SLQR/SLQG: An LQR/LQG Theory for Systems with Saturating Actuators

C. Gökçek, P. T. Kabamba, and S. M. Meerkov

Department of Electrical Engineering and Computer Science
University of Michigan, Ann Arbor, MI 48109-2122 USA
cgokcek@eecs.umich.edu, kabamba@engin.umich.edu, smm@eecs.umich.edu

Abstract

An extension of the LQR/LQG methodology to systems with saturating actuators, referred to as SLQR/SLQG, is obtained. The development is based on the method of stochastic linearization. Using this method and the Lagrange multiplier technique, solutions to the SLQR and SLQG problems are derived. These solutions are given by Riccati and Lyapunov equations coupled with two transcendental equations. It is shown that, under standard stabilizability and detectability conditions, these equations have a unique solution, which can be found by a simple bisection algorithm. When the level of saturation tends to infinity, these equations reduce to their standard LQR/LQG counterparts.

1. INTRODUCTION

The LQR/LQG methodology is one of the main techniques for control systems design. It is applicable, as the abbreviation implies, to linear systems with quadratic performance index. In practice, however, even if the plant can be viewed as linear (or linearized, when the controller forces its operation close to an operating point), the actuator is often nonlinear due to power and material limitations. One such fundamental limitation is amplitude saturation in actuators.

In these situations, the appropriate model of the system is a linear plant and a saturating actuator. The LQR/LQG methodology is not applicable in this case. In this paper, the LQR/LQG theory is extended to systems with saturating actuators. The resulting technique is referred to as SLQR/SLQG, where S stands for saturating.

The approach of this paper is based on a quasilinearization technique known as stochastic linearization [1]–[4]. According to this technique, the saturation nonlinearity is replaced by its equivalent gain, which is a function of the variance of the signal at its input. As a result, the closed-loop system is described by a linear dynamic equation and a nonlinear static relationship, which characterizes the equivalent gain.

Using this structure and the Lagrange multiplier technique, solutions of the SLQR and SLQG problems are derived in terms of Riccati and Lyapunov equations coupled with two transcendental equations, which define the variance of the signal at the input of saturation and the Lagrange multiplier associated with the constrained minimization problem. It is shown that, under standard stabilizability and detectability conditions, these equations have a unique solution, which can be found by a simple bisection algorithm.

It should be pointed out that stochastic linearization has been used in the context of the LQR/LQG theory in the past [5], [6]. However, their frameworks are different from that of this work. Moreover, unlike the present work, [5], [6] do not provide algorithms for synthesis of controllers. In addition, systems with saturating actuators have been studied in numerous publications (e.g., see [7]–[15]), but the problem of designing controllers that minimize a quadratic performance index remains open. The present work is intended to contribute to this end.

The outline of this paper is as follows: Section 2 gives the problem formulation. Section 3 presents a brief review of the stochastic linearization technique. Sections 4 and 5 are devoted to SLQR and SLQG theories, respectively. Section 6 applies the developed results to a simple example. Section 7 gives the concluding remarks. Due to the space limitation, the proofs are omitted and can be found in [16].

2. PROBLEM FORMULATION

Consider the system shown in Figure 2.1, where $P(s)$ is the plant, $C(s)$ is the controller, $\varphi(u)$ is the static saturation nonlinearity

$$\varphi(u) = \beta \operatorname{sat}\left(\frac{u}{\alpha}\right), \quad \alpha > 0, \beta > 0, \tag{2.1}$$

$A(s)$ describes the dynamics of the actuator, $F_1(s)$ and $F_2(s)$ are coloring filters, and $H_1(s)$ and $H_2(s)$ are weighting filters. Signals $u, v, y \in \mathbb{R}$ are the commanded control, actual control and measured output, respectively, w_1, w_2 $\in \mathbb{R}$ are uncorrelated standard white noise processes, and z_1, $z_2 \in \mathbb{R}$ are the controlled outputs. Assume that the system excluding the controller has the state-space representation

$$\begin{aligned} \dot{x}_G &= Ax_G + B_1 w + B_2 \varphi(u), \\ z &= C_1 x_G + D_{12} u, \\ y &= C_2 x_G + D_{21} w, \end{aligned} \tag{2.2}$$

where $x_G = [x_P^T \; x_A^T \; x_{F_1}^T \; x_{F_2}^T \; x_{H_1}^T \; x_{H_2}^T]^T \in \mathbb{R}^n$, $w = [w_1 \; w_2]^T$ and $z = [z_1 \; z_2]^T$.

