On the Problem of General Structural Assignments of Linear Systems Through Sensor/Actuator Selection

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Abstract

A systematic method is developed for determining an output matrix C for a given matrix pair (A, B) such that the resulting linear system characterized by the matrix triple (A, B, C) has the pre-specified system structural properties, such as the finite and infinite zero structure and the invertibility structures. Since the matrix C describes the locations of the sensors, the procedure of choosing C is often referred to as sensor selection. The method developed in this paper for sensor selection can be applied to the dual problem of actuator selection, where, for a given matrix pair (A, C), a matrix B is to be determined such that the resulting matrix triple (A, B, C) has the pre-specified structural properties.

Keywords: Linear systems, system structures, invariant indices, sensor selection, actuator selection.

1 Introduction and Problem Statement

As is well known in the literature, the structural properties of linear systems, such as the finite and infinite zero structures and the invertibility structures, have played very important roles in many linear systems and control areas. One of the major difficulties in applying the useful multivariable control synthesis techniques, e.g., H_2 and H_∞ control techniques, to actual design is the inadequate study of the linkage between control performance and design implementation involving hardware selection, e.g., appropriate sensors suitable for robustness and performance. This linkage provides a foundation upon which trade-offs can be incorporated at the preliminary design stage. Thus, one can introduce careful control design considerations into the overall engineering design process in an early stage. This is what motivated the work to be reported in this paper. Our objective is to study the flexibility on the structural properties that one can assign to a given linear system, and to identify sets of sensors which would yield desirable structural properties.

It is appropriate to trace a short history of the development of the techniques related to structural assignments of linear systems. To the best of our knowledge, most results in the open literature are related to invariant zero or transmission zero (i.e., finite zero structure) assignments (see for example, Emami-Naeini and van Dooren, 1982; Karcanias et al, 1988; Kouvaritakis and MacFarlane, 1976; Patel, 1978; Vardulakis, 1980; and Syrmos and Lewis, 1993). It is important to point out that all the results reported in the literature so far, including the ones mentioned above, deal solely with the assignments of the finite zeros. The infinite zero structure and other structures such as invertibility structures of the resulting system are either fixed or of not much concern. Only recently had Chen and Zheng (1995) proposed a technique, which is capable of assigning both finite and infinite zero structures simultaneously. However, up to date, to the best of our knowledge, there still does not exist any method that deals with the assignment of complete system structures, including finite and infinite zero structures and invertibility structures. We propose in this paper a technique which is capable of assigning all these structural properties. More specifically, we consider a linear time-invariant system characterized by the following state space equation,

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input. The problem of structural assignments or sensor selection is to find a measurement output,

$$y = Cx, \tag{2}$$

such that the resulting system characterized by the matrix triple (A, B, C) would have the pre-specified desired structural properties, including finite and infinite zero structures and invertibility structures. We note that this technique can be applied to solve the dual problem of actuator selection, i.e., to find a matrix B, for a given matrix pair A and C, such that the resulting system again characterized by the triple (A, B, C) would have the pre-specified desired structural properties.

The outline of this paper is as follows. We recall in Section 2 some background materials. Section 3 gives our main results, i.e., algorithms or procedures for selecting sensors to result in some pre-specified desired structural properties. In Section 4, we use a benchmark flexible mechanical system to illustrate the main results of this paper. Finally, we make our concluding remarks in Section 5.

Throughout the paper, I_k denotes the identity matrix of dimension $k \times k$. With a slight abuse of notation, I_k with $k \leq 0$ is treated as an empty matrix. Also, $\star$ denotes some constant matrix which is of less interest in the context. A set of complex scalars, $\mathcal{W}$, is said to be self-conjugate if, for any $w \in \mathcal{W}$, its complex conjugate $\bar{w} \in \mathcal{W}$.

2 Background Materials

In this section, we recall two structural decomposition techniques of linear systems, i.e., the controllability structural decomposition(CSD) for a matrix pair (A, B), which was discovered by Luenberger (1967) and Brunovsky (1970), and the special coordinate basis decomposition(SCB) for a matrix triple (A, B, C) (see Sannuti and Saberi, 1987).

