Fundamental Limits for Sensitivity Reduction in Multiple-Input Multiple-Output Plants

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Abstract

By applying known operator theoretic tools for the minimisation of the weighted $\mathcal{H}_\infty$ norm of sensitivity, explicit, closed form expressions involving Blaschke products are obtained for fundamental limits on the output feedback control of linear multivariable systems. These depend on both right-hand plane poles and zeroes, and their relative alignment.

Using the operator theoretic tools developed in the $\mathcal{H}_\infty$ literature, we extend known results on fundamental limits by developing closed form expressions for the minimum achievable $\mathcal{H}_\infty$ norms of sensitivity functions for linear multivariable plants. The problem of minimising the $\mathcal{H}_\infty$ norm of a given closed loop transfer function [7, 13, 21, 30, 27, 31, 32] has mathematical roots as far back as the early twentieth century [6]. The theory generalises to multivariable, two and four block problems and irrational plants [11, 12, 20]. Historically, the aim has been the development of synthesis algorithms, in contrast to fundamental performance analysis.

However, there is also much literature devoted to solving problems which, because of their simplicity, are high on insight into the fundamental limits of controlled performance. Bode's work [2] was one of the earliest, but others include

1. Frequency domain integrals [9, 14, 16],
2. Time domain integrals [17, 23],
3. Limiting $\mathcal{H}_2$ performance [4, 10, 26, 29],
4. Limiting $\mathcal{H}_\infty$ performance [5, 18, 28].

Performance limits are a central issue in control and filtering. Their analysis may aid in the construction of sensible optimisation criteria on which to base controller design. For example, prudent $\mathcal{H}_\infty$ weighting design will not unecessarily penalise a direction in which the sensitivity is fundamentally limited.

Recent independent work [18], present some results given here, derived from interpolation constraints. Other works [28], tackle the state feedback problem but do not give explicit closed form solutions. Using normalised coprime factorisations elegant characterisation of $\mathcal{H}_\infty$ limits for generalised sensitivity are available [15]. Integral constraints [14, 23] give rise to *lower bounds* on the achievable $\mathcal{H}_\infty$ norm of the (complementary) sensitivity function. The journal version of results presented here is to appear [5]. Recent results express $\mathcal{H}_\infty$ limits in terms of the maximum singular value of matrices with entries dependent on the pole and zero location and directions [8], [9].

We apply operator theoretic methods leading to closed form expressions which have previously been given in terms of the maximum singular value of a Hankel matrix [12, 21, 30]. By looking a little closer at the algebraic structure of the result, extra insight into the the relationship between the plant characteristics and the optimal $\mathcal{H}_\infty$ bound may be obtained, particularly in some special multivariable cases. We are thus not claiming a contribution to the $\mathcal{H}_\infty$ field per se, but believe that the existing $\mathcal{H}_\infty$ formalism can be applied to yield further insight into interesting and fundamental questions.

We use standard notation, although for brevity we define $\lfloor z \rceil \equiv \mathrm{Re}\{z\}$ non-standardly. Full Blaschke products are denoted as $b_Z(s) = \prod_{k=1}^{n_z} b_{zk}(s)$, and $b_P(s) = \prod_{i=1}^{n_p} b_{pl}(s)$ where $b_{zk}(s) :\overset{\text{def}}{=} \frac{z_k - s}{z_k^* + s}$, and $b_{pl}(s) :\overset{\text{def}}{=} \frac{p_l - s}{p_l^* + s}$ are (nonstandard) partial Blaschke products.

1 Mathematical Preliminaries

We consider the standard single-block $\mathcal{H}_\infty$ problem for a linear, time invariant (LTI), multiple-input multiple-output (MIMO) system $G(s)$ using a single degree of freedom (linear) feedback controller structure (Figure 1). Transfer functions of interest [14], include the output sensitivity and complementary senstivity: respectively $S_O(s) :\overset{\text{def}}{=} (I + GK)^{-1}$ and $T_O(s) :\overset{\text{def}}{=} GK(I + GK)^{-1}$, as well as input sensitivities. We then seek the value, $\inf_K \|W_L(s) S_O(s) W_R(s)\|_\infty$, etc. where $W_L(s), W_R(s)$ are weighting transfer matrices. These represent fundamental limitations on the plant performance because they are the *best* that can be achieved whilst satisfying a minimal stability requirement.

