# Delay-dependent stabilization of time-delay systems with saturating actuators

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## Abstract

Sufficient delay-dependent conditions for the stabilization of linear continuous-time systems with time-delay in the state, additive bounded disturbances and limited actuators are given. From these conditions, a region inside which the stability of the closed-loop saturated system is assured and a saturating state feedback control law are obtained.

## 1 Introduction

During the last years a large amount of attention has been paid to the problem of stabilization of linear systems with state delays [7], [10]. Time delays are frequently encountered in control systems and can be the reason of a poor performance behavior or closed-loop instability [16]. The existing results in continuous-time time delay systems are mainly concerned with delay independent conditions, i.e. the time delays are allowed to be arbitrarily large [5], [15], [20]. However, these conditions can be conservative, especially in situations where the existing time delays are small. On the other hand, delay-dependent stabilization conditions can be found in [3], [6], [12], but these strategies show to be very conservative for systems allowing unlimited size time-delays.

Concerning the subject of constrained control of linear systems, a great effort has been spent during the last decade to take into account saturating controls in linear systems control design. In fact, this is an important practical constraint usually disregarded in classical control design methods. See, for example, the two special issues [1], [21] and the references therein for an overview on this subject.

In the context of continuous-time systems with both time-delay and saturating controls, some delay-independent results treating local as well as global stabilization via memoryless feedback control laws have been proposed. See, for example, [18], [22], [24], [4]. In [25], the necessity to explicitly define a set of admissible initial conditions associated to the problem of stabilization has been clearly underlined.

This paper is concerned with the stabilization of a continuous-time linear system subject to time-delays in the state and saturating controls. Additive bounded disturbances are also considered in the system model. A saturating state feedback control law is derived from a Lyapunov-Krasovskii approach [14] which provides delay-dependent conditions. A region in which the stability of the closed-loop saturated system is ensured is also determined.

First, difference inclusions are used to describe the saturated closed-loop system via a locally equivalent polytopic model. Then, the synthesis of the suitable gain matrix with the associated set of admissible initial conditions is carried out by using *S*-procedure and quadratic stabilization results. Some relaxation schemes resulting in LMI optimization problems which can be efficiently solved [2] are also discussed.

## 2 Problem statement

### 2.1 Notation

$\Re^+$ is the set of non-negative real numbers. For any vector $x \in \Re^n$, $x_{(i)}$ denotes the $i$th component of $x$. $A_{(i)}$ denotes the $i$th row of matrix $A$. For two symmetric matrices, $A$ and $B$, $A > B$ means that $A - B$ is positive definite. $A'$ and $trace(A)$ denote the transpose and the trace of $A$, respectively. $I_m$ denotes the m-order identity matrix. $1_m$ denotes in $\Re^m$ the vector $\left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right]'$. $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote respectively the maximal and minimal eigenvalues of matrix $P$. $\mathrm{diag}(B_1, B_2, \dots, B_n)$ denotes a block diagonal matrix composed by the blocks $B_1$, $B_2$, $\dots$, $B_n$. $co\{.\}$ denotes a convex hull. $\mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \Re^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\Re^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. $\|\phi\|_c = \sup_{-\tau \leq t \leq 0} \|\phi(t)\|$ stands for the norm of a function $\phi \in \mathcal{C}_\tau$. When the delay is finite then “sup” can be replaced by “max”. $\mathcal{C}_\tau^\nu$ is the set defined by $\mathcal{C}_\tau^\nu = \{\phi \in \mathcal{C}_\tau \,;\, \|\phi\|_c < \nu,\ \nu > 0\}$. For any vector $\nu \in \Re^m$, one defines $sat(\nu_{(i)}) = sign(\nu_{(i)}) \min(u_{0(i)}, |\nu_{(i)}|)$, $u_{0(i)} > 0$, $i = 1, \dots, m$.