Theorem 2.1 (The Controllability Structural Decomposition) Consider a pair of constant matrices (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that B is of full rank. Then, there exist nonsingular state and input transformations T_s and T_i such that $(\tilde{A}, \tilde{B}) := (T_s^{-1}AT_s, T_s^{-1}BT_i)$ has the following form,

$$\left(\begin{bmatrix} A_o & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ \star & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ \star & \star & \star & \cdots & \star & \star \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix} \right), \tag{3}$$

where $k_i > 0$, $i = 1, \cdots, m$, A_o is of dimension $n_o := n - \sum_{i=1}^{m} k_i$ and its eigenvalues are the uncontrollable modes of the pair $(A,\ B)$. Moreover, the set of integers, $\mathcal{C}(A, B) := \{\, n_o,\ k_1,\ \cdots,\ k_m \,\}$, is referred to as the *controllability index* of (A, B). □

Theorem 2.2 (The Special Coordinate Basis.) Consider a linear time-invariant system $\Sigma(A, B, C)$. Without loss of generality, we assume that both B and C are of full rank. There exist nonsingular state, input and output transformations $\Gamma_s \in \mathbb{R}^{n\times n}, \Gamma_o \in \mathbb{R}^{p\times p}$ and $\Gamma_i \in \mathbb{R}^{m\times m}$ such that

$$\tilde{A} := \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{aa} & L_{ab}C_b & 0 & L_{ad}C_d \\ 0 & A_{bb} & 0 & L_{bd}C_d \\ B_cE_{ca} & L_{cb}C_b & A_{cc} & L_{cd}C_d \\ B_dE_{da} & B_dE_{db} & B_dE_{dc} & A_{dd} \end{bmatrix}, \tag{4}$$

$$\tilde{B} := \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}, \tag{5}$$

$$\tilde{C} := \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}, \tag{6}$$

where the pair (A_{bb}, C_b) is observable, (A_{cc}, B_c) is controllable, and the triple (A_{dd}, B_d, C_d) is observable and controllable. They have the special form

$$A_{dd} = \begin{bmatrix} 0 & I_{q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{q_{m_d}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} + L_{dd}C_d + B_dE_{dd} \tag{7}$$

$$B_d = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \tag{8}$$

for some $\mu_i > 0$, $i = 1,\ 2, \ldots, p_b$; $\ell_i > 0$, $i = 1,\ 2, \ldots, m_c$; $q_i > 0$, $i = 1,\ 2, \ldots, m_d$, and some appropriate dimensional matrices L_{bb}, E_{cc}, E_{dd} and L_{dd}. □

In order to display various multiplicities of invariant zeros, let X_a be a nonsingular transformation matrix such that A_{aa} can be transformed into a Jordan canonical form, i.e.,

$$X_a^{-1} A_{aa} X_a = J = \text{blkdiag}\left\{ J_1, J_2, \cdots, J_k \right\}, \tag{9}$$

where J_i, $i = 1, 2, \cdots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_i = \text{diag}\left\{ \alpha_i, \alpha_i, \cdots, \alpha_i \right\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}. \tag{10}$$

For any given $\alpha \in \lambda(A_{aa})$, let there be τ_α Jordan blocks of A_{aa} associated with α. Let $n_{\alpha,1},\ n_{\alpha,2},\ \cdots,\ n_{\alpha,\tau_\alpha}$ be the dimensions of these Jordan blocks. Then, we say α is an invariant zero of Σ with multiplicity structure $S_\alpha^\star(\Sigma)$ (see also Saberi et al, 1991),

$$S_\alpha^\star(\Sigma) = \left\{ n_{\alpha,1}, n_{\alpha,2}, \cdots, n_{\alpha,\tau_\alpha} \right\}. \tag{11}$$

The geometric multiplicity of α is then simply given by τ_α, and the algebraic multiplicity of α is given by $\sum_{i=1}^{\tau_\alpha} n_{\alpha,i}$. Here we should note that the invariant zeros together with their structures of Σ are related to the structural invariant indices list $\mathcal{I}_1(\Sigma)$ of Morse (1973).

The infinite zero structure of Σ is given by

$$S_\infty^\star(\Sigma) = \left\{q_1, q_2, \cdots, q_{m_d}\right\}. \tag{12}$$

That is, each q_i corresponds to an infinite zero of order q_i.

In order to display the structural invariant index list $\mathcal{I}_2(\Sigma)$ of Morse(1973), we transform the controllable pair (A_{cc}, B_c) into the controllability structural decomposition (see Theorem 2.1). $\mathcal{I}_2(\Sigma)$ is the controllability index of (A_{cc}, B_c), $\{\ell_1, \cdots, \ell_{m_c}\}$, i.e.

$$\mathcal{I}_2(\Sigma) = \left\{\ell_1, \cdots, \ell_{m_c}\right\}. \tag{13}$$

Similarly, we have

$$\mathcal{I}_3(\Sigma) = \left\{\mu_1, \cdots, \mu_{p_b}\right\}, \tag{14}$$

where $\left\{\mu_1, \cdots, \mu_{p_b}\right\}$ is the controllability index of the controllable pair (A'_{bb}, C'_b).