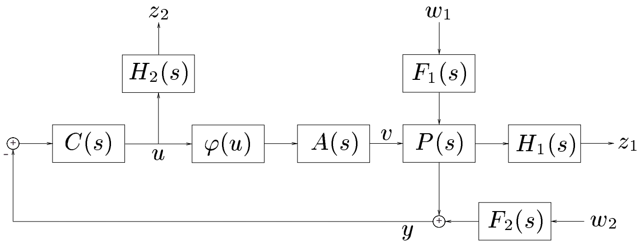

Figure 2.1: System model.

In the following, we will be interested in the (steady-state) variance of z, which is denoted by σ_z^2, and address the problem: Given (2.2), find a controller $C(s)$ that stabilizes the system when $w \equiv 0$ and minimizes σ_z^2, using the method of stochastic linearization.

The above normalized white noise assumption does not restrict generality since such normalization can be carried out by appropriately scaling B_1 and D_{21}. Moreover, the assumption that the signals u, v, y, w_1, w_2, z_1, z_2 are all scalar is made for the sake of simplicity, and the theory developed below can be extended to the vector case.

3. STOCHASTIC LINEARIZATION

When a linear time-invariant system is driven by white noise, calculation of the output variance is relatively easy and requires only the solution of a Lyapunov equation. For a nonlinear system, however, the situation is quite different. Specifically, calculation of the output variance requires the solution of the Fokker-Planck equation [17]. Unfortunately, exact solution of this equation is available only in a few special cases. Numerical solution of the Fokker-Planck equation, on the other hand, requires extremely intensive computational effort and has limited value especially from a design point of view. Therefore, several approximate methods, such as stochastic linearization, moment closure and functional series, have been developed to overcome these difficulties (e.g., see [4] and the references therein). Among these methods, due to its simplicity and reasonable accuracy, the most widely used is the method of stochastic linearization.

This method is essentially the stochastic analogue of the well-known describing function method [3], [18]. The basic idea behind it is to replace the nonlinear element with a linear time-invariant element that minimizes the mean-square error between the outputs of these two systems. For an isolated nonlinear element $\varphi(u)$ driven by a zero-mean wide-sense stationary Gaussian process u, it can be shown [3], [4] that the optimal linearization is a constant gain of value

$$N = E\{\varphi'(u)\}, \tag{3.1}$$

where E is the expectation operator. However, when the nonlinear element $\varphi(u)$ appears in a feedback configuration driven by white noise as in Figure 2.1, its input u is no longer Gaussian and is not equal to the corresponding signal $\hat{u}$ of the linearized system. Nevertheless, if u is approximately Gaussian due to low-pass filtering in the loop, it can be shown [3], [4] that a suboptimal linearization of $\varphi(u)$ is

$$N = E\{\varphi'(\hat{u})\}. \tag{3.2}$$

Note that since $\hat{u}$ is Gaussian, N depends only on the variance of $\hat{u}$.