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### 2.2 Problem statement

Consider the linear continuous-time delay[1] system:

$$\dot{x} = Ax(t) + A_\tau x(t-\tau) + Bu(t) + Dw(t) \qquad (1)$$

with the initial conditions

$$x(t_0+\psi) = \phi(\psi), \forall\psi \in [-\tau, 0],\ t_0, \phi \in \Re_+ \times \mathcal{C}_\tau^v \qquad (2)$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the control and $w \in \Re^d$ is the disturbance. Matrices $A$, $A_\tau$, $B$ and $D$ are real constant matrices of appropriate dimensions. The system (1)-(2) is supposed to verify the following assumptions:

**A1.** $(A + A_\tau, B)$ stabilizable.

**A2.** The input vector is subject to amplitude constraints, i.e. $u \in \mathcal{U}_0 \subset \Re^m$ with

$$\mathcal{U}_0 = \{u \in \Re^m \,; -u_{0(i)} \leq u_{(i)} \leq u_{0(i)}, u_{0(i)} > 0, i = 1 \cdots m\} \qquad (3)$$

**A3.** The disturbance vector $w$ is subject to amplitude constraints, i.e. $w \in \mathcal{W}_0 \subset \Re^d$ with

$$\mathcal{W}_0 = \{w \in \Re^d \,; w'w \leq w_0^{-1}, w_0 > 0\} \qquad (4)$$

The control design problem addressed in this paper consists in determining a control law which stabilizes system (1)-(2) under assumptions A1, A2 and A3 and is detailed below.

**Problem 1 (Control Design Problem)**. *Find matrix $K \in \Re^{m\times n}$ and a set of initial conditions $\mathcal{S}_0 \subset \Re^n$ such that the closed-loop system given by*

$$\dot{x}(t) = Ax(t) + A_\tau x(t-\tau) + Bsat(Kx(t)) + Dw(t) \qquad (5)$$

*exhibits the following conditions:*

- *when $w = 0$, system (5) is locally asymptotically stable $\forall\phi(\psi) \in \mathcal{S}_0$, $\forall\psi \in [-\tau, 0]$,*

- *when $w \neq 0$, the trajectories of system (5) remain bounded $\forall\phi(\psi) \in \mathcal{S}_0$, $\forall\psi \in [-\tau, 0]$, and $\forall w \in \mathcal{W}_0$.*

In the disturbance-free case (i.e. $w = 0$), given a stabilizing state feedback gain $K$, the resulting nonlinear closed-loop system (5) possesses a basin of attraction of the equilibrium point $x_e = 0$ [19], [23]. When $w \neq 0$ it is not possible to strictly define one equilibrium point for the closed-loop system with the time-varying disturbance $w(t)$. At a given instant $t$ with $w(t) = w_e$, however, a corresponding equilibrium point $x(t) = x_e$, $\dot{x}(t) = 0$, could be computed, implying that, associated to any disturbance $w_e \in \mathcal{W}_0$, there exists a set of equilibrium points. In this sense, one could say that the closed-loop system (5) exhibits a basin of attraction of this set of equilibrium points. The determination of this basin is a very hard (if not impossible) task, even in the case where $\tau = 0$ (no delay) and $w = 0$ (no disturbance). An interesting way to overcome this difficulty is to determine a suitable set of initial conditions from which the stability relative to the trajectories of the closed-loop system (5) is ensured [13]. A natural objective is then to maximize the size of this set, which could be addressed through another optimization problem.

The global asymptotic stability of system (5) can be addressed only if the open-loop system is stable. In this case, the set $\mathcal{S}_0$ solution to Problem 1 is equal to $\Re^n$ (see [18]). Throughout this paper, no assumption on the open-loop system stability is made. In this sense, Problem 1 and the initial region maximization sub-problem are referred to be local stabilization problems, that is, $\mathcal{S}_0 \neq \Re^n$.

## 3 Stability results

### 3.1 Saturation nonlinearities representation

A locally equivalent polytopic representation for the nonlinear system (5) based on difference inclusions [17], [25] is used. In fact, system (5) can be equivalently written as

$$\dot{x}(t) = (A + B\Gamma(\alpha(x))K)x(t) + A_\tau x(t-\tau) + Dw(t) \qquad (6)$$

where $\Gamma(\alpha(x))$ is a diagonal matrix whose diagonal elements are defined for $i = 1, \ldots, m$ as:

$$\alpha_{(i)}(x) = \begin{cases} \frac{u_{0(i)}}{K_{(i)}x(t)} \text{ if } K_{(i)}x(t) > u_{0(i)} \\ 1 \text{ if } -u_{0(i)} \leq K_{(i)}x(t) \leq u_{0(i)} \\ -\frac{u_{0(i)}}{K_{(i)}x(t)} \text{ if } K_{(i)}x(t) < -u_{0(i)} \end{cases}$$