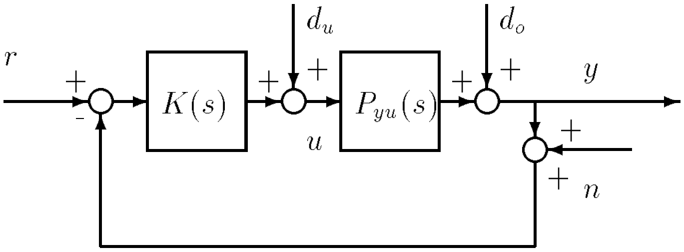

Figure 1: Single Degree of Freedom Controller Structure

Consider a coprime factorisation [31] of $G(s) = M(s)N(s)^{-1} = \tilde{M}(s)^{-1}\tilde{N}(s)$ and factorise $N(s) = L(s)N_o(s)$ with $L(s)$ inner and $N_o(s)$ outer and $\tilde{M}(s) = \tilde{M}_o(s)\tilde{F}(s)$ with $\tilde{M}_o(s)$, $\tilde{F}(s)$ co-outer and co-inner [10, 33], respectively. We now define $\hat{L}(s)$, $\hat{W}_L(s)$ from the inner-outer factorisation $\hat{L}(s)\hat{W}_L(s)$ of $W_L(s)L(s)$, and similarly, factorise $\tilde{F}(s)W_R(s)$ into co-outer-co-inner $\hat{W}_R(s)\hat{F}(s)$.

1.1 $\mathcal{H}_\infty$-norm Function Approximation

The following result, which evaluates the minimum mismatch of an $\mathcal{H}_\infty$ transfer matrix model matching problem is standard [7], [31].

Theorem 1.1 *Define* $\gamma^\star = \inf_{Q \in \mathcal{H}_\infty} \|F(s) - b_\Lambda(s)Q(s)\|_\infty$, *where* $F(s)$ *is in* $\mathcal{H}_\infty^{m\times m}$ *and* $b_\Lambda(s)$ *is a Blaschke product with simple zeros* $\lambda_i, i = 1 \ldots n_\lambda$. *Then* $\gamma^\star = \sigma_{\max}(A_N)$, *where* $A_N = R^{-*}F_\Lambda R^*$, F_Λ *is the diagonal "interpolation" matrix* $F_\Lambda =$ *block diag*$\{F(\lambda_1), F(\lambda_2), \ldots, F(\lambda_n)\}$ *and* R *is an upper triangular Cholesky factor of* $\Gamma_\Lambda \in \mathbb{C}^{n_\lambda m \times n_\lambda m}$ *where the block matrix* $[\Gamma_\Lambda]_{(m(i-1)+r, m(j-1)+t)} = (\lambda_i^* + \lambda_j)^{-1}\delta_r^t$.

Proof: As for discrete time [31]. ■

Lemma 1.1 *A complex matrix* Γ_Λ *given by* $[\Gamma_\Lambda]_{ij} = (\lambda_i^* + \lambda_j)^{-1}$ *factorises as* $R^*R = \Gamma_\Lambda$ *where* $r_{ij} = \frac{\sqrt{2\lfloor\lambda_i\rceil}}{\lambda_i^*+\lambda_j}\prod_{k=1}^{i-1}\left(\frac{\lambda_k-\lambda_j}{\lambda_k^*+\lambda_j}\right)$ *for* $i \le j$ *and is zero otherwise. If* R *is partitioned as* $R = \begin{bmatrix} R_{11}^{n_1\times n_1} & R_{12}^{n_1\times n_2} \\ 0 & R_{22}^{n_2\times n_2} \end{bmatrix}$ *then* $R_{22} = ED$ *where*

$$D = \mathrm{diag}\left\{\prod_{i=1}^{n_1}\left(\frac{\lambda_i - \lambda_{n_1+1}}{\lambda_i^* + \lambda_{n_1+1}}\right), \ldots, \prod_{i=1}^{n_1}\left(\frac{\lambda_i - \lambda_{n_2}}{\lambda_i^* + \lambda_{n_2}}\right)\right\},$$

and E *is given by* $e_{ij} = \frac{\sqrt{2\lfloor\lambda_{n_1+i}\rceil}}{\lambda_{n_1+i}^* + \lambda_{n_1+j}}\prod_{k=n_1+1}^{n_1+i-1}\left(\frac{\lambda_k - \lambda_{n_1+j}}{\lambda_k^* + \lambda_{n_1+j}}\right)$ *for* $i \le j$ *and zero otherwise. The inverse* R^{-1} *has* $r_{ij}^{inv} = \frac{2\lfloor\lambda_i\rceil\sqrt{2\lfloor\lambda_j\rceil}}{\lambda_i^* + \lambda_j}\prod_{k=1,k\neq i}^{j}\left(\frac{\lambda_k^* + \lambda_i}{\lambda_k - \lambda_i}\right)$ *if* $i \le j$, *and zero otherwise.*

Proof: The expressions for R, R^{-1}, R_{22} can be verified by by backsubstitution. ■

We now give the main result.

