality region for a perturbed closed-loop system matrix of F16 aircraft approximation model. These results may be applied to analysis and design of flight control systems for aircraft, shuttles and space station to have a robust good ride quality.

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