yielding $0 < \alpha_{(i)}(x) \leq 1$, $i = 1, \ldots, m$, $\forall x \in \Re^n$. Note that the scalar $\alpha_{(i)}(x)$ can be considered as an indicator of the saturation degree of the $i$th entry of $u_{(i)}$. Hence, the smaller is $\alpha_{(i)}(x)$, the farther is $x$ from the region of linearity $S(u_0, 1_m)$ defined by

$$S(u_0, 1_m) = \left\{x \in \Re^n; -u_{0(i)} \leq K_{(i)}x(t) \leq u_{0(i)}, i = 1 \cdots m\right\} \qquad (7)$$

Since this paper addresses the problem of local stabilization a limit is imposed on $x$ yielding a lower bound for $\alpha_{(i)}(x)$. Thus, the following lemma can be presented:

**Lemma 1** *Given any compact set $\mathcal{S}_c \subset \Re^n$ and the saturated system (5), assume that $x(t) \in \mathcal{S}_c$. Then $\dot{x}(t)$ can be determined from the following polytopic model*

$$\dot{x}(t) = \sum_{j=1}^{2^m} \xi_j \mathbb{A}_j x(t) + A_\tau x(t-\tau) + Dw(t) \qquad (8)$$

*with* $\sum_{j=1}^{2^m} \xi_j = 1$, $\xi_j \geq 0$.

**Proof.** For $x \in \mathcal{S}_c$ the components of vector $\alpha(x)$ admit a lower bound denoted

$$\alpha_{\ell(i)} = \min\{\alpha_{(i)}(x); x \in \mathcal{S}_c\},\ i = 1, \ldots, m \qquad (9)$$

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[1] For simplicity, the development is presented for the case of one single delay.

meaning that for any $x \in \mathcal{S}_c$ one gets

$$0 < \alpha_{\ell(i)} \leq \alpha_{(i)}(x) \leq 1,\ \forall i = 1, \dots, m \tag{10}$$

which allow us to define the vector $\alpha_\ell$ and the following vertex matrices:

$$\mathbb{A}_j = A + B\Gamma_j(\alpha_\ell)K,\ j = 1, \dots, 2^m \tag{11}$$

In (11), $\Gamma_j(\alpha_\ell)$ is a diagonal matrix whose diagonal elements take the values 1 or $\alpha_{\ell(i)}$, $i = 1, \dots, m$ (in order to describe all the vertices). Hence, if $x(t) \in \mathcal{S}_c$ then the differencial inclusion results [17] allow to determine $\dot{x}(t)$ from the polytopic model given by (8). □

Actually, the set $S(u_0, \alpha_\ell)$:

$$S(u_0, \alpha_\ell) = \{x \in \Re^n; -\frac{u_{0(i)}}{\alpha_{\ell(i)}} \leq K_{(i)}x(t) \leq \frac{u_{0(i)}}{\alpha_{\ell(i)}}, i = 1 \cdots m\} \tag{12}$$

contains by definition $\mathcal{S}_c$ and corresponds to the maximal set in which model (8) represents system (5).

### 3.2 Disturbance free case ($w = 0$)

The stability of system (5) in the disturbance free case ($w = 0$) can be assessed by the following proposition.

**Proposition 1** *For a given $\tau > 0$, if there exist matrices $Y \in \Re^{m \times n}$, $W = W' > 0 \in \Re^{n \times n}$, $R = R' > 0 \in \Re^{n \times n}$, $Q = Q' > 0 \in \Re^{n \times n}$, $\alpha_\ell \in \Re^m$ and a positive scalar $\gamma$ satisfying*

$$\begin{bmatrix} \Upsilon & \tau WA' + \tau Y'\Gamma_j(\alpha_\ell)B' & \tau WA'_\tau \\ \tau AW + \tau B\Gamma_j(\alpha_\ell)Y & -\tau Q & 0 \\ \tau A_\tau W & 0 & -\tau R \end{bmatrix} < 0,\ \forall j = 1, \dots, 2^m \tag{13}$$