Theorem 1.2 *Given an* m*-input,* p*-output,* $p \le m$ *LTI multivariable strictly proper plant* $G(s)$ *with co-prime factorisation as above, define* $\gamma_{S_O}^\star \overset{\text{def}}{:=} \inf_{K\in\mathcal{K}} \|W_L(s)S(s)W_R(s)\|_\infty$. *Then* $\gamma_{S_O}^\star = \max\{\sigma_{\max}(\Psi), \sigma_{\max}(W_L(\infty)W_R(\infty))\}$, $\Psi = E\bar{T}^*DE^{-1}$.

Each element of E *is given by* $e_{m(i-1)+r,m(j-1)+t} = \delta_r^t\frac{\sqrt{2\lfloor z_i\rceil}}{z_i^* + z_j}\prod_{k=1}^{i-1} b_{zk}(z_j)$ *for* $i \le j, r, t \le m$ *and is zero otherwise.* $D = \mathrm{diag}\{b_P^{-1}(z_1)I_m, \ldots, b_P^{-1}(z_{nz})I_m\}$ *and* $T = \mathrm{diag}\{O_{n_pm}, \bar{T}\}$, *where* $\bar{T} = \mathrm{diag}\left\{\hat{T}(z_1), \hat{T}(z_2), \ldots, \hat{T}(z_{nz})\right\}$, $\hat{T}(s) = \hat{W}_L(s)L^{adj}(s)\tilde{F}^{adj}(s)\hat{W}_R(s)$. *The* n_p *RHP poles of* $g(s)$ *are denoted* p_l *and* $b_{zk}(s)$ *and* b_P *are zero and pole Blaschke products.*

Proof: All achievable stable sensitivity functions can be expressed as $S_O(s) = Y(s)\tilde{M}(s) - N(s)Q(s)\tilde{M}(s)$, where $X(s)Y(s)^{-1}$ is a right coprime factorisation of a stabilising controller satisfying the Bezout identity [31]. We can pre- and post- multiply the expression for $W_L(s)S_O(s)W_R(s)$ by the adjugates $\hat{L}^{\mathrm{adj}}(s)$ and $\hat{F}^{\mathrm{adj}}(s)$ of $\hat{L}(s)$ and $\hat{F}(s)$ respectively and hence reduce [31] the problem to $\inf_{\bar{Q}\in\mathcal{H}_\infty} \|\hat{L}^{\mathrm{adj}}W_LY\tilde{M}_o\hat{W}_Rb_P + b_Z(s)b_P\bar{Q}\|_\infty$.

By the Bezout identity $\hat{L}^{\mathrm{adj}}W_LY\tilde{M}_o\hat{W}_Rb_P + b_Z\hat{W}_RN_o\tilde{X}W_R\hat{F}^{\mathrm{adj}} = \hat{W}_LL^{\mathrm{adj}}\tilde{F}^{\mathrm{adj}}\hat{W}_R$. We apply Theorem 1.1, where the interpolation matrix is $F_\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \bar{T} \end{bmatrix}$ and $b_\Lambda = b_Zb_P$. Since $G(s)$ is strictly proper, $\gamma^\star = \max\{|A_N|, |W_L(\infty)W_R(\infty)|\}$.

We partition Γ, R defined in Theorem 1.1 compatibly with F_Λ and factorise Γ_{22} into Cholesky factors as E^*E. $R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$, $A_N = \begin{bmatrix} 0 & 0 \\ R_{22}^{-*}\bar{T}R_{12}^* & R_{22}^{-*}\bar{T}R_{22}^* \end{bmatrix}$, and $\sigma_{\max}(A_N) = \sigma_{\max}\left(E\bar{T}^*R_{22}^{-1}\right)$. The elements of E are give by Lemma 1.1. ■

2 Multi-Input Multi-Output Results

The following focusses exclusively on unweighted sensitivity minimisation since the weighted problem is necessarily more complicated and the resulting insights therefore diminished. Although the choice of weighting function can be used to de-emphasise a particular *direction* with a *small magnitude* weight, reducing zero and pole interaction by modifying their *directions* is futile (see $\hat{T}$ in Theorem 1.1). We also focus exclusively on output sensitivity functions noting that input sensitivity analysis follows analogously.

