

Averaging with respect to arbitrary closed sets: closeness of solutions for systems with disturbances

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Abstract

We consider two different definitions of “average” for systems with disturbances: the “strong” and “weak” averages that were introduced in [7]. Our definitions are more general than those in [7] as we use the distance to an arbitrary closed set \mathcal{A} instead of the Euclidean norm for states in the definitions of averages. This generalization allows us to deal with more general cases of averaging for systems with disturbances, such as partial averaging. Under appropriate conditions, the solutions of a time-varying system with disturbances are shown to converge uniformly on compact time intervals to the solutions of the system’s average as the rate of change of time increases to infinity.

1 Introduction

Averaging is an important approximate method for analysis of time-varying systems. In its classical form (see, for example, [1, 2, 4, 9, 12] and references therein) it applies to ordinary differential equations of the form

$$\dot{x} = f(t/\epsilon, x, \epsilon) \quad (1)$$

where $\epsilon > 0$ and where f has an average f_{av} satisfying a condition like

$$f_{av}(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} f(\tau, x, 0) d\tau \quad \forall t \geq 0.$$

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Classical averaging results state that, under appropriate smoothness assumptions on f , the solutions of (1) converge uniformly on compact time intervals to the solutions of

$$\dot{x} = f_{av}(x) \quad (2)$$

as ϵ tends to zero.¹ Moreover, if the system (2) has an exponentially stable equilibrium point p that is an equilibrium point of (1) for small $\epsilon > 0$ then p is an exponentially stable equilibrium point of (1) for small $\epsilon > 0$.

Except for [7], we are not aware of any results on averaging that consider systems with exogenous disturbances. However, systems with disturbances occur frequently in control theory. Recently, in [7], two different definitions of average for systems with disturbances (“strong” and “weak” average) were introduced and results were presented on deducing input-to-state stability (ISS) for a system from input-to-state stability for the system’s strong or weak average. These results generalize, in a sense, the exponential stability result mentioned above, as well as more recent stability results based on averaging, like in [10].

Given an arbitrary closed set \mathcal{A} , which is not necessarily compact, in this note we study systems with disturbances that possess an \mathcal{A} -weak or \mathcal{A} -strong average (see Definitions 1 and 2, respectively), that is averages defined with respect to the set \mathcal{A} . The definitions that we use in this paper are more general than those used in [7], since in [7] we only considered the case of \mathcal{A} -weak and \mathcal{A} -strong averages when $\mathcal{A} = \{0\}$.

¹Sometimes averaging is applied to systems of the form $\frac{dx}{d\tau} = \epsilon f(\tau, x, \epsilon)$ instead of the form (1). In this case, the convergence of solutions is established on intervals of length proportional to $1/\epsilon$. This is seen to be an equivalent result by scaling time as $\tau = t/\epsilon$.

The generalized definitions allow us for instance to deal with *partial averaging* of systems with disturbances (for some classical results on partial averaging see [3, pp. 190-195]; for some recent results on partial averaging of disturbance-free ordinary differential equations see [8]; for partial averaging of disturbance-free functional differential equations, see [6] and references therein).

More precisely, we study systems of the form

$$\dot{x} = f(t/\epsilon, x, w(t), \nu) \quad x(t_0) = x_0 \quad (3)$$

where $w(\cdot)$ is the exogenous disturbance and ν is a parameter vector with small norm. We are interested in conditions that guarantee that the solutions of (3) are close, on compact time intervals, to the solutions of (3)'s \mathcal{A} -average which is taken when $\nu = 0$. The average system has the form

$$\dot{y} = f_{av}^A(y, w(t)) \quad y(t_0) = y_0 \quad (4)$$

where f_{av}^A is assumed to be locally Lipschitz. We will not impose any stability assumptions on the \mathcal{A} -average system. Moreover, for f we will only assume that it is continuous in (x, w, ν) uniformly in t . In particular, we do not assume uniqueness of solutions for (3). We assume that w belongs to a set of functions that is equi-bounded (see Definition 4) for the case of \mathcal{A} -strong averages, or equi-bounded and equi-uniformly continuous (see Definition 5) for the case of \mathcal{A} -weak averages. We will show, among other things, that when a trajectory for the \mathcal{A} -averaged system is defined on a given compact time interval, the trajectories of the actual system converge to that average trajectory uniformly on the compact time interval.