*where*

$$\Upsilon \triangleq W(A + A_\tau)' + (A + A_\tau)W + Y'\Gamma_j(\alpha_\ell)B' + B\Gamma_j(\alpha_\ell)Y + \tau A_\tau Q A'_\tau + \tau A_\tau R A'_\tau$$

$$\begin{bmatrix} W & \alpha_{\ell(i)}Y'_{(i)} \\ \alpha_{\ell(i)}Y_{(i)} & \gamma u^2_{0(i)} \end{bmatrix} \geq 0,\ \forall i = 1, \dots, m \tag{14}$$

$$0 < \alpha_{\ell(i)} \leq 1,\ \forall i = 1, \dots, m \tag{15}$$

*Then, the feedback gain $K = YW^{-1}$ and the set $\Phi_0$ defined as*

$$\Phi_0 = \Big\{\phi; \|\Phi\|_c^2 \leq \frac{1}{\gamma(\lambda_{max}(W^{-1}) + \frac{3\tau^2}{2}\lambda_{max}(A'_\tau R^{-1} A_\tau) + \frac{\tau^2}{2}\max_j \lambda_{max}(\mathbb{A}'_j Q^{-1} \mathbb{A}_j))}\Big\} \tag{16}$$

*solve Problem 1.*

**Proof.** If (14) is satisfied, then the ellipsoid

$$\mathfrak{E}(P, \gamma^{-1}) = \{x \in \Re^n; x(t)'Px(t) \leq \gamma^{-1}\}$$

with $P = W^{-1}$ is included in the set $S(u_0, \alpha_\ell)$ defined as (12) where the vector $\alpha_\ell$ verifies (15). Then, using the results of Lemma 1, $\dot{x}(t)$ can be computed from the polytopic system (8) with $w(t) = 0$.

Now, the aim is to prove that if (13) is satisfied then the closed-loop system (8) with $w = 0$ is locally asymptotically stable $\forall \phi(\theta) \in \Phi_0$, $\forall \theta \in [-2\tau, 0]$. For that, consider the Leibniz-Newton formula

$$\int_a^b \dot{\nu}(\beta)d\beta = \nu(b) - \nu(a) \tag{17}$$

Then, for $t \geq \tau$ and from the polytopic model (8) with $w = 0$, one has

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{-\tau}^0 \dot{x}(t+\beta)d\beta \\ &= x(t) - \int_{-\tau}^0 \left[\sum_{j=1}^{2^m} \xi_j \mathbb{A}_j x(t+\beta) + A_\tau x(t-\tau+\beta)\right] d\beta \end{aligned} \tag{18}$$

implying that

$$\begin{aligned} \dot{x}(t) \quad = \quad & [\sum_{j=1}^{2^m} \xi_j \mathbb{A}_j + A_\tau]x(t) - \int_{-\tau}^0 A_\tau \sum_{j=1}^{2^m} \xi_j \mathbb{A}_j x(t+\beta)d\beta \\ & - \int_{-\tau}^0 A_\tau A_\tau x(t-\tau+\beta)d\beta \end{aligned} \tag{19}$$

The state trajectories of (8) with $w = 0$ also satisfy the equivalent system (19) with initial data on $[-2\tau, 0]$. Thus, the stability of the polytopic system (8) can be studied through the analysis of system (19).

Introduce the Lyapunov-Krasovskii functional

$$V(x_t) = x(t)'Px(t) + S(x_t) \tag{20}$$

where $P$ is a symmetric positive definite matrix and $S(x_t)$ is an appropriate positive definite quadratic form to be determined. Furthermore, $x_t$ denotes the restriction of $x$ to the interval $[t - 2\tau, t]$, $\forall t \geq t_0$, translated to $[-2\tau, 0]$, i.e., $x_t(\psi) = x(t + \psi)$, $\forall \psi \in [-2\tau, 0]$.

