

Improved stabilising conditions for model predictive control

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Abstract

It is known that stability of a model predictive control system is ensured if the terminal conditions of the optimal control problem solved online satisfy certain criteria. The usual requirement is that the terminal cost function is a control Lyapunov function defined on the terminal constraint set. Conventionally the terminal cost function is chosen, when the system being controlled is linear, to be the value function for the infinite horizon *unconstrained* optimal control problem and the terminal constraint set is chosen to be the output admissible set for the closed-loop system using the optimal unconstrained controller $u = -Kx$. The purpose of this paper is to relax these terminal conditions thereby facilitating online solution of the optimal control problem. Using some recent results, we present alternative conditions that employ, as the terminal cost, the infinite horizon cost resulting from a *nonlinear* controller $u = -\text{sat}(Kx)$ and, as the terminal constraint set, the set in which this controller is optimal for the infinite horizon *constrained* optimal control problem. It is shown that this solution provides a considerably larger terminal constraint set.

1 Introduction

This paper is concerned with closed-loop stability of constrained linear systems when model predictive control is employed. Model predictive control is a form of control in which the current control is obtained by solving, at each sampling instant, a finite horizon open-loop optimal control problem and applying the first element of the optimal control sequence so obtained. Obviously, model predictive control of constrained systems

is nonlinear so that stability is, in general, a nontrivial issue. After the pioneering work of Chen and Shaw [1], and of Keerthi and Gilbert [5], the value function of the finite-horizon optimal control problem has been used, almost universally, as a Lyapunov function for analysing closed-loop stability [6].

Several ‘ingredients’ of the online optimal control problem directly affect closed-loop stability; these are: the terminal cost $F(\cdot)$, the terminal constraint set \mathcal{X}_f (both of which are employed in the optimal control problem solved online), and the local controller $\kappa_f(\cdot)$ that ensures existence of feasible solutions to the optimal control problem (see, e.g., [1, 5, 6]). Ideally, the terminal cost $F(\cdot)$ is the infinite horizon value function $V_\infty^0(\cdot)$ (for the constrained optimal control problem), in which case, the finite horizon value function is $V_N^0(\cdot) = V_\infty^0(\cdot)$. With this choice, online optimisation is unnecessary and the advantages of an infinite horizon problem automatically accrue. However, constraints generally render this approach impossible. Usually, then, \mathcal{X}_f is chosen to be an appropriate neighbourhood of the origin in which $V_\infty^0(\cdot)$ is exactly (or approximately) known, and $F(\cdot)$ is set equal to $V_\infty^0(\cdot)$ or its approximation. When the system being controlled is linear, $F(\cdot)$ is often chosen to be the value function of the infinite horizon, *unconstrained* optimal control problem, $\kappa_f(\cdot)$ is chosen to be the optimal controller ($\kappa_f(x) = -Kx$) for this problem, and \mathcal{X}_f the *maximal output admissible set* \mathcal{O}_∞ (defined in [3]) for the closed-loop system using the local controller $\kappa_f(\cdot)$. In this case, $V_\infty^0(x) = F(x) = x^T P x$ for all $x \in \mathcal{X}_f$ (P is the solution of an algebraic Riccati equation) [8].

The purpose of this paper is to provide *new* terminal ingredients for model predictive control of constrained linear systems. The ingredients are an improvement

over those previously used in that the terminal constraint set \mathcal{X}_f is strictly larger than \mathcal{O}_∞ , thus facilitating the solution of the optimal control problem. To obtain the improved terminal conditions we employ recent results [2] that show that the nonlinear controller $\kappa_{nl}(x) = -\text{sat}(Kx)$ is optimal in a region which includes the maximal output admissible set \mathcal{O}_∞ . The proposed terminal cost function $F(\cdot)$ is the infinite horizon value function $V_\infty^0(\cdot)$ which is proved, for stable plants, to be finitely parameterisable in \mathcal{X}_f . However, the proposed terminal cost function $F(\cdot)$, while still convex (thus ensuring solvability of the optimal control problem) is no longer quadratic (which implies that the problem is not a quadratic program, and needs to be solved using convex programming or conventional nonlinear programming). Some of these results are established for the case of stable plants. However examples indicate that the results may hold more generally.