For the saturation function defined in (2.1), it follows from (3.2) that

$$N = \frac{\beta}{\alpha} \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}\sigma_{\hat{u}}}\right), \tag{3.3}$$

where erf is the error function

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{+x} \exp(-t^2)\, dt. \tag{3.4}$$

Note that N is a decreasing function of $\sigma_{\hat{u}}$. Moreover, when $\sigma_{\hat{u}}/\alpha$ is very small $N \approx \beta/\alpha$ and when $\sigma_{\hat{u}}/\alpha$ is very large $N \approx \sqrt{2/\pi}\,\beta/\sigma_{\hat{u}}$. As a result, with the equivalent gain N given in (3.3), the stochastic linearization of (2.2) is

$$\begin{aligned} \dot{\hat{x}}_G &= A\hat{x}_G + B_1 w + B_2 N \hat{u}, \\ \hat{z} &= C_1 \hat{x}_G + D_{12} \hat{u}, \\ \hat{y} &= C_2 \hat{x}_G + D_{21} w. \end{aligned} \tag{3.5}$$

The method of stochastic linearization is used throughout this paper by assuming that it is reasonably accurate. However, except for a few special cases, there seems to be no analytical results that predict its accuracy [4]. Nevertheless, by numerous simulations (e.g., see [3]–[6], and the references therein), it has been shown that the method yields reasonable accuracy in predicting variances, usually within 10% of actual values.

The following simple example illustrates the accuracy of stochastic linearization in a case where exact analytic calculation of the output variance is possible.

Example 3.1: Consider the system

$$\begin{aligned} \dot{x}_G &= -x_G + w + \varphi(u), \\ z &= x_G, \\ y &= x_G, \end{aligned} \tag{3.6}$$

where $x_G \in \mathbb{R}$, w is standard white noise, and assume that $\alpha = 1$, $\beta = 1$. With

$$u = -ky, \quad k \geq 0, \tag{3.7}$$

the variance of z can be calculated analytically by solving the corresponding Fokker-Planck equation. The result is shown in Figure 3.1 by the solid curve. Application of stochastic linearization yields the dashed curve for $\sigma_{\hat{z}}^2$. The error between the exact value of σ_z^2 and that predicted by stochastic linearization is less than 5.54% for all $k \geq 0$. Moreover, as k gets smaller, this error approaches zero. These results give some assurance concerning the accuracy of stochastic linearization. For comparison, Figure 3.1 also shows the variance of z by the dash-dot curve when the saturation is ignored.

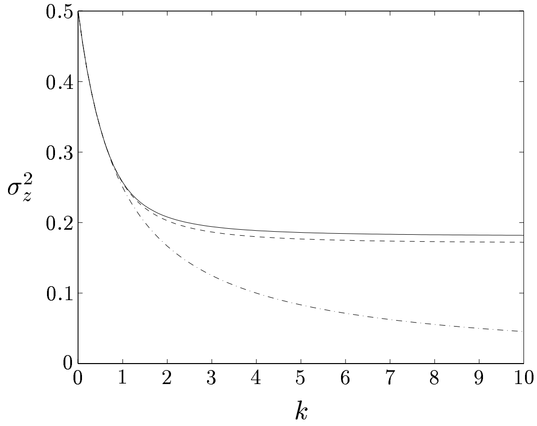

Figure 3.1: The variance of z. ■

4. SLQR

In this section, we present the SLQR control theory. To this end, consider the open-loop system

$$\begin{aligned} \dot{x}_G &= Ax_G + B_1 w + B_2 \varphi(u), \\ z &= C_1 x_G + D_{12} u \end{aligned} \tag{4.1}$$

with the state feedback controller

$$u = Kx_G, \tag{4.2}$$

and assume the following

Assumption 4.1:
(a) A has no eigenvalues in the open right-half plane,
(b) (A, B_2) is stabilizable,
(c) (C_1, A) is detectable,
(d) $D_{12} = \begin{bmatrix} 0 \\ \sqrt{\rho} \end{bmatrix}$, $\rho > 0$,
(e) $D_{12}^T C_1 = 0$. ■

Remark 4.1: These assumptions, except for (a) are standard in the LQR theory. Assumption (a) is necessary for the existence of a semi-globally stabilizing control law for linear systems with saturating control. ■

The closed-loop system (4.1), (4.2) is governed by

$$\begin{aligned} \dot{x}_G &= Ax_G + B_2 \varphi(Kx_G) + B_1 w, \\ z &= (C_1 + D_{12}K)x_G, \\ u &= Kx_G. \end{aligned} \tag{4.3}$$