The time derivative of $V(x_t)$ along the trajectories of system (8) with $w = 0$ is given by

$$\dot{V}(x_t) = \dot{x}(t)'Px(t) + x(t)'P\dot{x}(t) + \dot{S}(x_t) = \sum_{j=1}^{2^m} \xi_j \mathcal{L}_j \tag{21}$$

with $\sum_{j=1}^{2^m} \xi_j = 1$, $\xi_j \geq 0$ and

$$\begin{aligned} \mathcal{L}_j = x(t)' \left[(\mathbb{A}_j + A_\tau)'P + P(\mathbb{A}_j + A_\tau)\right] x(t) \\ + \nu_j(x_t) + \eta_j(x_t) + \dot{S}_j(x_t) \end{aligned} \tag{22}$$

with

$$\nu_j(x_t) = -2\int_{-\tau}^0 x(t)'PA_\tau \mathbb{A}_j x(t+\beta)d\beta \tag{23}$$

$$\eta_j(x_t) = -2\int_{-\tau}^0 x(t)'PA_\tau A_\tau x(t-\tau+\beta)d\beta \tag{24}$$

Using the property $-2u'v \leq u'Xu + v'X^{-1}v$ where $X$ is any symmetric positive definite matrix and $u$ and $v$ are vectors of appropriate dimensions one gets

$$\nu_j(x_t) \leq \tau x(t)'PA_\tau Q A'_\tau Px(t)$$

$$+\int_{-\tau}^{0} x(t+\beta)'\mathbb{A}_j' Q^{-1}\mathbb{A}_j x(t+\beta)d\beta \tag{25}$$

$$\eta_j(x_t) \leq \tau x(t)' P A_\tau R A_\tau' P x(t)$$
$$+\int_{-\tau}^{0} x(t-\tau+\beta)' A_\tau' R^{-1} A_\tau x(t-\tau+\beta)d\beta \tag{26}$$

Define

$$S_j(x_t) = \int_{-\tau}^{0}\int_{t+\beta}^{t} x(\theta)'\mathbb{A}_j' Q^{-1}\mathbb{A}_j x(\theta)d\theta d\beta$$
$$+\int_{-\tau}^{0}\int_{t-\tau+\beta}^{t} x(\theta)' A_\tau' R^{-1} A_\tau x(\theta)d\theta d\beta \tag{27}$$

and its time-derivative is given by

$$\dot{S}_j(x_t) = \tau x(t)'\mathbb{A}_j' Q^{-1}\mathbb{A}_j x(t)$$
$$-\int_{-\tau}^{0} x(t+\beta)'\mathbb{A}_j' Q^{-1}\mathbb{A}_j x(t+\beta)d\beta + \tau x(t)' A_\tau' R^{-1} A_\tau x(t)$$
$$-\int_{-\tau}^{0} x(t-\tau+\beta)' A_\tau' R^{-1} A_\tau x(t-\tau+\beta)d\beta \tag{28}$$

Now, taking into account (25), (26) and (28), it follows

$$\mathcal{L}_j \quad \leq \quad x(t)'\Psi_j x(t) \tag{29}$$

where

$$\begin{array}{c}\Psi_j = (\mathbb{A}_j + A_\tau)' P + P(\mathbb{A}_j + A_\tau) + \tau P A_\tau Q A_\tau' P \\ +\tau P A_\tau R A_\tau' P + \tau\mathbb{A}_j' Q^{-1}\mathbb{A}_j + \tau A_\tau' R^{-1} A_\tau\end{array} \tag{30}$$

Using Schur's complement and the change of variable $W = P^{-1}$ it can be verified that relation (13) ensures $W\Psi_j W < 0$ for all $j = 1,\ldots,2^m$. Thus, from (21), there exists $\pi_3$ such that $\dot{V}(x_t) \leq -\pi_3 \mid\mid x(t) \mid\mid^2$ and therefore one gets $V(x_t) \leq V(x_{t_0})$ provided that the model (8) with $w = 0$ is valid, that is, provided that $x(t) \in S(u_0, \alpha_\ell)$.