2.1 Output Sensitivity

Theorem 2.1 *Given open loop $G(s)$ with one RHP zero z with output direction v^*, and poles p_l, with $\tilde{F}(s)$ is as in Section 1, then* $\gamma_{S_o}^{\star} \overset{\text{def}}{:=} \inf_{K(s) \in \mathcal{K}} \|S_O\|_\infty = \|v^* \tilde{F}(z)^{-1}\|_2$.

Proof: In the notation of Section 1.1, $T = V \operatorname{diag}\{1, 0, \ldots, 0\} V^* \tilde{F}^{\text{adj}}(z)$, and $\sigma_{\max}\left(R_{11} T R_{11}^{-1} \begin{bmatrix} I & -R_{12} R_{22}^{-1} \end{bmatrix}\right) = \sigma_{\max}(T)\, \sigma_{\max}\left(\begin{bmatrix} I & -R_{12} R_{22}^{-1} \end{bmatrix}\right)$, $\sigma_{\max}(T) = \|v^* \tilde{F}(z)^{-1}\|_2 \cdot |b_Z(p)|$ and $\sigma_{\max}\left(\begin{bmatrix} I & -R_{12} R_{22}^{-1} \end{bmatrix}\right) = \left|b_Z(p)^{-1}\right|$. The second equality follows since R_{11} is a scaled identity matrix and the matrices in the product have diagonal block structure. The fourth equality follows from the SISO results for the single pole case (Lemma 1.1). ■

2.1.1 Direction of worst case gain:

One stable sensitivity function that satisfies both the interpolation conditions $S(s) = Q_1(s) \tilde{F}(s) = I - L(s) Q_2(s)$, $Q_1(s), Q_2(s) \in \mathcal{H}_\infty$ and has the infimal $\mathcal{H}_\infty$ norm, is given by $S^\star(s) = v v^* \tilde{F}(z)^{-1} \tilde{F}(s)$. This "optimal" sensitivity function has rank one: all disturbances orthogonal to the direction of the non-zero singular value may be rejected (almost) perfectly by allowing the controller gain to become infinite in some directions. The worst case input direction is a function of frequency $u(j\omega) = \tilde{F}(j\omega)^{-1} \tilde{F}(z) v$ with the worst case output direction constant at v.

Since the expression for $\gamma_{S_o}^{\star}$ depends on the relative alignment of the pole and zero directions, we present below two extreme cases of relative pole alignment: perfectly aligned and mutually orthogonal.

With one zero, and multiple poles having all directions parallel, [10], $L(s) =$

$$F \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & I \end{bmatrix} \operatorname{diag}\left\{\tfrac{z-s}{z+s}, I\right\} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & I \end{bmatrix} F^*,$$

and $\tilde{F}(s) = F \operatorname{diag}\left\{\prod_{i=1}^{n_z} \frac{p_i - s}{p_i + s}, I\right\} F^*$,for some constant unitary F, where θ is the alignment angle between the pole and zero directions. For this case ${\gamma_{S_o}^{\star}}^2 = |B_P(z)|^{-2} \cos^2\theta + \sin^2\theta$. An expression for an optimal sensitivity function is $S^\star(s) =$

$$F \begin{bmatrix} B_P(s)^2 B_P(z)^{-2} \cos^2\theta & \cos\theta \sin\theta & O \\ B_P(s)^2 B_P(z)^{-2} \cos\theta \sin\theta & \sin^2\theta & O \\ O & O & O \end{bmatrix} F^*.$$

With multiple poles with mutually orthogonal directions and one zero, $L(s) = \begin{bmatrix} v & \bar{V} \end{bmatrix} \operatorname{diag}\left\{\frac{z-s}{z+s}, I\right\} \begin{bmatrix} v^* \\ \bar{V}^* \end{bmatrix}$, $\tilde{F}(s) = F \operatorname{diag}\left\{\frac{p_1 - s}{p_1 + s}, \ldots, \frac{p_{np} - s}{p_{np} + s}, I\right\} F^*$, [10] for constant unitary matrices $\begin{bmatrix} v & \bar{V} \end{bmatrix}$ and F. This gives ${\gamma_{S_o}^{\star}}^2 = \sum_{j=1}^{n_p} |B_{pj}(z)|^{-2} \cos^2 \angle(v, f_{\cdot j}) + \sum_{j=n_p+1}^{r} \cos^2 \angle(v, f_{\cdot j})$, where $f_{\cdot j}$ is the j_{th} column of F. Note that $\sum_{j=1}^{r} \cos^2 \angle(v, f_j) = 1$.