2 Definitions and notation

The system considered is

$$x(k+1) = Ax(k) + Bu(k) \quad (1)$$

or, more concisely, $x^+ = f(x, u) := Ax + Bu$; where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are, respectively, the current state and control and x^+ is the successor state. The pair (A, B) is assumed controllable. The control is required to satisfy the constraint

$$u(k) \in \Omega \quad (2)$$

for all k , where $\Omega := [-1, 1]$. The following notation will be employed. The solution of (1) at time k , when the initial state is x at time i and the control sequence is \mathbf{u} , is $x^{\mathbf{u}}(k; x, i)$; to simplify notation, $x^{\mathbf{u}}(k; x) := x^{\mathbf{u}}(k; x, 0)$, i.e. the initial time is dropped when it is zero. $\mathcal{I} := \{0, 1, 2, \dots\}$ is the set of non-negative integers, $\mathcal{I}^+ := \{1, 2, 3, \dots\}$ the set of positive integers, and $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$. For all $\epsilon > 0$, $B_\epsilon := \{x \mid |x| \leq \epsilon\}$. For any set X in, say, \mathbb{R}^n , X^c denotes the complement of X (in \mathbb{R}^n). For each $\Delta \in \mathbb{R}^+$, the function $\text{sat}_\Delta(\cdot)$ is defined by

$$\text{sat}_\Delta(u) := \begin{cases} u & \text{if } |u| \leq \Delta \\ \Delta & \text{if } u > \Delta \\ -\Delta & \text{if } u < -\Delta \end{cases} \quad (3)$$

The function $\text{sat}(\cdot)$ is defined to be $\text{sat}_1(\cdot)$. In the sequel we consider both a linear controller $u = -Kx$ and a nonlinear controller $u = \text{sat}(-Kx) = -\text{sat}(Kx)$. The closed-loop satisfies $x^+ = \phi_\ell(x)$ when the linear controller is used and $x^+ = \phi_{nl}(x)$ when the nonlinear controller is employed, where the mappings $\phi_\ell : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and, $\phi_{nl} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are defined by

$$\phi_\ell(x) := A_K x, \quad A_K := A - BK \quad (4)$$

$$\phi_{nl}(x) := Ax - B\text{sat}(Kx) \quad (5)$$

In the sequel \circ denotes concatenation, i.e. $(a \circ b)(x) := a(b(x))$, $a^0(x) := x$ and, for all $i \in \mathcal{I}^+$, $a^i(x) := (a^{i-1} \circ a)(x) = (a \circ a^{i-1})(x)$. For all $i \in \mathcal{I}^+$, the function $\delta_i(\cdot)$ is defined by

$$\delta_i(x) := Kx - \text{sat}_{\Delta_i}(Kx) \quad (6)$$

where the saturation bounds Δ_i are defined by

$$\Delta_1 := 1, \quad \Delta_i := 1 + \sum_{j=0}^{i-2} |KA^j B|, \quad i = 2, 3, \dots \quad (7)$$

Hence, $\phi_{nl}(x) = A_K x + B\delta_1(x)$. The set $X_0 := \{x \mid \delta_1(x) = 0\} = \{x \mid |Kx| \leq 1\}$ and, for each $i \in \mathcal{I}^+$, the set $X_i \subset \mathbb{R}^n$ is defined by

$$X_i := \{x \mid \delta_i(A^{i-1} A_K x) = 0\} = \{x \mid |\bar{K}_i x| \leq \Delta_i\} \quad (8)$$

where

$$\bar{K}_i := KA^{i-1} A_K \quad (9)$$

We also require the sets Y_i and Z_i defined by

$$Y_0 := Y_1 := \mathbb{R}^n \quad (10)$$

$$Y_i := X_1 \cap X_2 \dots \cap X_{i-1}, \quad i = 2, 3, \dots \quad (11)$$

$$Z_1 := Y_1 = \mathbb{R}^n \quad (12)$$

$$Z_i := \{x \mid \phi_{nl}^k(x) \in Y_{i-k}, k = 0, \dots, i-2\}, \quad (13)$$

$i = 2, 3, \dots$; so that $Z_1 = Y_1$, $Z_2 = Y_2$, $Z_3 = \{x \mid x \in Y_3, \phi_{nl}(x) \in Y_2\}$, $Z_4 = \{x \mid x \in Y_4, \phi_{nl}(x) \in Y_3, \phi_{nl}^2(x) \in Y_2\}$, etc. The set sequence $\{Z_i\}$ is monotonically non-increasing (with respect to inclusion), i.e.