The paper is organized as follows: In Section 2 we present some preliminary definitions including the definitions of \mathcal{A} -weak and \mathcal{A} -strong average that generalize the definitions of strong and weak averages from [7]. Our main results are stated formally in Theorems 1 and 2 of Section 3. In the last Section we provide the proof of main results.

2 Preliminaries

For our purposes, a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- \mathcal{KL} if it is nondecreasing from zero in its first argument and converging to zero in its second argument. Given a measurable function $w(t)$, we define its infinity norm $\|w\|_{\infty} := \text{ess sup}_{t \geq 0} |w(t)|$. If we have $\|w\|_{\infty} < \infty$, then we write $w \in \mathcal{L}_{\infty}$. If $w(t)$ is absolutely continuous, its derivative is defined almost everywhere and we can write $w(t) - w(t_0) = \int_{t_0}^t \dot{w}(\tau) d\tau$. Given

an arbitrary set $\mathcal{A} \subset \mathbb{R}^n$, we define the distance of a point $x \in \mathbb{R}^n$ to the set \mathcal{A} as:

$$|x|_{\mathcal{A}} := \inf_{s \in \mathcal{A}} |x - s|,$$

where $|x|$ is the Euclidean norm of x . We say that a function $f(\cdot, w)$ is \mathcal{A} -locally Lipschitz if given any triple of strictly positive real numbers (R_1, R_2, R_3) there exists $L > 0$, such that

$$|f(x, w) - f(y, w)| \leq L|x - y|,$$

for all $|x|_{\mathcal{A}} \leq R_1$, $|y|_{\mathcal{A}} \leq R_2$, $|w| \leq R_3$. The following two definitions of \mathcal{A} -strong and \mathcal{A} -weak average for a time-varying system with exogenous disturbances generalize the definitions that were introduced in [7]:

Definition 1 (\mathcal{A} -weak average) *Let $\mathcal{A} \subset \mathbb{R}^n$ be an arbitrary closed set. An \mathcal{A} -locally Lipschitz function $f_{wa}^A : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be the \mathcal{A} -weak average of $f(t, x, w, \nu)$ if there exist $\beta \in \mathcal{KL}$ and $T^* > 0$ such that $\forall T \geq T^*$ and $\forall t \geq 0$ we have²*

$$\left| f_{wa}^A(x, w) - \frac{1}{T} \int_t^{t+T} f(s, x, w, 0) ds \right| \leq \beta \left(\max\{|x|_{\mathcal{A}}, |w|, 1\}, T \right). \quad (5)$$

The \mathcal{A} -weak average of system (3) is then defined as

$$\dot{y} = f_{wa}^A(y, w). \quad (6)$$

If $\mathcal{A} = \{0\}$, then \mathcal{A} -weak average is referred to simply as weak average. \square

Definition 2 (\mathcal{A} -strong average) *Let $\mathcal{A} \subset \mathbb{R}^n$ be an arbitrary closed set. An \mathcal{A} -locally Lipschitz function $f_{sa}^A : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be the \mathcal{A} -strong average of $f(t, x, w, \nu)$ if there exist $\beta \in \mathcal{KL}$ and $T^* > 0$ such that $\forall w \in \mathcal{L}_{\infty}$, $\forall T \geq T^*$ and $\forall t \geq 0$ the following holds:*

$$\left| \frac{1}{T} \int_t^{t+T} \left[f_{sa}^A(x, w(s)) - f(s, x, w(s), 0) \right] ds \right| \leq \beta \left(\max\{|x|_{\mathcal{A}}, \|w\|_{\infty}, 1\}, T \right). \quad (7)$$

The \mathcal{A} -strong average of system (3) is then defined as

$$\dot{y} = f_{sa}^A(y, w). \quad (8)$$

If $\mathcal{A} = \{0\}$, then \mathcal{A} -strong average is referred to simply as strong average. \square

²Note that w in the integral is a constant vector.