2.1.2 Case of a Single Pole:

For a single pole p with input direction $F \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T$ and a single zero z with output direction $\begin{bmatrix} \cos\phi & \sin\phi & \ldots & 0 \end{bmatrix} F^*$ then the infimal norm is ${\gamma_{S_o}^{\star}}^2 = \left|B_Z(p)^2\right| \cos^2\phi + \sin^2\phi$. This is identical to the lower bound given in [19] (equation (25)), showing that this latter bound is actually tight. An optimal sensitivity function, with respect to the pole direction, is $S^\star(s) =$

$$F \begin{bmatrix} \left(\frac{p-s}{p+s}\right)\left(\frac{z+p}{z-p}\right) \cos^2\phi & \cos\phi \sin\phi & O \\ \left(\frac{p-s}{p+s}\right)\left(\frac{z+p}{z-p}\right) \cos\phi \sin\phi & \sin^2\phi & O \\ O & O & O \end{bmatrix} F^*.$$

2.1.3 Case of Two Poles:

Assume we have a single zero z with direction v^*, and two poles p_1, p_2, with inner factor directions f_1, f_2, given by $v = \begin{bmatrix} \cos\theta & 0 & \sin\theta \end{bmatrix}^* U$, for some constant unitary U, and $f_1 = U^* \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \end{bmatrix}$, $f_2 = U^* \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & 0 \end{bmatrix}$,

The angle θ is that between the zero direction transpose v and the plane $\mathcal{P}$ defined by the two pole directions f_1, f_2. The relationship between the true input pole directions and f_1, f_2 is given in [10]. Application of Theorem 2.1 yields ${\gamma_{S_o}^{\star}}^2 =$

$$\begin{aligned} &\left\{ \left[B_{p1}(z)^2 \cos^2\alpha + \sin^2\alpha\right] \left[B_{p2}(z)^2 \cos^2\beta + \sin^2\beta\right] \right. \\ &\quad - \left[B_{p2}(z)^2 - 1\right] \left[2 B_{p1}(z) \cos\alpha \cos\beta \sin\alpha \sin\beta \right. \\ &\qquad \left. \left. + (\cos^2\beta - \sin^2\beta) \sin^2\alpha\right] \right\} \cos^2\theta + \sin^2\theta. \end{aligned}$$

2.2 Output Complementary Sensitivity

For the output complementary sensitivity function with one pole p and multiple zeroes z_k, $\tilde{F}(s) = F \operatorname{diag}\left\{\frac{p-s}{p+s}, I\right\} F^*$, and $L(s)$ is as in Section 1. We then have $\inf_{k \in \mathcal{K}} \|T_o(s)\|_\infty = \|L(p)^{-1} f\|_2$, where f is the pole output direction, the first column of F.

3 Single-Input Single-Output Results

We next specialise the MIMO results to the single-input single-output (SISO) case, calculating explicit expressions for the infimum of the SISO weighted sensitivity function norm $\|w(s) S(s)\|_\infty$. For scalar systems, in contrast to MIMO, the input and output sensitivity functions $S(s)$ are equal, as are the input and output complementary sensitivity functions $T(s)$. The main result for this section, specialised to the scalar case, is the following theorem.

3.1 Scalar Sensitivity Minimisation

Theorem 3.1 *Consider a strictly proper open loop plant* $g(s)$ *with all RHP poles and zeroes with multiplicity of, at most, one. Define* $\gamma^\star \overset{\mathrm{def}}{:=} \inf_{k\in\mathcal{K}} \|S(s)\|_\infty$. *Then*

$$\gamma^\star = \max\left\{\sigma_{\max}(\Psi), 1\right\} \quad \text{where} \quad \psi_{ij} = 2\sqrt{\lfloor z_i\rceil\lfloor z_j\rceil}\sum_{k=i}^{j} b_P^{-1}(z_k)w(z_k)\left(\frac{2\lfloor z_k\rceil}{(z_i^*+z_k)(z_k^*+z_j)}\prod_{l=i,l\neq k}^{j}\left(\frac{z_l^*+z_k}{z_l-z_k}\right)\right)$$

for $i \leq j$ *and zero otherwise.*

Proof: See [5] ■

3.1.1 Zero-Pole Symmetry: The following shows that interchanging RHP poles and zeroes, results in the same minimal *unweighted* sensitivity.