$$Z_{i+1} \subset Z_i, \quad \forall i \in \mathcal{I}^+ \quad (14)$$

and

$$Z_{i+1} = Y_{i+1} \cap \{x \mid \phi_{nl}(x) \in Z_i\}, \quad \forall i \in \mathcal{I}^+ \quad (15)$$

3 Model predictive control

In model predictive control, a finite horizon optimal control problem $\mathcal{P}_N(x)$ defined below is repeatedly solved. Because of time invariance, the initial time in the optimal control problem may be taken to be zero. Thus $\mathcal{P}_N(x)$ is defined by

$$\mathcal{P}_N(x) : \quad V_N^0(x) = \min_{\mathbf{u}} V_N(x, \mathbf{u}) \quad (16)$$

subject to the control constraint $\mathbf{u} \in \Omega^N$, and the terminal constraint $x(N) \in \mathcal{X}_f$, where

$$\mathbf{u} := \{u(0), u(1), \dots, u(N-1)\} \quad (17)$$

is a sequence of N control actions,

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k), u(k)) + F(x(N)) \quad (18)$$

$$\ell(x, u) := |x|_Q^2 + |u|_R^2 \quad (19)$$

and $x(k) := x^u(k; x)$, $k = 0, 1, \dots, N$. We assume that Q and R are positive definite. At event (x, k) (at state x , time k), problem $\mathcal{P}_N(x)$ is solved yielding the optimal control sequence $\mathbf{u}^0(x) = \{u^0(0; x), u^0(1; x), \dots, u^0(N-1; x)\}$, the optimal state sequence $\mathbf{x}^0(x) = \{x^0(0; x), x^0(1; x), \dots, x^0(N; x)\}$ (where $x^0(0; x) = x$, the initial state), and the value function $V_N^0(x) = V_N(x, \mathbf{u}^0(x))$. The first control $u^0(0; x)$ is applied to the plant so that the (implicit) model predictive control law is

$$u = \kappa_N(x) := u^0(0; x) \quad (20)$$

If $F(\cdot)$ and \mathcal{X}_f are chosen appropriately (see, for example, [6]), the control law can be shown to be stabilising. For any function $\theta: \mathbb{R}^n \rightarrow \mathbb{R}^n$, let θ^* be defined by

$$\theta^*(x, u) := \theta(f(x, u)) - \theta(x) \quad (21)$$

where $f(x, u) = Ax + Bu$. We have [6]

Theorem 1 *Suppose the terminal cost function $F: \mathcal{X}_f \rightarrow \mathbb{R}$, the terminal constraint set \mathcal{X}_f and the local control law $\kappa_f: \mathcal{X}_f \rightarrow \mathbb{R}$ satisfy:*

A1: \mathcal{X}_f is closed and $0 \in \mathcal{X}_f$,

A2: $\kappa_f(x) \in \Omega$, $\forall x \in \mathcal{X}_f$ (control constraint satisfied in \mathcal{X}_f),

A3: \mathcal{X}_f is positively invariant for the system, $x^+ = f(x, \kappa_f(x))$,

A4: $[\bar{F} + \ell](x, \kappa_f(x)) \leq 0$, $\forall x \in \mathcal{X}_f$ ($F(\cdot)$ is a local Lyapunov function).

Then

$$[V_N^0 + \ell](x, \kappa_N(x)) \leq 0$$

for all $x \in \mathcal{X}_N$, the (compact, convex) set of states steerable to \mathcal{X}_f by an admissible control in time N or less. Also \mathcal{X}_N is positively invariant for the closed-loop system $x^+ = f(x, \kappa_N(x))$ where $\kappa_N(\cdot)$ is the model predictive control law.

