

Self-Bounded (A, B) -Invariant Polyhedra of Discrete-time Systems

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Abstract

This work extends the concept of self-bounded (A, B) -invariant subspaces to convex polyhedral sets. Self-bounded (A, B) -invariant polyhedra are defined and characterized. Necessary and sufficient conditions under which a given polyhedron is self-bounded are established in the form of linear matrix relations. It is then shown that the class of self-bounded sets contained in a given region has an infimum, that is, a self-bounded set which is contained in any set of this class. The infimal set is characterized and a numerical method is proposed for its computation in the polyhedral case. It is also shown how these results can be extended to systems subject to control constraints and bounded additive disturbances. A numerical illustrative example is finally presented.

1 Introduction

Linear systems subject to pointwise-in-time constraints are an object of great interest for both theoreticians and practitioners in control systems, as constraints often arise from physical limitations on input and/or output variables. In particular, the *positive invariance* approach has been successfully used to solve a large number of problems on constrained dynamical systems. A set in the state space is *positively invariant* if any trajectory originated from this set does not leave it. An overview of the literature concerning positively invariant sets and their application to the analysis and synthesis of control systems can be found in [1].

A key concept of this approach is that of (A, B) -invariant (or *controlled invariant*) sets, which are sets that can be made positively invariant through the choice of a suitable control law [2, 3] (see also [4, 5]). A typical objective in constrained control problems is to

force the state trajectory to evolve inside a given region. A possible solution is then to guarantee that the initial state belongs to an (A, B) -invariant set contained in the aforementioned region, and to apply a control law such that this set is positively invariant. In general, a controlled state trajectory can leave an (A, B) -invariant set to reach another one. However, there is a class of (A, B) -invariant sets which cannot be exited by trajectories contained in the given region. Such sets characterize a situation of constrained controllability and are known as *self-bounded* (A, B) -invariant sets.

The concept of self-boundedness was first introduced in [6], but limited to subspaces. The present work aims at extending this concept to polyhedral sets. This choice is motivated by the fact that physical limitations inherent to the operation of actual systems very often result in linear constraints on their variables. Self-bounded convex sets are first geometrically characterized. Then, necessary and sufficient conditions under which a given polyhedral set is self-bounded are established in the form of linear matrix relations. The infimal self-bounded set contained in a given set is then characterized and a numerical algorithm is proposed for its computation in the polyhedral case. The study is extended to systems subject to bounded additive disturbances and to control constraints.

Notation: In mathematical expressions, the symbol “.” stands for “such that”. 0 represents a null matrix of appropriate dimension. $\text{Im}(M)$ and $\text{ker}(M)$ represent respectively the image and null spaces of matrix M . The columns of a matrix M form a *generating set* of a polyhedral cone \mathcal{R} if and only if there exists a non-negative vector ξ such that $x = M\xi$, $\forall x \in \mathcal{R}$. Each column of M is then called a *generator* of \mathcal{R} . A generating set of \mathcal{R} is said to be a *minimal generating set* if it is defined by the smallest number of generators.

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2 Preliminaries

Consider the linear, time-invariant, discrete-time system described by:

$$x(k+1) = Ax(k) + Bu(k), \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and k is a nonnegative integer.

A nonempty closed set $\Omega \subset \mathbb{R}^n$ is said to be *positively invariant* with respect to a dynamic system $x(k+1) = f(x(k))$ if for all initial state $x(0) \in \Omega$ the state trajectory remains in Ω .

A nonempty closed set $\Omega \subset \mathbb{R}^n$ is said to be *(A, B)-invariant* with respect to system (1) if for all initial state $x(0) \in \Omega$ there exists a control sequence $\{u(k)\}$ such that $x(k) \in \Omega \forall k > 0$. Therefore, an *(A, B)-invariant set* is a set which can be made positively invariant through a suitable control action.

It can be shown that, given a convex set Ω , there exists an *(A, B)-invariant set* which contains all those *(A, B)-invariant sets* contained in Ω :

$\mathcal{C}^\infty(\Omega) \triangleq$ supremal *(A, B)-invariant set* contained in Ω .

(A, B)-invariance of polyhedra for discrete-time systems was studied in e.g. [2, 3], where conditions under which a given polyhedron is *(A, B)-invariant*, as well as numerical methods for computation of $\mathcal{C}^\infty(\Omega)$ were established.