Remark 1 Definitions 1 and 2 generalize the definitions introduced in [7], since in [7] the set \mathcal{A} was always chosen to be the origin in \mathbb{R}^n . This generalization allows us, for instance, to state results on partial averaging of systems with disturbances. Indeed, consider:

$$\dot{\tilde{x}} = \tilde{f}(t/\epsilon, t, \tilde{x}, w) . \quad (9)$$

If we want to average \tilde{f} only in the first argument and leave the dependence on t in the second argument, then the average system is time-varying and we have partially averaged \tilde{f} . Our results apply to this situation in the following way. Introduce a new state variable $p = t$ and rewrite (9) as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(t/\epsilon, p, \tilde{x}, w) \\ \dot{p} &= 1 . \end{aligned} \quad (10)$$

Introducing $x := (\tilde{x}^T \ p)^T$ and $f := (\tilde{f}^T \ 1)^T$, we can rewrite (10) as $\dot{x} = f(t/\epsilon, x, w)$. Let the closed non-compact set be defined as $\mathcal{A} := \{x : \tilde{x} = 0\}$. Then, partial weak or strong averaging results for (9) can be recovered by using \mathcal{A} -weak or \mathcal{A} -strong averaging results respectively. \square

Remark 2 It has been shown in [7] that functions f that have a strong average are, in essence, functions of the form $f(t, x, w, 0) = \tilde{f}(t, x) + g(x, w)$ where $\tilde{f}(t, x)$ has a well-defined (weak) average. \square

We also need definitions of forward completeness, equi-boundedness and equi-uniform continuity.

Definition 3 Let \mathcal{F} be a set of locally essentially bounded functions, \mathcal{A} an arbitrary closed set and let f_{av} be a continuous function. The system

$$\dot{x} = f_{av}(x, w) \quad x(0) = x_0 \quad (11)$$

is said to be \mathcal{FA} -forward complete if for each $r > 0$ and $T > 0$ there exists $R \geq r$ such that, for all $|x_0|_{\mathcal{A}} \leq r$ and $w(\cdot) \in \mathcal{F}$, the solutions $x(t)$ of (11) exist and $|x(t)|_{\mathcal{A}} \leq R$ for all $t \in [0, T]$. If $\mathcal{A} = \{0\}$, and the system is \mathcal{FA} -forward complete, then we simply say that the system is \mathcal{F} -forward complete.

Definition 4 Let \mathcal{F} be a set of locally essentially bounded functions. The set \mathcal{F} is equi-(essentially) bounded if there exists a strictly positive real number Ω such that, for all $w(\cdot) \in \mathcal{F}$, $\|w\|_{\infty} \leq \Omega$;

Definition 5 Let \mathcal{F} be a set of locally essentially bounded functions. The set \mathcal{F} is equi-uniformly continuous if for each $\rho > 0$ there exists $\delta > 0$ such that, for all $w(\cdot) \in \mathcal{F}$ and all $t \geq 0$,

$$\tau \in [0, \delta] \quad \implies \quad |w(t + \tau) - w(t)| \leq \rho .$$

Remark 3 A sufficient condition for \mathcal{F} to be equi-uniformly continuous is that all $w(\cdot) \in \mathcal{F}$ are absolutely continuous (on $[0, \infty)$) and there exists a strictly positive real number Ω_1 such that, for all $w(\cdot) \in \mathcal{F}$, $\|w\|_{\infty} \leq \Omega_1$, i.e., $w(\cdot)$ is Lipschitz. \square

3 Main Results

Our main results give conditions under which the solutions of (3) are close to the solutions of (3)'s \mathcal{A} -weak or \mathcal{A} -strong average, when these averages exist.

Theorem 1 (Closeness to \mathcal{A} -weak average)
Suppose an arbitrary closed set \mathcal{A} is given and also:

1. the function $f(t, x, w, \nu)$ is:
 - (a) measurable in t for each (x, w, ν) ,
 - (b) continuous in x uniformly in t for each pair of (w, ν) ,
 - (c) for any $R > 0$, continuous in (w, ν) , uniformly in t and $x \in \{x : |x|_{\mathcal{A}} \leq R\}$,
 - (d) for every $R > 0$ there exists $B > 0$ such that $|f(t, x, 0, 0)| \leq B$ for all t and $x \in \{x : |x|_{\mathcal{A}} \leq R\}$;
2. the set \mathcal{F} is equi-(essentially) bounded and equi-uniformly continuous;
3. the \mathcal{A} -weak average of the system (3) exists and is \mathcal{FA} -forward complete.