Corollary 1 *Suppose $Q > 0$ and $R > 0$, that $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ satisfy **A1–A4** and that, in addition, there exists a finite c such that $F(x) \leq c|x|^2$ for all $x \in \mathcal{X}_f$. Then the origin is exponentially stable for the closed-loop system $x^+ = f(x, \kappa_N(x))$ with a region of attraction \mathcal{X}_N .*

Proof: Since $Q > 0$ it follows from Theorem 1 that there exists a finite positive constant a such that $V_N^0(x) \geq a|x|^2$, $\forall x \in \mathcal{X}_N$ and $V_N^0(x, \kappa_N(x)) \leq -a|x|^2$, $\forall x \in \mathcal{X}_N$. Next $V_N^0(x) \leq F(x)$ for all $x \in \mathcal{X}_f$

[4]. This is easily shown. Let x is an arbitrary point in \mathcal{X}_f and let $\{x^f(k; x), k = 0, 1, 2, \dots\}$ be the state sequence resulting from initial state x and controller $\kappa_f(\cdot)$. Then, by **A4**

$$F(x) \geq \sum_{k=0}^{N-1} \ell(x^f(k; x), \kappa_f(x^f(k; x))) + F(x^f(N; x))$$

where (by **A3**) $x^f(k; x) \in \mathcal{X}_f$ for all $k = 0, 1, \dots, N$ and (by **A2**) $\kappa_f(x^f(k; x)) \in \Omega$ for all $k = 0, 1, \dots, N-1$. But, by optimality (since $x^f(N; x) \in \mathcal{X}_f$),

$$V_N^0(x) \leq \sum_{k=0}^{N-1} \ell(x^f(k; x), \kappa_f(x^f(k; x))) + F(x^f(N; x))$$

Hence $V_N^0(x) \leq F(x) \leq c|x|^2$ for all $x \in \mathcal{X}_f$. Exponential stability, with a region of attraction \mathcal{X}_N , follows. ■

For future reference we define the *constrained infinite horizon* optimal control problem as follows:

$$\mathcal{P}_\infty(x): \quad V_\infty^0(x) = \min_{\mathbf{u}} V_\infty(x, \mathbf{u}) \quad (22)$$

subject to the control constraint $u(k) \in \Omega$, $\forall k \in \mathcal{I}$; where, now, \mathbf{u} is the infinite sequence $\{u(k) \mid k \in \mathcal{I}\}$ and $V_\infty(x, \mathbf{u}) := \sum_{k=0}^{\infty} \ell(x(k), u(k))$, with $x(k) := x^u(k; x)$. Note that $\mathcal{P}_\infty(\cdot)$ does not have either a terminal cost nor a terminal constraint; both are irrelevant since, if a solution to the problem exists, $x^0(k; x) \rightarrow 0$ as $k \rightarrow \infty$.

4 Terminal conditions

4.1 Preliminaries

A triple $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ satisfying **A1–A4** and $F(x) \leq c|x|^2$ for all $x \in \mathcal{X}_f$ ensures exponential stability as shown above. A useful choice of terminal conditions [8] for the problem considered is to choose $F(\cdot)$ to be the value function $V_{uc}^0(\cdot)$ for the *unconstrained* infinite horizon optimal control problem $\mathcal{P}_{uc}(x)$ for the same system (1) and cost

$$V_{uc}(x, \mathbf{u}) := \sum_{k=0}^{\infty} \ell(x(k), u(k)) \quad (23)$$

where $\ell(x, u) = |x|_Q^2 + |u|_R^2$ as before. Thus

$$F(x) = x^T P x \quad (24)$$

where $P > 0$ is the (unique positive semi-definite) solution of the algebraic Riccati equation

$$P = A^T P A + Q - K^T \bar{R} K \quad (25)$$

where

$$K := \bar{R}^{-1} B^T P A, \quad \bar{R} := R + B^T P B \quad (26)$$

The local controller is defined by

$$\kappa_f(x) := -Kx \quad (27)$$

and is, therefore, the optimal controller for the unconstrained infinite horizon problem $\mathcal{P}_{uc}(\cdot)$. The set \mathcal{X}_f is usually taken to be the maximal output admissible set \mathcal{O}_∞ defined in [3], i.e.

$$\mathcal{O}_\infty := \{x \mid KA_K^j x \in \Omega, j \in \mathcal{I}\}. \quad (28)$$

An interesting consequence of this choice for $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ is that $V_\infty^0(x) = F(x)$ for all x in \mathcal{X}_f and that $V_N^0(x) = V_\infty^0(x)$ for all $x \in \mathcal{X}_N$ such that the terminal constraint is not active (i.e. $x^0(N; x)$ lies in the interior of \mathcal{X}_f); if N is so chosen, the terminal constraint may be omitted from $\mathcal{P}_N(\cdot)$.