3 Structural Assignments of Linear Systems

Having been familiarized with all the structural properties of linear systems, we are now ready to present the main results of this paper.

Theorem 3.1 Consider the linear system (1), i.e.,

$$\dot{x} = Ax + Bu, \tag{15}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Assume that matrix B is of full column rank, the controllability index of the pair (A, B) is given by $\mathcal{C}(A, B) = \{n_o, k_1, \dots, k_m\}$, and the uncontrollable modes of (A, B), if any, are given by $\Delta = \{u_1, \dots, u_{n_o}\}$. Let

$$\Lambda_2 := \{\ell_1, \ell_2, \dots, \ell_{m_c}\} \subset \mathcal{C}^* =: \{k_1, k_2, \cdots, k_m\}, \tag{16}$$

$$\mathcal{C}^* \setminus \Lambda_2 := \{\omega_1, \omega_2, \cdots, \omega_{m_d}\}, \tag{17}$$

$$m_d = m - m_c, \;\; \omega_1 \le \omega_2 \le \cdots \le \omega_{m_d}, \tag{18}$$

$$\Lambda_4 := \{q_1, q_2, \cdots, q_{m_d}\}, \quad q_i \le \omega_i, \;\; i = 1, 2, \cdots, m_d. \tag{19}$$

Moreover, we let a set of complex scalars

$$\Lambda_1 = \Theta_c \cup \Delta_1 := \{z_1, \dots, z_{s_1}\} \cup \Delta_1, \tag{20}$$

where Θ_c is self-conjugate and so is $\Delta_1 \subset \Delta$. Because of space limitation, we assume that the entries of $\Delta_2 = \Delta \setminus \Delta_1$ are distinct. Furthermore, s_1 is chosen such that

$$s_1 \le n - \sum_{i=1}^{m_c} \ell_i - \sum_{i=1}^{m_d} q_i - n_o. \tag{21}$$

Finally, let

$$\Lambda_3 := \{\mu_1, \mu_2, \dots, \mu_{p_b}\} \tag{22}$$

be a set of positive integers with $\mu_1 \le \mu_2 \le \cdots \le \mu_{p_b}$, which satisfy the following constraint:

$$s_1 + n_o + \sum_{i=1}^{p_b} \mu_i + \sum_{i=1}^{m_c} \ell_i + \sum_{i=1}^{m_d} q_i = n. \tag{23}$$

Then, there exists a non-empty set $\mathbf{\Omega} \subset \mathbb{R}^{(m_d+p_b)\times n}$ such that for any $C \in \mathbf{\Omega}$, the resulting system characterized by the matrix triple (A, B, C) has the following properties: its invariant zeros are given by Λ_1, and their invariant indices $\mathcal{I}_2 = \Lambda_2$, $\mathcal{I}_3 = \Lambda_3$ and $\mathcal{I}_4 = \Lambda_4$, or equivalently, the infinite zero structure of the triple (A, B, C) is given by Λ_4, and its invertibility structures are respectively given by Λ_2 and Λ_3. □

Proof. We will give a constructive proof that would yield a desired set $\mathbf{\Omega}$. We first introduce the following key lemma, which is crucial to the proof of Theorem 3.1. For simplicity, we omit the proof of this lemma.

Lemma 3.1 Consider a linear system $\tilde{\Sigma}$ characterized by a matrix triple $(\tilde{A}, \tilde{B}, \tilde{C})$. Without loss of generality, we assume that it is already in the form of the special coordinate basis of Theorem 2.2. Let

$$\breve{A} := \begin{bmatrix} A_{aa} & M_{ab} & 0 & M_{ad} \\ 0 & A_{bb} & 0 & M_{bd} \\ B_c E_{ca} & B_c E_{cb} & A_{cc} & M_{cd} \\ B_d E_{da} & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix}, \tag{24}$$

with any constant submatrices M_{ab}, M_{ad}, M_{bd} and M_{cd} of appropriate dimensions. Then, the triple $(\breve{A}, \tilde{B}, \tilde{C})$ has the same structural invariant indices $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ as those of $\tilde{\Sigma}$. □