Then, for each triple (T, δ, r) of strictly positive real numbers there exists a triple (ϵ^*, ν^*, μ) of strictly positive real numbers such that, for each $\epsilon \in (0, \epsilon^*)$, $|\nu| < \nu^*$, $t_0 \geq 0$, $|y_0|_{\mathcal{A}} \leq r$, $w \in \mathcal{F}$ and each x_0 such that $|x_0 - y_0| \leq \mu$, each solution $x_{\epsilon, \nu}(t, t_0, x_0, w)$ of (3) and the solution $y(t - t_0, y_0, w)$ of the weak average satisfy

$$|x_{\epsilon, \nu}(t, t_0, x_0, w) - y(t - t_0, y_0, w)| \leq \delta \quad (12)$$

for all $t \in [t_0, t_0 + T]$. \square

Without the assumption that \mathcal{F} is equi-uniformly continuous, the conclusion of Theorem 1 is not correct, in general. This is demonstrated by the system $\dot{x} = -0.25x^3 + \cos(t/\epsilon)x^3w$, which was discussed in detail in [7]. There it was shown that the weak average of this system is $\dot{y} = -0.25y^3$ but the system under the input $w(t) = \cos(t/\epsilon)$ exhibits finite escape time.

The assumption that \mathcal{F} is equi-uniformly continuous can be removed when the strong average exists and is \mathcal{F} -forward complete:

Theorem 2 (Closeness to \mathcal{A} -strong average)

Suppose an arbitrary closed set \mathcal{A} is given and also:

1. the function $f(t, x, w, \nu)$ is:
 - (a) measurable in t for each (x, w, ν) ,
 - (b) continuous in x uniformly in t for each pair of (w, ν) ,
 - (c) for any $R > 0$ continuous in (w, ν) , uniformly in t and $x \in \{x : |x|_{\mathcal{A}} \leq R\}$,
 - (d) for every $R > 0$ there exists $B > 0$ such that $|f(t, x, 0, 0)| \leq B$ for all t and $x \in \{x : |x|_{\mathcal{A}} \leq R\}$;
2. the set \mathcal{F} is equi-(essentially) bounded;
3. the \mathcal{A} -strong average of the system (3) exists and is $\mathcal{F}\mathcal{A}$ -forward complete.

Then, for each triple (T, δ, r) of strictly positive real numbers there exists a triple (ϵ^*, ν^*, μ) of strictly positive real numbers such that, for each $\epsilon \in (0, \epsilon^*)$, $|\nu| < \nu^*$, $t_o \geq 0$, $|y_o|_{\mathcal{A}} \leq r$, $w \in \mathcal{F}$ and each x_o such that $|x_o - y_o| \leq \mu$, each solution $x_{\epsilon, \nu}(t, t_o, x_o, w)$ of (3) and the solution $y(t - t_o, y_o, w)$ of the strong average satisfy

$$|x_{\epsilon, \nu}(t, t_o, x_o, w) - y(t - t_o, y_o, w)| \leq \delta \quad (13)$$

for all $t \in [t_o, t_o + T]$. \square

The above results can also be applied to average systems that are not \mathcal{F} -forward complete (see, for instance, [11] for the appropriate modifications needed for the case when $\mathcal{A} = \{0\}$).

4 Proof of Theorem 1

Since the proof of Theorem 2 follows exactly the same steps as the proof of Theorem 1 with the appropriate changes, we present below only the proof of Theorem 1 (for more details on the proof of Theorem 2 for the case of $\mathcal{A} = \{0\}$ see [11]).

Step 1: Definition of ϵ^*, ν^* and μ

The triple (T, δ, r) is given. Without loss of generality, assume $\delta < 1$. Let $R \geq r$ come from $\mathcal{F}\mathcal{A}$ -forward completeness of the weak average (Definition 1) and let Ω come from equi-(essential) boundedness of \mathcal{F} (Definition 4). From the definition of \mathcal{A} -weak average (in particular because $f_{wa}^{\mathcal{A}}$ is \mathcal{A} -locally Lipschitz - see Definition 1) it follows that there exists $L > 0$ such that, for all (x, y) satisfying $|x|_{\mathcal{A}} \leq R + 1$, $|y|_{\mathcal{A}} \leq R$, and for all w satisfying $|w| \leq \Omega$ we have

$$|f_{wa}^{\mathcal{A}}(x, w) - f_{wa}^{\mathcal{A}}(y, w)| \leq L|x - y|. \quad (14)$$