Furthermore, the Lyapunov functional defined in (20) satisfies

$$\pi_1 \mid\mid x(t) \mid\mid^2 \leq V(x_t) \leq \pi_2 \mid\mid x_t \mid\mid_c^2$$

with $\pi_1 = \lambda_{min}(P)$ and

$$\pi_2 = \lambda_{max}(P) + \frac{\tau^2}{2}\max_j \lambda_{max}(\mathbb{A}_j' Q^{-1}\mathbb{A}_j) + \frac{3\tau^2}{2}\lambda_{max}(A_\tau' R^{-1} A_\tau)$$

Hence, for $\phi(\theta) \in \Phi_0$, $\theta \in [-2\tau, 0]$, one gets

$$x(t)' P x(t) \leq V(x_t) \leq V(x_{t_0}) \leq \gamma^{-1},\ \forall t \geq t_0$$

Therefore, for any initial condition in $\Phi_0$, the system (5) verifies the conditions of the Krasovskii Theorem [11] and $V(x_t)$ is a local strictly decreasing Lyapunov function. Thus, the asymptotic stability of system (5) is ensured. □

### 3.3 Disturbance case ($w \neq 0$)

A solution to Problem 1 in the case $w \neq 0$ is presented now.

**Proposition 2** *For a given $\tau > 0$, if there exist matrices $Y \in \Re^{m\times n}$, $W = W' > 0 \in \Re^{n\times n}$, $R = R' > 0 \in \Re^{n\times n}$, $Q = Q' > 0 \in \Re^{n\times n}$, $Z = Z' > 0 \in \Re^{n\times n}$ $\alpha_\ell \in \Re^m$ and three positive scalars $\gamma$, $\omega$, $\mu$ satisfying equations (13), (14), (15) and*

$$\begin{bmatrix} \Theta & \tau W A' + \tau Y'\Gamma_j(\alpha_\ell)B' & \tau W A_\tau' & D & 0 \\ \tau A W + \tau B\Gamma_j(\alpha_\ell)Y & -\tau Q & 0 & 0 & 0 \\ \tau A_\tau W & 0 & -\tau R & 0 & 0 \\ D' & 0 & 0 & -\omega I_d & \tau D' \\ 0 & 0 & 0 & \tau D & -\tau Z \end{bmatrix} \leq 0,\ \ \forall j = 1,\ldots,2^m \tag{31}$$

*where*

$$\begin{array}{c}\Theta \triangleq W(A + A_\tau)' + (A + A_\tau)W + Y'\Gamma_j(\alpha_\ell)B' + B\Gamma_j(\alpha_\ell)Y \\ +\tau A_\tau Q A_\tau' + \tau A_\tau R A_\tau' + \tau A_\tau Z A_\tau' + \mu W\end{array}$$

$$-\mu w_0 + \omega\gamma \leq 0 \tag{32}$$

$$-w_0 + \frac{\tau}{2}\omega\gamma \leq 0 \tag{33}$$

*Then, the feedback gain $K = YW^{-1}$ and the set $\Phi_0$ defined as*

$$\Phi_0 = \Big\{\phi; \mid\mid \Phi \mid\mid_c^2 \leq \frac{w_0 - \frac{\tau}{2}\omega\gamma}{\gamma w_0(\lambda_{max}(W^{-1}) + \frac{3\tau^2}{2}\lambda_{max}(A_\tau' R^{-1} A_\tau)} \cdots$$
$$\overline{\qquad + \frac{\tau^2}{2}\max_j \lambda_{max}(\mathbb{A}_j' Q^{-1}\mathbb{A}_j))}\Big\} \tag{34}$$

*solve Problem 1.*

**Proof.** Consider the Lyapunov functional

$$V(x_t, w) = x(t)' P x(t) + S(x_t, w) \tag{35}$$

where $P = P' > 0$ and $S(x_t, w)$ has to be suitably determined.

Since the case $w = 0$ has been addressed through Proposition 1 and therefore from the satisfaction of equations (13), (14) and (15), in the case $w \neq 0$ it must be shown that $\dot{V}(x_t, w) \leq 0$ for any $x$ such that $x(t)' P x(t) \geq 0$ and any $w$ such that $w(t)' w(t) \leq w_0^{-1}$. Thus, by using the $S$-procedure [2], it must be proven that there exist two positive scalars $\mu$ and $\omega$ satisfying

$$\dot{V}(x_t, w) + \mu(x(t)' P x(t) - \gamma^{-1}) + \omega(w_0^{-1} - w(t)' w(t)) \leq 0 \tag{36}$$

By using equation (17), in the case $w \neq 0$ one gets:

$$\dot{x}(t) = \Big[\sum_{j=1}^{2^m}\xi_j\mathbb{A}_j + A_\tau\Big] x(t) + D w(t)$$
$$-\int_{-\tau}^{0} A_\tau\Big[\sum_{j=1}^{2^m}\xi_j\mathbb{A}_j x(t+\beta) + D w(t+\beta)\Big] d\beta$$
$$-\int_{-\tau}^{0} A_\tau A_\tau x(t-\tau+\beta)d\beta \tag{37}$$

Therefore, $\dot{V}(x_t, w)$ can be computed along the trajectories of system (37).