Now, we are ready to give a proof to Theorem 3.1. It follows from Theorem 2.1 that there exist nonsingular state and input transformations T_0 and T_i such that the transformed pair

$$(A_1, B_1) := \left(T_0^{-1} A T_0, T_0^{-1} B T_i\right) \tag{25}$$

is in the CSD form of (3) with its controllability index being as $\mathcal{C}(A, B) = \{n_o, k_1, \dots, k_m\}$. In view of the properties of the special coordinate basis, it is simple to see that each input channel in B_1 could either be assigned to the state variables associated with x_c or x_d of the resulting system. However, if we assign a particular input channel to be a member of x_c of the desired system, we will have to assign the whole block associated with this particular channel to it. This is because of the following reasons: 1) the whole block is completely controllable by the input channel; and 2) both dynamics of x_a and x_b cannot be controlled by input channels associated with x_c. On the other hand, there is no such a constraint for the structure associated with x_d, i.e., the infinite zero structure.

Let Λ_2 and Λ_4 be given respectively as in (16) and (19), and let

$$n_c=\sum_{i=1}^{m_c}\ell_i \quad \text{and} \quad n_d=\sum_{i=1}^{m_d}q_i. \tag{26}$$

It is simple to verify that there exist permutation transformations P_1 and P_{i1} such that

$$A_2=P_1^{-1}A_1P_1=\begin{bmatrix} A_o & 0 & 0 \\ B_c\cdot\star & A_{cc} & B_c\cdot\star \\ \tilde{B}_d\cdot\star & \tilde{B}_d\cdot\star & A_* \end{bmatrix}, \tag{27}$$

$$B_2=P_1^{-1}B_1P_{i1}=\begin{bmatrix} 0 & 0 \\ B_c & 0 \\ 0 & \tilde{B}_d \end{bmatrix}, \tag{28}$$

where

$$A_{cc}:=\begin{bmatrix} 0 & I_{\ell_1-1} & \dots & 0 & 0 \\ \star & \star & \dots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{\ell_{m_c}-1} \\ \star & \star & \dots & \star & \star \end{bmatrix}, \tag{29}$$

$$B_c=\begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}, \tag{30}$$

and

$$\tilde{B}_d=\begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}, \quad A_*:=\begin{bmatrix} 0 & I_{\omega_1-q_1-1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{q_1-1} & \dots & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \dots & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & I_{\omega_{m_d}-q_{m_d}-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_{q_{m_d}-1} \\ \star & \star & \star & \star & \dots & \star & \star & \star & \star \end{bmatrix}, \tag{31}$$

and where $\star$s are submatrices of less interest.

Next, it is simple to see that there exists another pair of permutation matrices P_2 and P_{i2} such that the transformed pair $(A_3,B_3):=(P_2^{-1}A_2P_2,P_2^{-1}B_2P_{i2})$ has the following form,

$$A_3=\begin{bmatrix} A_o & 0 & 0 & 0 \\ 0 & A_{ab}^* & 0 & \star \\ B_c\cdot\star & B_c\cdot\star & A_{cc} & B_c\cdot\star \\ B_d\cdot\star & B_d\cdot\star & B_d\cdot\star & A_{dd}^*+B_d\cdot\star \end{bmatrix}, \tag{32}$$

$$B_3=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}, \tag{33}$$

where

$$A_{dd}^*=\begin{bmatrix} 0 & I_{q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{q_{m_d}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{34}$$

$$B_d=\begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}, \tag{35}$$

and

$$A_{ab}^*=\begin{bmatrix} 0 & I_{\omega_1-q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{\omega_{m_d}-q_{m_d}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{36}$$

Let us define

$$C_d=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{37}$$

which is in conformity with the structures of A_{dd}^* and B_d in (35), and we further define

$$C_3=[\,0 \quad 0 \quad 0 \quad C_d\,], \tag{38}$$

which is in conformity with structures of A_3 and B_3 in (33). The result of Lemma 3.1 implies that there exists a nonsingular state transformation T_3 such that

$$A_4=T_3^{-1}A_3T_3=\begin{bmatrix} A_{ab} & 0 & L_{abd}C_d \\ B_c\cdot\star & A_{cc} & L_{cd}C_d \\ B_d\cdot\star & B_d\cdot\star & A_{dd}^*+B_d\cdot\star \end{bmatrix}, \tag{39}$$

$$B_4=T_3^{-1}B_3=\begin{bmatrix} 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}, \tag{40}$$

and

$$C_4=C_3T_3=C_3=[\,0 \quad 0 \quad C_d\,], \tag{41}$$

where

$$A_{ab}=\begin{bmatrix} A_o & 0 \\ 0 & A_{ab}^* \end{bmatrix}, \quad L_{abd}=\begin{bmatrix} 0 \\ L_{abd}^* \end{bmatrix}. \tag{42}$$