Then define

$$\mu := \exp\left(-\frac{1}{2}LT\right) \frac{\delta}{2} \quad (15)$$

and let ν^* be such that

$$\left|f\left(\frac{t}{\epsilon}, x, w, \nu\right) - f\left(\frac{t}{\epsilon}, x, w, 0\right)\right| \leq \frac{0.125 L \delta^2}{e^{LT} - 1} \quad (16)$$

for all $|\nu| \leq \nu^*$, $|w| \leq \Omega$, $|x|_{\mathcal{A}} \leq R + 1$, $t \geq 0$. Condition 1(c) of Theorem guarantees that such ν^* always exists. In preparation for defining ϵ^* , let $\beta \in \mathcal{KL}$ and $T^* > 0$ come from Definition 1 and let $\tilde{T} \geq T^*$ satisfy

$$\beta\left(\max\{R + 1, \Omega\}, \tilde{T}\right) \leq \frac{\delta^2}{12T \exp(LT)}. \quad (17)$$

According to 1(c) and 1(d) conditions of Theorem and Definition 1, the quantity

$$B := \sup_{\left\{ \begin{array}{l} t \geq 0, \\ |x|_{\mathcal{A}} \leq R + 1, \\ |y|_{\mathcal{A}} \leq R, \\ |w| \leq \Omega, \\ |\nu| \leq \nu^* \end{array} \right\}} \max\left\{|f(t, x, w, \nu)|, |f_{wa}^{\mathcal{A}}(y, w)|\right\} \quad (18)$$

is finite. Define

$$g(t/\epsilon, \tilde{w}) := \tilde{w}_1^T [f(t/\epsilon, \tilde{w}_2, \tilde{w}_3, 0) - f_{wa}^{\mathcal{A}}(\tilde{w}_2, \tilde{w}_3)] \quad (19)$$

the \tilde{w}_i being components, of appropriate dimension, of a vector \tilde{w} . Define $\tilde{\mathcal{F}}$ to be the set of functions

$$\tilde{w}(\cdot) = \begin{bmatrix} \tilde{w}_1(\cdot) \\ \tilde{w}_2(\cdot) \\ \tilde{w}_3(\cdot) \end{bmatrix} \quad (20)$$

such that $\tilde{w}_3(\cdot) \in \tilde{\mathcal{F}}$, and $\tilde{w}_1(\cdot)$, $\tilde{w}_2(\cdot)$ are absolutely continuous with $\|\tilde{w}_1\|_\infty \leq 1$, $\|\tilde{w}_2\|_\infty \leq 2B$, $\|\tilde{w}_3\|_\infty \leq R+1$, and $\|\tilde{w}_1\|_\infty \leq B$. Let $\rho > 0$ be such that, for all $\tilde{w} \in \tilde{\mathcal{F}}$ and all $t_i \geq 0$, if $s \in [t_i, t_i + \rho]$ then

$$|\exp[L(t_i - s)]g(s/\epsilon, \tilde{w}(s)) - g(s/\epsilon, \tilde{w}(t_i))| \quad (21)$$

is less than $\frac{\delta^2}{12T \exp(LT)}$. This ρ exists since g is continuous in \tilde{w} uniformly in t , $\tilde{\mathcal{F}}$ is equi-uniformly continuous and for $s = t_i$ the quantity being bounded in (21) is zero. Then define

$$\epsilon^* := \min \left\{ \frac{\rho}{\tilde{T}}, \frac{\delta^2}{12(2B\tilde{T} \exp(LT))} \right\}. \quad (22)$$

Step 2: Comparison of solutions

Let $\epsilon \in (0, \epsilon^*)$, $|\nu| < \nu^*$, $|y_0|_{\mathcal{A}} \leq r$, $t_0 \geq 0$, $w \in \mathcal{F}$ and consider any x_0 such that $|x_0 - y_0| \leq \mu$. Define

$$e_{\epsilon, \nu}(t) := x_{\epsilon, \nu}(t, t_0, x_0, w) - y(t - t_0, y_0, w) \quad (23)$$

and note that $|e_{\epsilon, \nu}(t_0)| \leq \mu < \frac{\delta}{2} \leq \frac{1}{2} < 1$. If $|e_{\epsilon, \nu}(t)| < 1$ for all $t \in [t_0, t_0 + T]$ then define $\bar{t} = t_0 + T$. Otherwise, define

$$\bar{t} = \inf \{t \in [t_0, t_0 + T] : |e_{\epsilon, \nu}(t)| = 1\}. \quad (24)$$