If inequality (36) holds, the trajectories of the saturated closed-loop system remain confined in the ellipsoid $\mathfrak{E}(P, \gamma^{-1})$. The term $\dot{V}(x_t, w)$ now reads:

$$\dot{V}(x_t, w) = \dot{x}(t)'Px(t) + x(t)'P\dot{x}(t) + \dot{S}(x_t, w) = \sum_{j=1}^{2^m} \xi_j \mathcal{L}_j \tag{38}$$

with $\sum_{j=1}^{2^m} \xi_j = 1$, $\xi_j \geq 0$, and

$$\mathcal{L}_j = x(t)' \left[ (\mathbb{A}_j + A_\tau)'P + P(\mathbb{A}_j + A_\tau) \right] x(t) + 2x(t)'PDw(t)$$

$$+ \dot{S}_j(x_t, w) + \nu_j(x_t) + \eta_j(x_t) + \zeta_j(x_t, w) \tag{39}$$

where the terms $\nu_j(x_t)$ and $\eta_j(x_t)$ are defined as in (23) and (24) whereas the term $\zeta_j(x_t, w)$ is given by

$$\zeta_j(x_t, w) = -2 \int_{-\tau}^{0} x(t)'PA_\tau Dw(t+\beta) d\beta \tag{40}$$

Furthermore, the terms $\nu_j(x_t)$ and $\eta_j(x_t)$ can be majorized as in (25) and (26), whereas $\zeta_j(x_t, w)$ verifies:

$$\zeta_j(x_t, w) \leq \tau x(t)'PA_\tau Z A_\tau' Px(t)$$

$$+ \int_{-\tau}^{0} w(t+\beta)'D'Z^{-1}Dw(t+\beta) d\beta \tag{41}$$

Hence, in this case, it suffices to define $S_j(x_t, w)$ as

$$S_j(x_t, w) = \int_{-\tau}^{0} \int_{t+\beta}^{t} x(\theta)'\mathbb{A}_j' Q^{-1} \mathbb{A}_j x(\theta) d\theta d\beta$$

$$+ \int_{-\tau}^{0} \int_{t+\beta-\tau}^{t} x(\theta)'A_\tau' R^{-1} A_\tau x(\theta) d\theta d\beta$$

$$+ \int_{-\tau}^{0} \int_{t+\beta}^{t} w(\theta)'D'Z^{-1}Dw(\theta) d\theta d\beta \tag{42}$$

yielding

$$\dot{S}_j(x_t, w) = \tau x(t)'\mathbb{A}_j' Q^{-1} \mathbb{A}_j x(t)$$

$$- \int_{-\tau}^{0} x(t+\beta)'\mathbb{A}_j' Q^{-1} \mathbb{A}_j x(t+\beta) d\beta + \tau x(t)'A_\tau' R^{-1} A_\tau x(t)$$

$$- \int_{-\tau}^{0} x(t-\tau+\beta)'A_\tau' R^{-1} A_\tau x(t-\tau+\beta) d\beta + \tau w(t) D'Z^{-1}Dw(t)$$

$$- \int_{-\tau}^{0} w(t+\beta) D'Z^{-1}Dw(t+\beta) d\beta \tag{43}$$

Therefore it follows that

$$\mathcal{L}_j + \mu x(t)'Px(t) - \omega w(t)'w(t) \ \leq \ x(t)'\Psi_j x(t)$$

$$+ \mu x(t)'Px(t) + 2x(t)'PDw(t) + \tau x(t)'PA_\tau Z A_\tau' Px(t)$$

$$+ \tau w(t)'D'Z^{-1}Dw(t) - \omega w(t)'w(t) \tag{44}$$

where $\Psi_j$ is defined in (30) and scalars $\mu$ and $\omega$ satisfy $-\mu\gamma^{-1} + \omega w_0^{-1} \leq 0$, that is, relation (32).