From now on, the study will be restricted to closed convex sets containing the origin, which are the most relevant for control purposes.

A trajectory of system (1) can be forced to belong to Ω if and only if the initial state belongs to an *(A, B)-invariant set* contained in Ω , hence in $\mathcal{C}^\infty(\Omega)$. Let Π be an *(A, B)-invariant set* contained in Ω . In general, for any initial state belonging to Π , it is possible not only to force the state to remain in Π but also to leave it with a trajectory in Ω and to reach another *(A, B)-invariant set* contained in Ω . On the contrary, there are *(A, B)-invariant sets* which cannot be exited by means of any trajectory on Ω . Such sets will be studied in next section.

3 Self-bounded (A, B)-invariant sets

Definition 3.1 An *(A, B)-invariant set* Π containing the origin and contained in a set Ω is said to be *self-bounded* with respect to Ω if $x(k) \in \Pi \forall k > 0, \forall x(0) \in \Pi$ and for any control sequence $\{u(k)\}$ such that $x(k) \in \Omega \forall k > 0$.

In words, Π is self-bounded with respect to Ω if for any $x(0) \in \Pi$, the state vector cannot leave Π through trajectories contained in Ω . That is, there is no control sequence $\{u(k)\}$ which drives the state outside Π while keeping the state in Ω .

This definition extends to convex sets the concept of self-bounded *(A, B)-invariant* (or controlled invariant) subspaces introduced by Basile and Marro [6, 7].

Consider now the following convex sets defined in the extended space $\mathbb{R}^n \times \mathbb{R}^m$:

$$\mathcal{I}(\Pi, \Omega) \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : x \in \Pi, Ax + Bu \in \mathcal{C}^\infty(\Omega) \right\}, \quad (2)$$

$$\mathcal{O}(\Pi) \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : Ax + Bu \in \Pi \right\}. \quad (3)$$

Theorem 3.1 *The convex set $\Pi \subset \Omega$ is self-bounded (A, B)-invariant with respect to Ω if and only if*

$$\mathcal{I}(\Pi, \Omega) \subset \mathcal{O}(\Pi). \quad (4)$$

Proof: Necessity is first proved: Since $\Pi \subset \mathcal{C}^\infty(\Omega)$ and $\mathcal{C}^\infty(\Omega)$ is *(A, B)-invariant*, then $\forall x(0) \in \Pi$, there always exists a control sequence $\{u\}(\mathcal{C}^\infty(\Omega)) = \{u(0), u(1), u(2), \dots\}$ such that $x(k) \in \mathcal{C}^\infty(\Omega) \forall k > 0$.

Suppose now that there exists $\begin{bmatrix} x_n \\ u_n \end{bmatrix} \in \mathcal{I}(\Pi, \Omega)$ such that $Ax_n + Bu_n \notin \Pi$. Then, it is clear that for $x(0) = x_n$ there exists a trajectory of the state completely contained in Ω , but which leaves Π , contradicting thereby the assumption that Π is self-bounded.

Sufficiency comes from the fact that if (4) is verified, then any trajectory starting from Π and completely contained in $\mathcal{C}^\infty(\Omega)$ (hence, in Ω) will not leave Π . \square

Corollary 3.1 *Let $\{u(k)\}(\mathcal{C}^\infty(\Omega))$ be any control sequence $\{u(0), u(1), u(2), \dots\}$ such that, $x(k) \in \mathcal{C}^\infty(\Omega), \forall x(0) \in \mathcal{C}^\infty(\Omega), \forall k > 0$. Then, any *(A, B)-invariant set* Π , self-bounded with respect to Ω , is such that $x(k) \in \Pi, \forall x(0) \in \Pi, \forall k > 0, \forall \{u(k)\}(\mathcal{C}^\infty(\Omega))$.*

Proof: It suffices to notice that any vector $\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$, corresponding to the state and the control in time k , where $x(0) \in \Pi, x(k) \in \mathcal{C}^\infty(\Omega)$ and $u(k)$ is in $\{u(k)\}(\mathcal{C}^\infty(\Omega))$, is such that $\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{I}(\Pi, \Omega)$.