Note that $\bar{t} > t_0$ and $e_{\epsilon, \nu}(\cdot)$ and $x_{\epsilon, \nu}(\cdot, t_0, x_0, w)$ are defined and absolutely continuous on $[t_0, \bar{t}]$. Let $\tilde{w}(\cdot) \in \tilde{\mathcal{F}}$ be such that, for all $t \in [t_0, \bar{t}]$,

$$\begin{bmatrix} \tilde{w}_1(t) \\ \tilde{w}_2(t) \\ \tilde{w}_3(t) \end{bmatrix} = \begin{bmatrix} e_{\epsilon, \nu}(t) \\ x_{\epsilon, \nu}(t, t_0, x_0, w) \\ w(t) \end{bmatrix}. \quad (25)$$

Such a $\tilde{w}(\cdot) \in \tilde{\mathcal{F}}$ exists since, for all $t \in [t_0, \bar{t}]$, $|e_{\epsilon, \nu}(t)| \leq 1$. Indeed, since $|y(t - t_0, y_0, w)|_{\mathcal{A}} \leq R$ for all $t \in [t_0, t_0 + T]$, it follows that $|x_{\epsilon, \nu}(t, t_0, x_0, w)|_{\mathcal{A}} \leq R + 1$ for all $t \in [t_0, \bar{t}]$. In turn, it follows from (18) that, for almost all $t \in [t_0, \bar{t}]$, $|\dot{e}_{\epsilon, \nu}(t)| \leq 2B$ and $|\dot{x}_{\epsilon, \nu}(t, t_0, x_0, w)| \leq B$. For almost all $t \in [t_0, \bar{t}]$ we have (dropping the arguments of signals for notational convenience)

$$\begin{aligned} \dot{e}_\epsilon &= f(t/\epsilon, x_{\epsilon, \nu}, w, \nu) - f_{wa}^A(y, w) \\ &= [f_{wa}^A(x_{\epsilon, \nu}, w) - f_{wa}^A(y, w)] \\ &\quad + [f(t/\epsilon, x_{\epsilon, \nu}, w, 0) - f_{wa}^A(x_{\epsilon, \nu}, w)] \\ &\quad + [f(t/\epsilon, x_{\epsilon, \nu}, w, \nu) - f(t/\epsilon, x_{\epsilon, \nu}, w, 0)]. \end{aligned} \quad (26)$$

For the scalar-valued function $V(t) := \frac{1}{2}e_{\epsilon, \nu}^T(t)e_{\epsilon, \nu}(t)$, which is also absolutely continuous on $[t_0, \bar{t}]$, we have $V(t_0) \leq \frac{1}{2}\mu^2 = \exp(-LT)\frac{\delta^2}{8}$ and, for almost all $t \in [t_0, \bar{t}]$,

$$\begin{aligned} \dot{V} &\leq LV \\ &\quad + e_{\epsilon, \nu}^T [f\left(\frac{t}{\epsilon}, x_{\epsilon, \nu}, w, 0\right) - f_{wa}^A(x_{\epsilon, \nu}, w)] \\ &\quad + \frac{L\delta^2}{8\{\exp(LT) - 1\}} \\ &= LV + g(t/\epsilon, \tilde{w}(t)) + \frac{L\delta^2}{8\{\exp(LT) - 1\}} \end{aligned} \quad (27)$$

where we have used the definition of \bar{t} , (16), (19) and (25). By standard comparison theorems it follows that for all $t \in [t_0, \bar{t}]$,

$$\begin{aligned} V(t) &\leq \exp(LT)V(t_0) + \frac{\delta^2}{8} + \\ &\quad \int_{t_0}^t \exp[L(t-s)]g(s/\epsilon, \tilde{w}(s)) ds \quad (28) \\ &\leq \frac{\delta^2}{4} + \int_{t_0}^t \exp[L(t-s)]g(s/\epsilon, \tilde{w}(s)) ds. \end{aligned}$$