Application of stochastic linearization to this system yields

$$\begin{aligned} \dot{\hat{x}}_G &= (A + B_2 NK)\hat{x}_G + B_1 w, \\ \hat{z} &= (C_1 + D_{12}K)\hat{x}_G, \\ \hat{u} &= K\hat{x}_G, \end{aligned} \tag{4.4}$$

where N is as in (3.3). Then under the above assumptions, we have

Theorem 4.1: The SLQR problem

$$\min_K \ \sigma_{\hat{z}}^2, \tag{4.5}$$

where the minimization is over all K such that $A + B_2 NK$ is Hurwitz, has a *unique* solution. Moreover, the minimum value of the cost is

$$\begin{aligned} \min_K \ \sigma_{\hat{z}}^2 &= \mathrm{tr}\{C_1 R C_1^T\} \\ &\quad + \rho \frac{N^2}{(\rho + \lambda)^2} B_2^T Q R Q B_2, \end{aligned} \tag{4.6}$$

and a state feedback gain K that achieves this minimum is

$$K = -\frac{N}{\rho + \lambda} B_2^T Q, \tag{4.7}$$

where (N, Q, R, λ) is the *unique* solution of the following system of equations

$$\lambda - \frac{\rho}{\dfrac{\sqrt{\pi}\alpha N \exp\left(\left[\mathrm{erf}^{-1}(\alpha N/\beta)\right]^2\right)}{2\beta \,\mathrm{erf}^{-1}(\alpha N/\beta)} - 1} = 0, \tag{4.8}$$

$$A^T Q + QA - \frac{N^2}{\rho + \lambda} Q B_2 B_2^T Q + C_1^T C_1 = 0, \tag{4.9}$$

$$\begin{aligned} (A - \frac{N^2}{\rho + \lambda} B_2 B_2^T Q) R + R (A - \frac{N^2}{\rho + \lambda} B_2 B_2^T Q)^T \\ + B_1 B_1^T = 0, \end{aligned} \tag{4.10}$$

$$\left(\frac{N^2}{\rho + \lambda}\right)^2 B_2^T Q R Q B_2 - \frac{\alpha^2 N^2}{2\left[\mathrm{erf}^{-1}(\alpha N/\beta)\right]^2} = 0. \tag{4.11}$$

In addition, if R is nonsingular, then the state-feedback gain K given above is unique. ■

The proof of this theorem (see [16]) suggests the following bisection algorithm to compute the SLQR state feedback gain.

Algorithm 4.1: Given A, B_1, B_2, C_1, D_{12}, α, β and a desired accuracy level $\epsilon > 0$,
(a) start with $N_1 = 0$ and $N_2 = \beta/\alpha$,
(b) let $N = (N_1 + N_2)/2$,

(c) calculate λ from (4.8),
(d) solve the Riccati equation (4.9) for Q,
(e) solve the Lyapunov equation (4.10) for R,
(f) calculate the left-hand side of (4.11) and call it δ,
(g) if $|\delta| < \epsilon$, then go to step (i),
(h) if $\delta < 0$, then let $N_1 = N$, else let $N_2 = N$, and go to step (b),
(i) calculate K from (4.7). ■

Remark 4.2: This algorithm terminates at the solution of the SLQR problem in a finite number of iteration with desired accuracy. ■

Remark 4.3: When $\alpha = \beta$ and $\alpha \to \infty$, i.e., the saturation is removed, N approaches 1^- and λ approaches 0^+ so that equations (4.6)–(4.11) reduce to the standard LQR equations. ■

Note that, with a very small ρ, Theorem 4.1 can also be used to calculate the achievable level of disturbance rejection measured by the variance of $\hat{z}$. Moreover, it follows from (4.11) that $N^2\sigma_{\hat{u}}^2 < 2\beta^2/\pi$, and therefore, the variance of $\hat{z}$ cannot be made arbitrarily small even when the plant is minimum phase. This result is expected because with a limited control authority the disturbance cannot be rejected completely.