Therefore, if condition (4) of self-boundedness is verified, then $x(k) \in \Pi \forall k \geq 0$. \square

This corollary states that any control law for which $\mathcal{C}^\infty(\Omega)$ is positively invariant is such that any self-bounded set contained in Ω is also positively invariant under this law.

3.1 Self-boundedness of general polyhedra

The study will now be restricted to convex polyhedra containing the origin, represented by sets of linear inequalities:

$$\begin{aligned} \Omega &= R[W, \zeta] = \{x : Wx \leq \zeta\}, \quad \zeta \geq 0, \\ \Pi &= R[G, \rho] = \{x : Gx \leq \rho\}, \quad \rho \geq 0. \end{aligned}$$

where W and G are matrices and ζ and ρ are vectors of appropriate dimensions. Let also the supremal (A, B) -invariant set contained in Ω be a convex polyhedron represented by:

$$\mathcal{C}^\infty(\Omega) = R[V, \nu] = \{x : Vx \leq \nu\}, \quad \nu \geq 0.$$

The hypothesis above is not always verified. $\mathcal{C}^\infty(\Omega)$ may be defined by an infinite number of linear inequalities. In practice, however, this is not a serious drawback: as shown in [2], under mild assumptions, $\mathcal{C}^\infty(\Omega)$ can be arbitrarily approximated by an (A, B) -invariant polyhedron $R[V, \nu]$.

Theorem 3.2 *The convex polyhedron $R[G, \rho] \subset R[W, \zeta]$ is self-bounded (A, B) -invariant with respect to $R[W, \zeta]$ if and only if there exist matrices L and M such that:*

$$\begin{aligned} LG + MVA &= GA, \\ MVB &= GB, \\ L\rho + M\nu &\leq \rho, \\ L \geq 0, M &\geq 0. \end{aligned} \quad (5)$$

Proof: It follows immediately from the application of Farkas' Lemma (see e.g. [8, 9]) to (4), with:

$$\mathcal{I}(\Pi, \Omega) = R \left[\begin{bmatrix} G & 0 \\ VA & VB \end{bmatrix}, \begin{bmatrix} \rho \\ \nu \end{bmatrix} \right], \quad (6)$$

$$\mathcal{O}(\Pi) = R \left[\begin{bmatrix} GA & GB \end{bmatrix}, \rho \right]. \quad (7)$$

□

The matrix relations in (5) are linear. Self-boundedness of a polyhedron can therefore be checked through the resolution of linear programming problems as for the classical positive invariance relations (see e.g. [8]).

3.2 Self-boundedness of symmetrical polyhedra

The case for which Ω and Π are 0-symmetrical polyhedra is now considered. Symmetrical polyhedra Ω , Π and $\mathcal{C}^\infty(\Omega)$ can be represented as:

$$\begin{aligned} \Omega &= S(W, \zeta) = \{x : |Wx| \leq \zeta\}, \quad \zeta \geq 0, \\ \Pi &= S(G, \rho) = \{x : |Gx| \leq \rho\}, \quad \rho \geq 0, \\ \mathcal{C}^\infty(\Omega) &= S(V, \nu) = \{x : |Vx| \leq \nu\}, \quad \nu \geq 0. \end{aligned}$$

Corollary 3.2 *The symmetrical convex polyhedron $S(G, \rho) \subset S(W, \zeta)$ is self-bounded (A, B) -invariant with respect to $S(W, \zeta)$ if and only if there exist matrices L and M such that:*

$$\begin{aligned} LG + MVA &= GA, \\ MVB &= GB, \\ |L|\rho + |M|\nu &\leq \rho. \end{aligned} \quad (8)$$

Proof: Symmetrical polyhedra $S(G, \rho)$ can be written in the form $R[\mathcal{G}, r]$ with $\mathcal{G} = \begin{bmatrix} G \\ -G \end{bmatrix}$, $r = \begin{bmatrix} \rho \\ \rho \end{bmatrix}$. Hence, according to Theorem 3.2, $S(G, \rho)$ is self-bounded if and only if there exist matrices L_1, L_2, M_1 and M_2 such that:

$$\begin{aligned} (L_1 - L_2)G + (M_1 - M_2)VA &= GA, \\ (M_1 - M_2)VB &= GB, \\ (L_1 + L_2)\rho + (M_1 + M_2)\nu &\leq \rho, \\ L_1, L_2, M_1, M_2 &\geq 0. \end{aligned} \quad (9)$$

Now, let $L = L_1 - L_2$ and $M = M_1 - M_2$, and consider the matrices L^+, L^-, M^+ and M^- whose elements from row i and column j are defined by: $L_{ij}^+ = \max\{L_{ij}, 0\}$, $L_{ij}^- = \max\{-L_{ij}, 0\}$, $M_{ij}^+ = \max\{M_{ij}, 0\}$, $M_{ij}^- = \max\{-M_{ij}, 0\}$.