4.2 Characterisation of the value function

It is the purpose of this paper to propose a larger \mathcal{X}_f , thus simplifying optimisation (or reducing N in those variants that omit the terminal constraint from the optimal control problem but increase the horizon N until this constraint is satisfied). To this end we employ some recent results [2] that show that the nonlinear controller $\kappa_{nl}(\cdot)$ defined by

$$\kappa_{nl}(x) := \text{sat}(-Kx) = -\text{sat}(Kx) \quad (29)$$

is optimal in a region which includes the output admissible set \mathcal{O}_∞ defined above.

We next state the main result of [2]:

Theorem 2 *The optimal value function $V_N^0(\cdot)$ for problem \mathcal{P}_N with $F(x) := x^T Px$ and $\mathcal{X}_f := \mathbb{R}^n$ satisfies*

$$V_N^0(x) = J^N(x) := x^T Px + \bar{R} \sum_{k=1}^N \delta_k (A^{k-1} x)^2, \quad (30)$$

for all $x \in Z_N$, and the optimal control law $\kappa_N(\cdot)$ satisfies

$$\kappa_N(x) = \kappa_{nl}(x) = -\text{sat}(Kx), \quad \forall x \in Z_N \quad (31)$$

Since the functions $x \mapsto \delta_k (A^{k-1} x)^2$ are convex, so is the value function $V_N^0(\cdot)$.

4.3 Specification of $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$

We cannot employ $(J^N(\cdot), Z_N, \kappa_{nl}(\cdot))$ for the triple $(F(\cdot), \mathcal{X}_f, \kappa_f)$ because, being the solution of a finite horizon optimal control problem (in a specified region Z_N), this triple does not satisfy conditions **A1–A4**. To obtain a suitable triple we allow N to tend to infinity. First we note

Proposition 1 *For all $i \in \mathcal{I}^+$, Y_i is closed and convex and contains the origin in its interior. If (K, A) is an observable pair and A_K non-singular, Y_i is a (convex, compact) polytope for all $i \geq n$.*

Proof: The first statement is obvious. The second follows from the linear independence of KA^i , $i = 1, 2, \dots, n$. ■

This result does not necessarily hold if Ω is not bounded (as it is for our problem). Let $J^\infty(\cdot), Y_\infty$ and Z_∞ be defined by

$$J^\infty(x) := x^T Px + \bar{R} \sum_{k=1}^{\infty} \delta_k (A^{k-1} x)^2 \quad (32)$$

$$Y_\infty := \bigcap_{j=1}^{\infty} X_j \quad (33)$$

$$Z_\infty := \{x \mid \phi_{nl}^k(x) \in Y_\infty, k \in \mathcal{I}\} \quad (34)$$

The function $J^\infty(\cdot)$ is well defined if A is asymptotically stable. The value function $V_\infty^0(\cdot)$ for problem $\mathcal{P}_\infty(\cdot)$ satisfies

$$V_\infty^0(x) = J^\infty(x), \quad \forall x \in Z_\infty \quad (35)$$

The set Z_∞ is a candidate for \mathcal{X}_f , but for this use it is necessary that it be *finitely determined*. To establish this we require:

Proposition 2 *Assume A is asymptotically stable (i.e. all of its eigenvalues lie in the open unit disc). Then Y_∞ is finitely determined ($Y_\infty = Y_{i_1}$ for some integer i_1), and Y_∞ is a (compact, convex) polytope that contains the origin in its interior.*