It can be shown [19] that the SLQR controller guarantees local asymptotic stability of the undisturbed nonlinear system. In addition, the region of attraction of the undisturbed closed-loop system controlled by the SLQR controller can be enlarged to include any given compact set by increasing ρ.

5. SLQG

In the above development, it is assumed that the state is available for feedback. Since this is usually an unrealistic assumption, we now develop the SLQG theory. For this purpose, consider the open-loop system

$$\begin{aligned} \dot{x}_G &= Ax_G + B_1 w + B_2\varphi(u), \\ z &= C_1 x_G + D_{12}u, \\ y &= C_2 x_G + D_{21} w \end{aligned} \tag{5.1}$$

with the output feedback controller

$$\begin{aligned} \dot{x}_C &= Mx_C - Ly, \\ u &= Kx_C, \end{aligned} \tag{5.2}$$

where $x_C \in \mathbb{R}^n$, and assume the following

Assumption 5.1:
(a) A has no eigenvalues in the open right-half plane,
(b) (A, B_2) is stabilizable and (C_2, A) is detectable,
(c) (A, B_1) is stabilizable and (C_1, A) is detectable,
(d) $D_{12} = \begin{bmatrix} 0 \\ \sqrt{\rho} \end{bmatrix}$, $\rho > 0$ and $D_{21} = \begin{bmatrix} 0 & \sqrt{\mu} \end{bmatrix}$, $\mu > 0$,
(e) $D_{12}^T C_1 = 0$ and $B_1 D_{21}^T = 0$. ■

Remark 5.1: These assumptions, except for (a) are standard in the LQG theory. Assumption (a) is necessary for the existence of a semi-globally stabilizing control law for linear systems with saturating control. ■

The closed-loop system (5.1), (5.2) is governed by

$$\begin{aligned} \dot{x}_G &= Ax_G + B_2\varphi(Kx_C) + B_1 w, \\ \dot{x}_C &= Mx_C - LC_2 x_G - LD_{21} w, \\ z &= C_1 x_G + D_{12} K x_C, \\ u &= Kx_C. \end{aligned} \tag{5.3}$$

Application of stochastic linearization to this system yields

$$\begin{aligned} \dot{\hat{x}}_G &= A\hat{x}_G + B_2 N K \hat{x}_C + B_1 w, \\ \dot{\hat{x}}_C &= M\hat{x}_C - LC_2\hat{x}_G - LD_{21} w, \\ \hat{z} &= C_1\hat{x}_G + D_{12}K\hat{x}_C, \\ \hat{u} &= K\hat{x}_C, \end{aligned} \tag{5.4}$$

where N is as in (3.3). Defining

$$\begin{aligned} &\tilde{A} = \begin{bmatrix} A & B_2 N K \\ -LC_2 & M \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_1 \\ -LD_{21} \end{bmatrix} \\ &\tilde{C} = \begin{bmatrix} C_1 & D_{12}K \end{bmatrix}, \ \tilde{K} = \begin{bmatrix} 0 & K \end{bmatrix}, \end{aligned} \tag{5.5}$$

the stochastically linearized state-space equations can be rewritten compactly as

$$\begin{aligned} \dot{\hat{x}} &= \tilde{A}\hat{x} + \tilde{B}w, \\ \hat{z} &= \tilde{C}\hat{x}, \\ \hat{u} &= \tilde{K}\hat{x}, \end{aligned} \tag{5.6}$$

where $\hat{x} = [\hat{x}_G^T \ \hat{x}_C^T]^T$. Then under the above assumptions, we have