Lemma 3.1 *Let* $g(s)$ *and* $\hat{g}(s) = g(s)^{-1}h(s)$ *be proper with at least, one of each of a RHP pole and zero, and with* $h(s)$ *stable minimum-phase. Let* $S(s)$ *be the sensitivity function for each* $k(s) \in \mathcal{K}$, *where* $\mathcal{K}$ *is the set of all stabilising controllers for* $g(s)$, *and define* $\hat{S}(s), \hat{k} \in \hat{\mathcal{K}}$ *analogously for* $\hat{g}(s)$. *Then* $\inf_{\hat{k}(s)\in\hat{\mathcal{K}}} \|\hat{S}(s)\|_\infty = \inf_{k(s)\in\mathcal{K}} \|S(s)\|_\infty$.

Proof: By comparison of matrices Ψ from Theorem 3.1 corresponding to each case. ■

The above lemma allows us to derive expressions for unweighted sensitivity minimisation for many RHP zeroes (and up to two poles) or for complementary sensitivity for many poles (and up to two zeroes). In contrast, previous results [14] have dealt with only a single RHP zero (sensitivity) or [23] a single pole (complementary sensitivity).

3.2 Core Results- SISO Case

3.2.1 Hard Bounds on achievable $\mathcal{H}_\infty$ Performance: The elements of Ψ are expressed as Blaschke products of RHP poles and zeroes, expressions which are reminiscent of orthonormal basis functions [1] evaluated at zeroes and poles. Since the matrix Ψ is upper triangular, it is easier to gain insight regarding its maximum singular value in comparison to that of the ratio of two square matrices, [12, 21, 30]. More insight may be gained by further specialising Theorem 3.1 to when there are two or fewer RHP poles or zeroes. The second part of the following is well-known.

Corollary 3.1 *If a strictly proper single-input single-output system* $g(s)$ *has only one RHP pole* p, *and any number of NMP zeros, then* unweighted *sensitivity* $S_{1p}(s)$ *minimisation gives* $\inf_{k(s)\in\mathcal{K}} \|S_{1p}(s)\|_\infty = \left|b_Z(p)^{-1}\right|$. *The optimal* unweighted *sensitivity function is* $S_{1p}^\star(s) = \dfrac{1 - b_Z(p)^{-1}b_Z(s)}{1 - b_Z(p)b_Z(s)}$.

Also, if $g(s)$ *has a single RHP zero* z, *and any number of unstable poles then the weighted sensitivity function* $w(s)S_{1z}(s)$ *satisfies* $\inf_{k(s)\in\mathcal{K}} \|w(s)S_{1z}(s)\|_\infty = \max\left\{\left|w(z)b_P(z)^{-1}\right|, w_\infty\right\}$.. *The "optimal" weighted sensitivity function* $S_{1z}^\star(s)$ *is* $S_{1z}^\star(s) = w(z)b_P(z)^{-1}b_P(s)w^{-1}(s)$.

Proof: Corollary of Theorem 3.1 and Lemma 3.1 and by back-substitution of the operator basis into the expression for the gain-maximising vector for A_N. ■

Note that these optimal sensitivity functions $S^\star(s)$ may require improper controllers but may be approached arbitrarily closely using proper controllers.

Proposition 3.1 *Given an open loop plant* $g(s)$ *with two RHP zeroes,* z_1 *and* z_2, *and an arbitrary (finite) number* n_p *of RHP poles* p_l, *then* $\gamma^\star \overset{\mathrm{def}}{:=} \inf_{k(s)\in\mathcal{K}} \|w(s)S(s)\|_\infty^2$ *is given by*

$$\gamma^{\star 2} = \tfrac{1}{2}\left(|H|^2+|K|^2+\zeta^2+\sqrt{\left(|H|^2-|K|^2\right)^2+2\zeta^2\left(|H|^2+|K|^2+\tfrac{1}{2}\zeta^2\right)}\right),$$

where $b_P(s) = \prod_{l=1}^{n_p} \frac{p_l-s}{p_l^*+s}$, $H = b_P^{-1}(z_1)w(z_1)$, $K = b_P^{-1}(z_2)w(z_2)$, *and* $\zeta^2 = \frac{4\lfloor z_1\rceil\lfloor z_2\rceil}{|z_1-z_2|^2}\left|b_P^{-1}(z_1)w(z_1) - b_P^{-1}(z_2)w(z_2)\right|^2$.