As in the proof of Proposition 1, using Schur's complement and the change of variable $W = P^{-1}$ it can be verified from (44) and (32) that inequality (36) holds provided the model (8) is valid, that is, provided $x(t) \in S(u_0, \alpha_\ell)$.

Furthermore, the Lyapunov functional defined in (35) satisfies

$$\pi_1 \parallel x(t) \parallel^2 \leq V(x_t, w) \leq \pi_2 \parallel x_t \parallel_c^2 + \pi_3 \parallel w_t \parallel_c^2$$

with $\pi_1 = \lambda_{min}(P)$, $\pi_3 = \frac{\tau}{2}\omega$ and

$$\pi_2 = \lambda_{max}(P) + \frac{\tau^2}{2} \max_j \lambda_{max}(\mathbb{A}_j' Q^{-1} \mathbb{A}_j) + \frac{3\tau^2}{2} \lambda_{max}(A_\tau' R^{-1} A_\tau)$$

Hence, for $\phi(\theta) \in \Phi_0$, $\theta \in [-2\tau, 0]$ and any admissible disturbance (that is, $w_0$ satisfying relations (32) and (33)), one gets

$$x(t)'Px(t) \leq V(x_t, w) \leq V(x_{t_0}, w) \leq \gamma^{-1}, \ \forall t \geq t_0$$

For any initial condition in $\Phi_0$ defined in (34) and any admissible disturbance satisfying assumption A2, system (5) verifies the conditions of the Krasovskii Theorem [11] and $V(x_t, w)$ is a local strictly decreasing Lyapunov function. Thus, the stability of system (5) is assured. □

The results derived from Proposition 2 are potentially conservative. This source of conservatism comes in particular from:

1. The polytopic representation used for the closed-loop saturated system.
2. The use of a single quadratic Lyapunov function to enforce the different specifications simultaneously.
3. The majorization used to express conditions on both admissible initial conditions and initial disturbances.
4. The decomposition of delay part $x(t-\tau)$ (see [9]).

Moreover, the presence of bilinearities in some decision variables may cause problems in obtaining a feasible solution. Indeed, relations (13), (14) and (31) are bilinear in the decision variable pairs $(Y, \alpha_\ell)$ and $(W, \mu)$. Relations (32) and (33) are bilinear in variables $\omega$ and $\gamma$. A way to overcome this difficulty consists in fixing alternately some variables and searching the others. Thus, some relaxations schemes based on the LMIs as described in [8] can be used.

In view of establishing the largest stability domain, it intuitively appears that greater is the state evolution domain, more easily it can absorb important disturbances. This intuitive statement can be numerically confirmed in the sense that if $\gamma$ is great (i.e. small state evolution domain) it can be difficult for a given $w_0$ (possibly small if the disturbance set is great) to verify (32) and (33).

Proposition 2 gives a sufficient solution for computing the gain $K$ and a set $\Phi(\gamma^{-1})$ of admissible initial conditions such that Problem 1 can be solved. An interesting way to orient the solutions for seeking the largest set $\Phi(\gamma^{-1})$ as possible is to consider some optimization problems. In this sense, depending on how the size of the set $\Phi(\gamma^{-1})$ is measured, there exist various candidate criteria to derive the largest set. The interested reader is referred to, for instance, [8].

Furthermore, it can be interesting to compute the maximal value of the delay $\tau$ from which relations of Proposition 2 are satisfied. In this sense, an optimization problem can be posed in which $\tau$ would be a decision variable, but different relaxation schemes should be considered since $\tau$ multiplies other decision variables.

## 4 Conclusion

Sufficient delay-dependent conditions for the local stabilization of continuous-time systems with time delays and saturating control have been established. The stability of the open-loop system is not required. The conditions obtained can be easily extended to deal with uncertain models in convex bounded domains.

Several questions can be considered as open problems:
- The influence of the decomposition of the delay part into the resulting solution to Problem 1.
- How this influence can be quantified.
- How to deal with the tradeoff between the size of the delay and the size of the sets of admissible initial conditions and disturbances.

Moreover, it should be interesting to study other types of control law, as for example dynamic output feedback or to investigate the use of different representations for the saturation nonlinearities.

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