In view of the properties of the special coordinate basis, it is simple to see that the triple (A_4, B_4, C_4) is in the form of the SCB with its structural invariant indices $\mathcal{I}_2 = \Lambda_2$ and $\mathcal{I}_4 = \Lambda_4$, $\mathcal{I}_3$ being empty and its invariant zeros being $\lambda(A_{ab})$.

Next, we define a new output matrix,

$$\check{C}_4 := C_4 + [\,K_c \quad 0 \quad 0\,] = [\,K_c \quad 0 \quad C_d\,], \tag{43}$$

where

$$K_c = [\,K_{c1} \quad K_{c2}\,], \tag{44}$$

which is partitioned in conformity with A_{ab} and L_{abd} in (42) with K_{c1} being an arbitrary matrix with appropriate dimension and K_{c2} being chosen such that

$$\Theta_c \subset \lambda\,(A_{ab}^* - L_{abd}^* K_{c2})\,, \tag{45}$$

and the remaining eigenvalues of $A_{ab}^* - L_{abd}^* K_{c2}$ are real and distinct. Moreover, these remaining eigenvalues of $A_{ab}^* - L_{abd}^* K_{c2}$ are distinct from the entries of Δ_2. This can be done because the pair (A_{ab}^*, L_{abd}^*) is completely controllable. Using the result of Chen et al (1992), we can show that there exists a state transformation T_4 such that

$$A_5 = T_4^{-1} A_4 T_4 = \begin{bmatrix} A_{ab} - L_{abd} K_c & 0 & \tilde{L}_{abd} C_d \\ B_c \cdot \star & A_{cc} & L_{cd} C_d \\ B_d \cdot \star & B_d \cdot \star & A_{dd} + B_d \cdot \star \end{bmatrix}, \tag{46}$$

$$B_5 = T_4^{-1} B_4 = \begin{bmatrix} 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}, \tag{47}$$

and

$$C_5 := \check{C}_4 T_4 = [\,0 \quad 0 \quad C_d\,]. \tag{48}$$

Again, the triple (A_5, B_5, C_5) is in the form of SCB and has the same structural indices $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ as the triple (A_4, B_4, C_4). Moreover, its invariant zeros are given by the eigenvalues of matrix $A_{ab} - L_{abd} K_c$, in which matrix $A_{ab} - L_{abd} K_c$ can be rewritten as,

$$A_{ab} - L_{abd} K_c = \begin{bmatrix} A_o & 0 \\ -L_{abd}^* K_{c1} & A_{ab}^* - L_{abd}^* K_{c2} \end{bmatrix}. \tag{49}$$

We next find a transformation T_{ab} such that $A_{ab} - L_{abd} K_c$ is transformed into the following form,

$$\tilde{A}_{ab} = T_{ab}^{-1}\,(A_{ab} - L_{abd} K_c)\,T_{ab} = \begin{bmatrix} A_{aa} & M_{ab} \\ 0 & A_{bb} \end{bmatrix}, \tag{50}$$

where $\lambda(A_{aa}) = \Lambda_1 = \Delta_1 \cup \Theta_c$ with Θ_c given in (20), and A_{bb} being a diagonal matrix. Let

$$T_5 = \begin{bmatrix} T_{ab} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{51}$$

Then, we have

$$A_6 = T_5^{-1} A_5 T_5 = \begin{bmatrix} A_{aa} & M_{ab} & 0 & L_{ad} C_d \\ 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c \cdot \star & B_c \cdot \star & A_{cc} & L_{cd} C_d \\ B_d \cdot \star & B_d \cdot \star & B_d \cdot \star & A_{dd} + B_d \cdot \star \end{bmatrix}, \tag{52}$$

$$B_6 = T_5^{-1} B_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}, \tag{53}$$

and

$$C_6 := C_5 T_5 = [\,0 \quad 0 \quad 0 \quad C_d\,]. \tag{54}$$

The remaining task is to assign the structural invariant indices $\mathcal{I}_3$ to coincide with the given set $\Lambda_3 = \{\mu_1, \cdots, \mu_{p_b}\}$, which can be done by choosing the following output matrix,

$$\tilde{C}_6 = \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}, \tag{55}$$