Theorem 5.1: There exists a *unique* controller $C(s) = K(sI - M)^{-1}L$ that solves the SLQG problem

$$\min_{K,L,M} \ \sigma_{\hat{z}}^2, \tag{5.7}$$

where the minimization is over all (K, L, M) such that $\tilde{A}$ is Hurwitz. Moreover, the minimum value of the cost is

$$\begin{aligned} \min_{K,L,M} \ \sigma_{\hat{z}}^2 &= \operatorname{tr}\{C_1(P+R)C_1^T\} \\ &\quad + \rho\,\frac{N^2}{(\rho+\lambda)^2} B_2^T QRQB_2, \end{aligned} \tag{5.8}$$

and a state-space realization for the controller is

$$\begin{aligned} K &= -\frac{N}{\rho+\lambda} B_2^T Q, \\ L &= -PC_2^T \frac{1}{\mu}, \\ M &= A + B_2 N K + LC_2, \end{aligned} \tag{5.9}$$

where (N, P, Q, R, S, λ) is the *unique* solution of the following system of equations

$$\lambda - \frac{\rho}{\dfrac{\sqrt{\pi}\alpha N \exp\left(\left[\mathrm{erf}^{-1}(\alpha N/\beta)\right]^2\right)}{2\beta\,\mathrm{erf}^{-1}(\alpha N/\beta)} - 1} = 0, \tag{5.10}$$

$$AP + PA^T - PC_2^TC_2P\frac{1}{\mu} + B_1B_1^T = 0, \tag{5.11}$$

$$A^TQ + QA - \frac{N^2}{\rho+\lambda}QB_2B_2^TQ + C_1^TC_1 = 0, \tag{5.12}$$

$$(A - \frac{N^2}{\rho+\lambda}B_2B_2^TQ)R + R(A - \frac{N^2}{\rho+\lambda}B_2B_2^TQ)^T + PC_2^TC_2P\frac{1}{\mu} = 0, \tag{5.13}$$

$$(A - PC_2^TC_2\frac{1}{\mu})^TS + S(A - PC_2^TC_2\frac{1}{\mu}) + \frac{N^2}{\rho+\lambda}QB_2B_2^TQ = 0, \tag{5.14}$$

$$\left(\frac{N^2}{\rho+\lambda}\right)^2 B_2^TQRQB_2 - \frac{\alpha^2N^2}{2\left[\mathrm{erf}^{-1}(\alpha N/\beta)\right]^2} = 0. \tag{5.15}$$

In addition, if R and S are nonsingular, then (5.9) is a minimal realization for $C(s)$. ■

The proof of this theorem (see [16]) suggests the following bisection algorithm to compute the SLQG controller.

Algorithm 5.1: Given A, B_1, B_2, C_1, C_2, D_{12}, D_{21}, α, β and a desired accuracy level $\epsilon > 0$,

(a) start with $N_1 = 0$ and $N_2 = \beta/\alpha$,
(b) let $N = (N_1 + N_2)/2$,
(c) calculate λ from (5.10),
(d) solve the Riccati equations (5.11) and (5.12) for P and Q, respectively,
(e) solve the Lyapunov equations (5.13) and (5.14) for R and S, respectively,
(f) calculate the left-hand side of (5.15) and call it δ,
(g) if $|\delta| < \epsilon$, then go to step (i),
(h) if $\delta < 0$, then let $N_1 = N$, else let $N_2 = N$, and go to step (b),
(i) calculate K, L, M from (5.9). ■

Remark 5.2: This algorithm terminates at the solution of the SLQG problem in a finite number of iteration with desired accuracy. ■

Remark 5.3: Similar to SLQR case, when $\alpha = \beta$ and $\alpha \to \infty$, i.e., the saturation is removed, equations (5.8)–(5.15) reduce to the standard LQG equations. ■

It can be shown [19] that the SLQG controller guarantees local asymptotic stability of the undisturbed nonlinear system. The global stability of the closed-loop undisturbed system, on the other hand, can be checked by using standard results like the Popov criterion [20].

6. EXAMPLE

Referring to Figure 2.1, consider the system

$$\dot{x}_G = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_G + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_1 + \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \varphi(u), \tag{6.1}$$

$$z_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x_G, \quad z_2 = \begin{bmatrix} \sqrt{\rho} \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix} x_G + \begin{bmatrix} 1 \times 10^{-4} \end{bmatrix} w_2,$$

and assume that $\alpha = 1$, $\beta = 1$. The open-loop variance of z_1 is 1. Suppose that the specification is to reduce the closed-loop variance of z_1 around 0.025.