Fix $t \in [t_0, \bar{t}]$ and define k to be the largest nonnegative integer such that $k \leq \frac{t-t_0}{\epsilon\tilde{T}}$. For $i = 0, \dots, k$, define $t_i = t_0 + i\epsilon\tilde{T}$ and note that, from the definition of k and (22), we have

$$\begin{aligned} t - t_k &\leq \epsilon\tilde{T} \leq \frac{\delta^2}{12(2B \exp(LT))}, \\ t_{i+1} - t_i &= \epsilon\tilde{T} \leq \rho. \end{aligned} \quad (29)$$

We split the interval of integration in (28) using the times t_i to obtain

$$\begin{aligned} V(t) &\leq \frac{\delta^2}{4} + \int_{t_k}^t \exp[L(t-s)]g(s/\epsilon, \tilde{w}(s)) ds \\ &\quad + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \exp[L(t-s)]g(s/\epsilon, \tilde{w}(s)) ds. \end{aligned} \quad (30)$$

It follows from the definition of $\tilde{\mathcal{F}}$, (18), (19) and (29) that the first integral on the right-hand side of (30) is bounded as

$$\begin{aligned} \int_{t_k}^t e^{L(t-s)} g\left(\frac{s}{\epsilon}, \tilde{w}(s)\right) ds &\leq 2\epsilon\tilde{T}B e^{LT} \\ &\leq \frac{\delta^2}{12}. \end{aligned} \quad (31)$$

To bound the second integral on the right-hand side of (30), we split the integrand into two pieces: one that will be used to exploit the two

time-scale behavior of g as a function of s and the other that will be used to exploit the continuity properties of g with respect to s . In the calculation that follows, we will use the bound (32), which is a result of the definition of $\tilde{\mathcal{F}}$, (5), (17), (19) and Hölder's inequality. Then we use the bound on (21) and inequality (29) and the fact that $k\epsilon\tilde{T} \leq t - t_o \leq T$ to obtain (33). Combining (30), (31) and (33), it follows that $V(t) \leq \frac{\delta^2}{2}$ for all $t \in [t_o, \bar{t}]$. Since $V(t) = \frac{1}{2}e_{\epsilon,\nu}^T(t)e_{\epsilon,\nu}(t)$, it follows that $|e_{\epsilon,\nu}(t)| \leq \delta < 1$ for all $t \in [t_o, \bar{t}]$. From the definition of \bar{t} , it follows that $\bar{t} = t_o + T$ so that $|e_{\epsilon,\nu}(t)| \leq \delta$ for all $t \in [t_o, t_o + T]$. This establishes the result. \square

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$$\begin{aligned} \left| \int_{t_i}^{t_{i+1}} g(s/\epsilon, \tilde{w}(t_i)) ds \right| &= \left| \tilde{w}_1^T(t_i) \int_{t_i}^{t_i+\epsilon\tilde{T}} \{f(s/\epsilon, \tilde{w}_2(t_i), \tilde{w}_3(t_i), 0) - f_{wa}^A(\tilde{w}_2(t_i), \tilde{w}_3(t_i))\} ds \right| \\ &\leq \epsilon\tilde{T}\beta \left(\max\{R+1, \Omega\}, \tilde{T} \right) \leq \epsilon\tilde{T} \frac{\delta^2}{12T \exp(LT)}. \end{aligned} \quad (32)$$

$$\begin{aligned} &\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, \tilde{w}(s)\right) ds \\ &= \sum_{i=0}^{k-1} \exp[L(t-t_i)] \left\{ \int_{t_i}^{t_{i+1}} g\left(\frac{s}{\epsilon}, \tilde{w}(t_i)\right) ds + \int_{t_i}^{t_{i+1}} \left\{ \exp[L(t_i-s)]g\left(\frac{s}{\epsilon}, \tilde{w}(s)\right) - g\left(\frac{s}{\epsilon}, \tilde{w}(t_i)\right) \right\} ds \right\} \\ &\leq k \exp(LT) \left\{ \epsilon\tilde{T} \frac{\delta^2}{12T \exp(LT)} + \epsilon\tilde{T} \frac{\delta^2}{12T \exp(LT)} \right\} \leq \frac{\delta^2}{6}. \end{aligned} \quad (33)$$

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