where

$$C_b = \begin{bmatrix} C_{b1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{bp_b} \end{bmatrix}, \tag{56}$$

and where C_{bi}, $i = 1, 2, \cdots, p_b$, is a $1 \times \mu_i$ vector with all its entries being nonzero. Utilizing the result of Lemma 3.1 one more time, we can show that the triple characterized by $(A_6, B_6, \tilde{C}_6)$ has its invariant zeros at $\lambda(A_{aa})$, and its structural invariant indices $\mathcal{I}_2 = \Lambda_2$, $\mathcal{I}_3 = \Lambda_3$ and $\mathcal{I}_4 = \Lambda_4$, respectively. Let $p = m_d + p_b$. We finally obtain the desired set,

$$\mathbf{\Omega} = \left\{ \Gamma_o \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix} (T_0 P_1 P_2 T_3 T_4 T_5)^{-1} : \Gamma_o \in \mathbb{R}^{p \times p} \text{ is nonsingular} \right\}. \tag{57}$$

This completes the proof of Theorem 3.1. ∎

4 An Illustrative Example

We study a benchmark problem for robust control of a two-mass-spring flexible mechanical system proposed by Wie and Bernstein (1990). We will identify sets of sensors that would yield the best performance under the H_∞ almost disturbance decoupling framework. The dynamic model of the system is given by

$$\dot{x} = Ax + Bu + Ew = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{58}$$

and

$$z = C_2 x = [\,0 \quad 0 \quad 1 \quad 0\,]\,x, \tag{59}$$

where x_1 and x_2 are respectively the positions of Mass 1 and Mass 2.

It is simple to verify that the subsystem (A, B, C_2) is of minimum phase and invertible. Hence, the disturbance w can be totally decoupled from the output to be controlled z under the full state feedback. Our objective next is to identify sets of measurement output or the locations of sensors that would yield the same performance as that of the state feedback case. It follows from the result of Chen (2000) that this can be made possible by choosing a measurement output, $y = C_1 x$, such that the resulting subsystem (A, E, C_1) is left invertible and of minimum phase.

We first transform the pair (A, E) into the controllability structural decomposition form of Theorem 2.1. Following the proof of Theorem 3.1, we obtain the following set of measurement matrices,

$$\mathbf{\Omega}_1 = \left\{ \Gamma_o \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \middle| \Gamma_o \in \mathbb{R}^{2\times 2}, \det(\Gamma_o) \neq 0 \right\}, \tag{60}$$

such that for any $C_1 \in \mathbf{\Omega}_1$, the resulting subsystem (A, E, C_1) is square invertible with two infinite zeros of order 2 and with no invariant zeros. Hence, it is of minimum phase. It is well known that higher orders of infinite zeros would yield higher controller gains, which is in general not desirable in practical situations. In what follows, we will identify a set of measurement matrices, $\mathbf{\Omega}_2$, such that for any $C_1 \in \mathbf{\Omega}_2$, the resulting subsystem (A, E, C_1) is of minimum phase and square invertible with two infinite zeros of order 1 and two invariant zeros at -1 and -1, respectively. The following $\mathbf{\Omega}_2$ is such a set obtained again using the procedure given in the proof of Theorem 3.1:

$$\mathbf{\Omega}_2 = \left\{ \Gamma_o \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \middle| \Gamma_o \in \mathbb{R}^{2\times 2}, \det(\Gamma_o) \neq 0 \right\}. \tag{61}$$

Thus, it is straightforward to verify that the H_∞ almost disturbance decoupling is achievable for the flexible mechanical system of (58)-(59) together with a measurement output $y = C_1 x$, where $C_1 \in \mathbf{\Omega}_1$ or $C_1 \in \mathbf{\Omega}_2$. In fact, we can show that the H_∞ almost disturbance decoupling for the system cannot be achieved if there is only one sensor allowed to be placed in the system, i.e., one would have to place two or more sensors in the system in order to decouple the disturbance (the frictions) to the position of the second mass. □

5 Conclusions

In this paper, we have for the first time considered the problem of assigning a complete structure to a linear system through the choice of the output matrix, or sensor locations. The complete structure of a linear system is captured by the so-called Morse's structural invariant indices, which describe various structural properties of a linear system, including finite and infinite zero structure and the invertibility structure. We proposed a systematic method for constructing a family of output matrices for a given matrix pair (A, B). Any matrix C from this family will result in a linear system (A, B, C) that has the pre-specified complete structure.

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