Proof: The sequence $\{\Delta_i\}$ is monotonically non-decreasing. Because A is asymptotically stable, the sequence $\{\Delta_i\}$ is bounded above and therefore converges to $\bar{\Delta} < \infty$ so that $\Delta_i \in [1, \bar{\Delta}]$ for all $i \in \mathcal{I}^+$. Secondly (following [3]), there exists a $c_1 \in \mathbb{R}^+$ such that the induced norm $\|\bar{K}_i\| < c_1$ for all $i \in \mathcal{I}^+$. Let $c_2 < 1/c_1$ so that $\|\bar{K}_i x\| < 1 \leq \Delta_i$ (which implies $x \in X_i$) for all $x \in B_{c_2}$, all $i \in \mathcal{I}^+$. Hence $B_{c_2} \subset Y_\infty$ so that Y_∞ contains the origin in its interior. To show that Y_∞ is finitely determined, let $c_3 \in \mathbb{R}^+$ be such that $Y_n \subset B_{c_3}$. Clearly $Y_\infty \subset Y_n \subset B_{c_3}$. There exists an integer $i_1 \geq n$ such that the induced norm $\|\bar{K}_i\| < 1/c_3$ for all $i \geq i_1$ and $\|\bar{K}_i x\| \leq (1/c_3)|x| \leq 1 \leq \Delta_i$ for all $x \in B_{c_3}$, all $i \geq i_1$; this implies $B_{c_3} \subset \bigcap_{i=i_1}^{\infty} X_i$. Hence $Y_n \subset \bigcap_{i=i_1}^{\infty} X_i$ and $Y_\infty = Y_n \cap (\bigcap_{i=n}^{i_1-1} X_i) \cap (\bigcap_{i=i_1}^{\infty} X_i) = Y_n \cap (\bigcap_{i=n}^{i_1-1} X_i) = Y_{i_1}$ and Y_∞ is finitely determined. ■

We also have

Proposition 3 *Assume A is asymptotically stable. Then: (i) The set Z_∞ is positively invariant for the system $x^+ = \phi_{nl}(x)$. (ii) $Z_\infty \subset Y_\infty$. (iii) Z_∞ is compact and contains the origin in its interior.*

Proof: (i) This follows from the definition (34) of Z_∞ if Z_∞ is not empty. Let $c > 0$ be such that the

level set $\mathcal{L} := \{x \mid x^T P x \leq c\} \subset \mathcal{O}_\infty \cap Y_\infty$; since both \mathcal{O}_∞ [3] and Y_∞ contain the origin in their interiors and since $P > 0$, such a c exists. Since $\kappa_{nl}(x) = -Kx$ and $\phi_{nl}(x) = A_K x$ in \mathcal{O}_∞ , and since \mathcal{L} is positively invariant for $x^+ = A_K x = \phi_{nl}(x)$, it follows that $\mathcal{L} \subset Z_\infty$. But \mathcal{L} contains the origin in its interior (since $P > 0$). (ii) From definition (34), $x \in Z_\infty$ implies $x \in Y_\infty$. (iii) Since Y_∞ is compact and $\phi_{nl}(\cdot)$ is continuous, each set $W_k := \{x \mid \phi_{nl}^k(x) \in Y_\infty\}$, $k \in \mathcal{I}$, is compact. Hence Z_∞ is compact. ■

Proposition 4 *Assume A is asymptotically stable. Then there exist a finite integer N_1 such that*

$$J^\infty(x) = J^{N_1}(x) := x^T P x + \bar{R} \sum_{i=1}^{N_1} \delta_i (A^{i-1} x)^2, \quad (36)$$

for all $x \in Z_\infty \subset Y_{i_1}$.

Proof: Arguing as in the proof of Proposition 2 we can establish the existence of a finite integer N_1 such that $|KA^{i-1}x| \leq 1 \leq \Delta_i$ (which implies $\delta_i(A^{i-1}x) = 0$) for all $i \geq N_1$, all $x \in Y_{i_1}$. ■

For all $j \in \mathcal{I}^+$, let W_j be defined by

$$W_j := \{x \mid \phi_{nl}^i(x) \in Y_\infty \text{ for } i = 1, 2, \dots, j-1 \text{ and } \phi_{nl}^j(x) \in \mathcal{L}\} \quad (37)$$

where \mathcal{L} is defined in the proof of Proposition 3.

Proposition 5 *Assume A is asymptotically stable. Then there exists an integer i_2 such that $Z_\infty = W_{i_2}$, i.e. Z_∞ is finitely determined.*

Proof: (i) Let

$$\max\{J^{N_1}(x) \mid x \in Y_\infty = Y_{i_1}\} = c_1 < \infty$$

By (35) and proposition 4, $V_\infty^0(\cdot) = J^\infty(\cdot) = J^{N_1}(\cdot)$ in Z_∞ so that there exists an $c_2 \in (0, \infty)$ such that $J^{N_1}(x, \kappa_{nl}(x)) = -\ell(x, \kappa_{nl}(x)) \leq -|x|_Q^2 \leq -c_2|x|^2$, for all $x \in Z_\infty$. There exists a $c_3 \in (0, \infty)$ such that

$J^{N_1}(x, \kappa_{nl}(x)) \leq -c_3$, for all $x \in Z_\infty \cap \text{closure}(\mathcal{L}^c)$.