First, using Theorem 4.1, with a very small ρ, the best achievable $\sigma^2_{\hat{z}_1}$ is calculated as 0.015. Then, with $\rho = 2.060 \times 10^{-2}$, Theorem 4.1 yields the SLQR controller

$$K = \begin{bmatrix} -0.228 & -5.044 & -4.822 \end{bmatrix}, \tag{6.2}$$

which achieves $\sigma^2_{\hat{z}_1} = 0.025$, $N = 0.827$ and ensures the absolute stability of (6.1) in the sector $[0, 1]$. Simulations show that $\sigma^2_{z_1}$ is approximately 0.026.

Next, with $\rho = 9.500 \times 10^{-3}$, Theorem 5.1 yields the SLQG controller

$$C(s) = \frac{616.672s^2 + 589.158s + 589.792}{s^3 + 325.093s^2 + 7.580 \times 10^3 s + 8.664 \times 10^3}. \tag{6.3}$$

This controller achieves $\sigma^2_{\hat{z}_1} = 0.025$, $N = 0.745$ and guarantees the absolute stability of (6.1) in the sector $[0, 1]$. Simulations show that the variance of z_1 is approximately 0.027.

Sample time histories of z_1 and u with the SLQR and SLQG controllers are shown in Figure 6.1 and Figure 6.2, respectively. In each case, the controller is off for the first 25 sec and on for the next 25 sec. Clearly, both controllers achieve the desired performance.

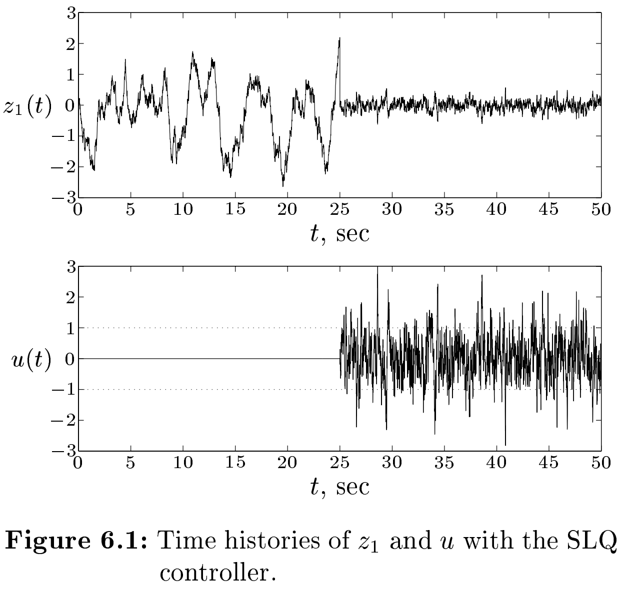

Figure 6.1: Time histories of z_1 and u with the SLQR controller.

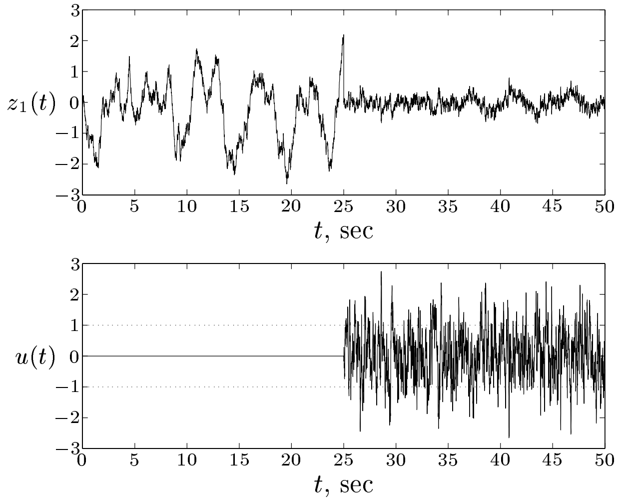

Figure 6.2: Time histories of z_1 and u with the SLQG controller.