The expression for two RHP poles, p_1 *and* p_2, *and* n_z *zeroes with* unity weighting, *is identical to the above but with* $b_Z(s) = \prod_{k=1}^{n_z}\left(\dfrac{z_k - s}{z_k^* + s}\right)$, $H = b_Z^{-1}(p_1)$, $K = b_Z^{-1}(p_2)$ *and* $\zeta^2 = \dfrac{4\lfloor p_1\rceil\lfloor p_2\rceil}{|p_1 - p_2|^2}\left|b_Z^{-1}(p_1) - b_Z^{-1}(p_2)\right|^2$.

Proof: Theorem 3.1, elementary algebra and application of Lemma 3.1. ■

Note: Section VI B of [32] also gives an elegant expression for the weighted two zero case, which is equivalent.

Remark 3.1 *For two conjugate zeroes,* z, z^*, *we have* $H = K^* = b_P(z)^{-1}$, $\xi^2 = (\tan\angle z)^{-2}$, *and* $\inf\|w(s)S(s)\|_\infty^2 = \left|b_P^{-1}(z)w(z)\right|^2 + 2\left(\frac{Im\{b_P(z)^{-1}w(z)\}}{\tan\angle z}\right)^2 + 2\left(\frac{Im\{b_P(z)^{-1}w(z)\}}{\tan\angle z}\right) \times \sqrt{\left|b_P(z)^{-1}w(z)\right|^2 + \left(\frac{Im\{b_P(z)^{-1}w(z)\}}{\tan\angle z}\right)^2}$. *A cor-*

responding expression for conjugate poles (unweighted) follows Lemma 3.1.

Remark 3.2 *The tight bound for n_p RHP poles and n_z RHP zeroes requires solving an n_{th} order polynomial, where $n = \min\{n_p, n_z\}$, for which there is no general closed form solution. Because the the $(n_p + n_z) \times (n_p + n_z)$ Nevanlinna matrix [31] interpolates zero for some columns (rows) due to constraints $T(z_k) = 0$, $S(p_l) = 0$, there exists an $n \times n$ matrix (Ψ) with the same spectrum.*

3.2.2 Lower Bounds on achievable $\mathcal{H}_\infty$ Performance: We are motivated to develop closed form lower bounds for the many RHP pole and zero cases. Expressions based on the diagonal entries of Ψ give previously known results.

Since $\sigma_{\max}(\Psi) \geq |\Psi_{ii}|$ for $i = 1 \ldots \dim(\Psi)$, then by Theorem 3.1, $\inf\limits_{k(s) \in \mathcal{K}} \|w(s)S(s)\|_\infty \geq \left|b_P^{-1}(z_k)w(z_k)\right|$, for each $k = 1 \ldots n_z$ and by Lemma 3.1 $\inf\limits_{k(s) \in \mathcal{K}} \|S(s)\|_\infty \geq \left|b_Z^{-1}(p_l)\right|$ for each $l = 1 \ldots n_p$. The first result is equivalent to those in [18] (equations (16) and (17)) based on interpolation constraints for each RHP zero in turn, as well as bounds from sensitivity integrals in [14]. When there is none or one RHP zero, these bounds may be approached arbitrarily closely and are thus tight.

Other expressions based on rows or columns of depend explicitly on all of the open loop plant RHP zeroes (poles). For any matrix Ψ, a lower bound on $\sigma_{\max}(\Psi)$ is given by the Euclidean norms of the columns (and rows) of Ψ. This fact gives $\|w(s)S^\star(s)\|_\infty^2 \geq \sum_{j=i}^{n_z} |\psi_{ij}|^2$, for each $i = 1 \ldots n_z$, (rows) which yields explicit expressions in terms of RHP poles and zeroes. A corresponding expression, of course, exists for columns. Reordering of the zeroes n_z and hence rearranging Ψ gives alternative lower bounds. The pole-zero symmetry results of Lemma 3.1 give still other bounds for unweighted $\|S^\star\|_\infty$. This lower bound on $\|w(s)S^\star\|_\infty$ is strictly increasing with the addition of each closed loop zero z_k.