Hence for all $x \in Z_\infty$ there exists an integer $i_2 \leq c_1/c_3$ such that $\phi_{nl}^{i_2}(x) \in \mathcal{L}$. Hence $x \in Z_\infty$ implies $x \in W_{i_2}$.

(ii) Suppose $x \in W_{i_2}$ so that $\phi_{nl}(x) \in Y_\infty$ for $i = 0, 1, \dots, i_2 - 1$ and $\phi_{nl}^{i_2}(x) \in \mathcal{L}$. Since \mathcal{L} is positively invariant for $x^+ = \phi_{nl}(x)$, $\phi_{nl}^j(x) \in \mathcal{L} \subset Y_\infty$ for $j = i_2, i_2 + 1, i_2 + 2, \dots$. Hence $x \in Z_\infty$. ■

We now establish that \mathcal{O}_∞ is a subset of Z_∞ .

Proposition 6 $\mathcal{O}_\infty \subset Z_\infty$.

Proof: By definition Z_∞ is the maximal positively invariant set in Y_∞ for the closed-loop system $x^+ = \phi_{nl}(x)$. The set \mathcal{O}_∞ is also a positively invariant set for $x^+ = \phi_{nl}(x)$ (since $\phi_{nl}(\cdot) = \phi_\ell(\cdot)$ in \mathcal{O}_∞). It suffices, therefore, to establish that $\mathcal{O}_\infty \subset Y_\infty$. Since $Y_\infty = Y_{i_1}$ we need only prove $\mathcal{O}_\infty \subseteq X_i$ for $i = 1, 2, \dots, i_1 - 1$. Assume, therefore, that $x \in \mathcal{O}_\infty$, so that

$$|KA_K^j x| \leq 1, \quad j \in \mathcal{I} \quad (38)$$

For any $i \in \mathcal{I}^+$

$$\begin{aligned} A_K^i &= (A - BK)A_K^{i-1} = AA_K^{i-1} - BKA_K^{i-1} \\ &= A(A - BK)A_K^{i-2} - BKA_K^{i-1} \\ &= A^2A_K^{i-2} - ABKA_K^{i-2} - BKA_K^{i-1} \\ &\vdots \\ &= A^{i-1}A_K - \sum_{j=0}^{i-2} A^j BKA_K^{i-1-j} \end{aligned}$$

which implies

$$KA^{i-1}A_K x = KA_K^i x + \sum_{j=0}^{i-2} KA^j BKA_K^{i-1-j} x \quad (39)$$

From (38) and (39), we obtain the inequality

$$|KA^{i-1}A_K x| \leq 1 + \sum_{j=0}^{i-2} |KA^j B| = \Delta_i$$

This implies $x \in X_i$ for all $i \in \mathcal{I}^+$ yielding the desired result. ■

Example 1 *In this example, the relative size of the sets \mathcal{O}_∞ , Y_∞ and Z_∞ is illustrated. Consider the system $x^+ = Ax + Bu$, $y = Cx$ with*

$$A = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ 0.08 \end{bmatrix}, \quad C = [0 \quad 1],$$

which is the zero-order hold discretisation, with a sampling period of 0.4 sec., of the double integrator $\dot{x}_1(t) = u(t)$, $\dot{x}_2(t) = x_1(t)$, $y(t) = x_2(t)$. In Eqs. (18)-(19) we take: $Q = I_{2 \times 2}$ and $R = 0.25$. The matrix P and the gain K were computed from (25)-(26).