7. CONCLUSIONS

In this paper, it is shown that the LQR/LQG design methodology admits an extension to systems with saturating actuators: the SLQR/SLQG theory. Results of this theory can be used for designing controller for systems with saturating actuators. The approach of this work can be used to extend the LQR/LQG theory to systems with other types of nonlinear actuators, for instance, those with deadzone, hysteresis, friction, quantization, etc.

REFERENCES

[1] R. C. Booton, "Nonlinear control systems with random inputs," *IRE Trans. Circuit Theory,* vol. 1, no. 1, pp. 9–18, 1954.

[2] I. E. Kazakov, "An approximate method for the statistical analysis of nonlinear systems," *Tr. VVIA,* no. 394, 1954.

[3] A. Gelb and W. E. Vander Velde, *Multiple-Input Describing Functions and Nonlinear System Design.* NY: McGraw-Hill, 1968.

[4] J. B. Roberts and P. D. Spanos, *Random Vibration and Statistical Linearization.* NY: John Wiley and Sons, 1990.

[5] W. M. Wonham and W. F. Cashman, "A computational approach to optimal control of stochastic saturating systems," *Int. J. Contr.,* vol. 10, no. 1, pp. 77–98, 1969.

[6] J. J. Beaman, "Nonlinear quadratic Gaussian control," *Int. J. Contr.,* vol. 39, no. 2, pp. 343–361, 1984.

[7] W. E. Schmitendorf and B. R. Barmish, "Null controllability of linear systems with constrained controls," *SIAM J. Contr. Optim.,* vol. 18, no. 4, pp. 327–345, 1980.

[8] R. L. Kosut, "Design of linear systems with saturating linear control and bounded states," *IEEE Trans. Automat. Contr.,* vol. 28, no. 1, pp. 121–124, 1983.

[9] E. D. Sontag, "An algebraic approach to bounded controllability of linear systems," *Int. J. Contr.,* vol. 39, no. 1, pp. 181–188, 1984.

[10] E. G. Gilbert and K. T. Tan, "Linear systems with state and control constraints: The theory and application of maximal output admissible sets," *IEEE Trans. Automat. Contr.,* vol. 36, no. 9, pp. 1008–1020, 1991.

[11] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Automat. Contr.,* vol. 39, no. 12, pp. 2411–2425, 1994.

[12] F. Tyan and D. S. Bernstein, "Anti-windup compensator synthesis for systems with saturation actuators," *Int. J. Robust and Nonlinear Contr.,* vol. 5, no. 5, pp. 521–537, 1995.

[13] A. Saberi, Z. Lin, and A. R. Teel, "Control of linear systems with saturating actuators," *IEEE Trans. Automat. Contr.,* vol. 41, no. 3, pp. 368–378, 1996.

[14] V. Kapila and W. M. Haddad, "Actuator amplitude saturation control for systems with exogenous disturbances," *Proc. Amer. Contr. Conf.,* Philadelphia, PA, pp. 1468–1472, 1998.

[15] V. Chellaboina, W. M. Haddad, and J. Oh, "Fixed-structure controller design for discrete-time systems with actuator amplitude and rate saturation constraints," *Proc. Amer. Contr. Conf.,* San Diego, CA, pp. 1992–1996, 1999.

[16] C. Gokcek, P. T. Kabamba, and S. M. Meerkov, "SLQR/SLQG: An LQR/LQG theory for systems with saturating actuators," *CGR-00-03*, EECS, University of Michigan, Ann Arbor, MI, 2000.

[17] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications.* Berlin: Springer-Verlag, 1989.

[18] D. P. Atherton, *Nonlinear Control Engineering.* London: Van Nostrand Reinhold, 1982.

[19] C. Gokcek, P. T. Kabamba, and S. M. Meerkov, "An LQR/LQG Theory for Systems with Saturating Actuators," *CGR-00-08,* EECS, University of Michigan, Ann Arbor, MI, 2000.

[20] H. K. Khalil, *Nonlinear Systems Analysis.* NJ: Prentice-Hall, 1996.