3.2.3 Equality of Bounds for S and T: By comparison of matrices Ψ of Theorem 3.1 corresponding to the unweighted closed loop sensitivity and *complementary* sensitivity functions, we obtain the well-known result that the tight lower bounds for the $\mathcal{H}_\infty$ norm of each are identical [22].

4 Complements

As is well known, if the open loop plant is controlled in a two degree of freedom **or** state feedback configuration [31] **or** if the open loop plant is stable, then the bound on the peak sensitivity minimisation is given by $\inf\limits_{K \in \mathcal{K}} \|S_O^x(s)\|_\infty = 1$ if $N(s)$ contains ORHP zeroes or is strictly proper and zero otherwise.

If the plant is minimum phase and unstable, then we have the well known result (see [21], [24]) that $\inf\limits_{K \in \mathcal{K}} \|S_O^{\mathrm{mp}}\|_\infty = 1$ if $G(s)$ is strictly proper and zero otherwise. Similar results hold for complementary sensitivity bounds.

Also, with minor exceptions, adding a closed loop constraint such that there must be zero steady state tracking error does not affect the $\mathcal{H}_\infty$ bounds.

4.1 Time Delays

For distributed systems, the analysis is similar to the rational case, [11, 12, 20, 25], but requires calculation of an infinite dimensional operator norm. When the irrational part $e^{-s\tau}$ of the openloop system arises from a time delay, a rational Padé approximation can be used and the time delay behaves like a rational non-minimum phase factor.

Proposition 4.1 *Let $g(s) = e^{-s\tau}n(s)m(s)^{-1}$ with $n(s)m(s)^{-1}$ rational, coprime and strictly proper with unstable pole p and at least one RHP zero. For unweighted sensitivity, $\inf\limits_{k \in \mathcal{K}} \|S(s)\|_\infty = e^{p\tau}b_Z^{-1}(p)$, and for weighted complementary sensitivity, $\inf\limits_{k \in \mathcal{K}} \|w(s)T(s)\|_\infty = w(p)e^{p\tau}b_Z^{-1}(p)$.*

Proof: The proof, omitted for brevity, essentially relies on convergence of the Padé approximation of the time delay in the finite complex plane: see [3]. The above result for unweighted sensitivity also follows trivially from results already given in [12] (Part III). ■

Remark 4.1 *Similarly for two poles, the expressions for minimum sensitivity and complementary sensitivity norms are given by the equation in Proposition 3.1 but with the substitutions $H = b_Z^{-1}(p_1)e^{p_1\tau}w(p_1)$, $K = b_Z^{-1}(p_2)e^{p_2\tau}w(p_1)$, and $\zeta^2 = \dfrac{4\lfloor p_1 \rceil \lfloor p_2 \rceil}{|p_1 - p_2|^2}\left|b_Z^{-1}(p_1)e^{p_1\tau}w(p_1) - b_Z^{-1}(p_2)e^{p_2\tau}w(p_2)\right|^2$. This expression is valid for complementary sensitivity for any proper weighting function $w(s)$ but for sensitivity, only for unity weighting.*

5 Conclusion

We have shown that well known frequency domain operator theoretic concepts from the $\mathcal{H}_\infty$ literature can be applied to expose fundamental limits for finite dimensional, linear, time invariant output feedback control systems. In the multivariable case, the minimum achievable $\mathcal{H}_\infty$ norm of the input and output (complementary) sensitivity functions depends upon the directional interaction of right hand plane poles and zeroes (see also [10]).

We demonstrate that some previously known bounds for sensitivity are, in fact, tight, and an expression for functions which achieve those lower bounds has been given. Explicit expressions for the optimal $\mathcal{H}_\infty$ norm of a multivariable sensitivity function have been found for the cases where there is a single non-minimum phase zero, and for multivariable complementary sensitivity functions where there is a single unstable pole.

For the scalar case, exchanging the role of poles and zeroes preserves the same tight bound. Furthermore, we gave explicit expressions for the optimum, involving Blaschke products, in the case of one or two unstable poles or zeroes. Lower bounds are given for more than two right hand plane poles (zeroes), which depend explicitly on each pole (zero).

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