In Figure 1 we show the sets: Y_∞ , the maximal output admissible set \mathcal{O}_∞ , and an ellipsoidal set \mathcal{E} which is positively invariant for $x^+ = \phi_{nl}(x)$ (the derivation of \mathcal{E} can be found in [2]). In this example $i_1 = 7$, so that $Y_\infty = Y_7$. An estimate of the set Z_∞ is given by:

$$\mathcal{O}_\infty \cup \mathcal{E} \subseteq Z_\infty \subseteq Y_\infty.$$

The trajectory of the closed-loop system with controller $\kappa_{nl}(x) = -\text{sat}(Kx)$ is also shown; this trajectory coincides with the optimal trajectory (Theorem 2). The

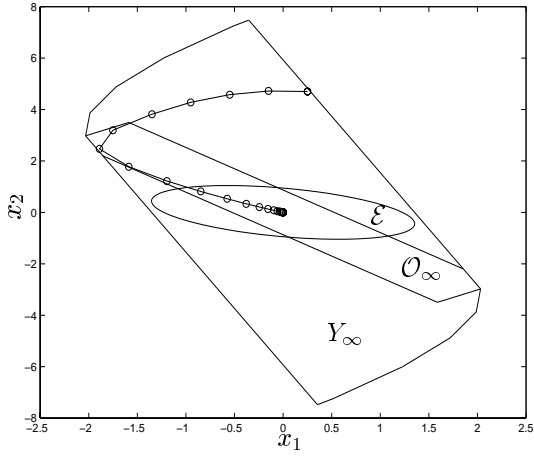


Figure 1: Set boundaries for the example.

initial condition, $x_0 = [0.25 \ 4.7]^T$, has been chosen so that a better estimate of the size of Z_∞ can be inferred from the simulated trajectory.

Notice that this example indicates that the assumption that A is asymptotically stable (made for the results of this paper) is not a necessary condition, and consequently, the results hold more generally than formally established here.

We can now state the main result of this paper.

Theorem 3 Assume A is asymptotically stable. Then the triple $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ where $F(\cdot) := J^\infty(\cdot) = J^{N_1}(\cdot)$, $\mathcal{X}_f := Z_\infty = W_{i_2}$ and $\kappa_f(\cdot) := \kappa_{nl}(\cdot)$, satisfies conditions **A1–A4**.

Proof: (i) It is established above that $\mathcal{X}_f := Z_\infty$ is closed and contains the origin in its interior, thus satisfying **A1**.

(ii) $\kappa_f(\cdot) := \kappa_{nl}(\cdot)$ satisfies **A2** by definition.

(iii) That $\mathcal{X}_f := Z_\infty$ is positively invariant for the system $x^+ = f(x, \kappa_f(x)) = \phi_{nl}(x)$ is established above.

(iv) Because $F(\cdot) := J^\infty(\cdot)$ is the value function $V_\infty^0(\cdot)$, and $\kappa_{nl}(\cdot)$ the optimal control law, for the infinite horizon optimal control problem $\mathcal{P}_\infty(\cdot)$, it satisfies

$$\bar{F}^*(x, \kappa_{nl}(x)) = -\ell(x, \kappa_{nl}(x))$$

for all $x \in \mathcal{X}_f$, thus satisfying **A4**. ■

4.4 Implementation

Since Z_∞ is not defined by linear inequalities and $F(\cdot)$ is not quadratic, problem $\mathcal{P}_N(\cdot)$ with $F(\cdot) = J^{N_1}(\cdot)$, $\mathcal{X}_f = Z_\infty$ and $\kappa_f(\cdot) = \kappa_{nl}(\cdot)$ is not a quadratic program. Indeed it is not even necessarily a convex program since there is no guarantee that Z_∞ is convex.

However, because Z_∞ is finitely specified, the variant of model predictive control in which the terminal constraint is omitted from $\mathcal{P}_N(\cdot)$ and N is chosen (either *a priori* or online) to ensure that the terminal constraint is satisfied (despite its omission from $\mathcal{P}_N(\cdot)$) is easily implemented. The resultant problem is convex because $F(\cdot)$ is convex and positive definite and may be solved using convex programming or, indeed, conventional non-linear programming (e.g., [7]).

5 Conclusions

We have shown how to obtain terminal ingredients $F(\cdot), \mathcal{X}_f, \kappa_f(\cdot)$ (for the optimal control problem employed in model predictive control) that ensure closed-loop stability. The ingredients provide a larger terminal constraint set than that provided by previous approaches thus facilitating online solution of the optimal control problem. Examples show that the new constraint set \mathcal{X}_f may be considerably larger than the output admissible set \mathcal{O}_∞ conventionally employed. The main result, though proven only for the case of stable systems, is believed to hold more generally.

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