Necessity of conditions (8) follows from the fact that these matrices are such that:

$$\begin{aligned} L^+ - L^- &= L = L_1 - L_2, \quad M^+ - M^- = M = M_1 - M_2, \\ (L^+ + L^-)\rho + (M^+ + M^-)\nu &= |L|\rho + |M|\nu \leq (L_1 + L_2)\rho + (M_1 + M_2)\nu \leq \rho. \end{aligned}$$

Sufficiency is shown as well, since matrices L^+, L^-, M^+ and M^- verify conditions (9). □

The preceding results are now specialized to subspaces, which are particular types of polyhedra:

Corollary 3.3 *The subspace $\mathcal{V} = \ker(G) \in \ker(W)$ is self-bounded (A, B) -invariant with respect to $\ker(W)$ if and only if \mathcal{V} is (A, B) -invariant and $\mathcal{B} \cap \mathcal{V}^* \subset \mathcal{V}$, where $\mathcal{B} = \text{Im}(B)$ and $\mathcal{V}^* = \ker(V)$ is the maximal (A, B) -invariant subspace contained in $\ker(W)$.*

Proof: From Corollary 3.2, \mathcal{V} is self-bounded (A, B) -invariant with respect to the subspace $\ker(W)$ if and only if there exist matrices L and M such that the first two expressions in (8) are verified. Therefore, there exists a matrix M such that $MVB = GB$, then $GBu = 0 \forall u : VBu = 0$, hence $Gx = 0 \forall x : Vx = 0$ and $x = Bu$. Therefore, $Gx = 0 \forall x \in \mathcal{B} \cap \mathcal{V}^*$, hence $\mathcal{B} \cap \mathcal{V}^* \subset \mathcal{V}$.

Now, since $\ker(V) = \mathcal{V}^*$ is (A, B) -invariant, then there exist matrices H and F such that $HV = VA + VBF$ [7]. Moreover, from (8), there exists a matrix L such that $LG + MVA = GA$. Then, $LG + M(HV - VBF) = LG + MHV - GBF = GA$, hence $LG + MHV = G(A + BF)$. Since $\ker(G) \subset \ker(V)$, then there exists a matrix N such that $V = NG$. Therefore, $(L + MHN)G = G(A + BF)$ which proves that $\ker(G)$ is (A, B) -invariant. □

The conditions stated in this Corollary are identical to those obtained in [6, 7].

4 The Infimal Self-bounded Set

The family of (A, B) -invariant sets contained in a given convex set, say Ω , is an upper semilattice with respect to the operation "convex hull of the union". This property guarantees the existence, in this family of the supremal set $\mathcal{C}^\infty(\Omega)$.

The most outstanding property of the family of self-bounded (A, B) -invariant sets contained in a given set is to be a lattice instead of a semilattice. Hence it admits both a supremum ($\mathcal{C}^\infty(\Omega)$) and an infimum. The existence of an infimum is guaranteed by the following property:

Property 4.1 *The family of all self-bounded (A, B) -invariant sets contained in a convex set Ω is closed under intersection.*

Proof: It follows immediately from the Definition 3.1. Let Π_1 and Π_2 be two self-bounded (A, B) -invariant sets contained in Ω . Then it is clear that any trajectory starting from $x(0) \in \Pi_1 \cap \Pi_2$ and contained in Ω can leave neither Π_1 nor Π_2 , hence $\Pi_1 \cap \Pi_2$. Therefore, $\Pi_1 \cap \Pi_2$ is self-bounded (A, B) -invariant. \square

Since Ω is closed by assumption, this property guarantees the existence, in the family of self-bounded (A, B) -invariant sets contained in Ω , of an infimal element (an element which is contained in all the other elements):

$$\mathcal{C}_\infty(\Omega) \triangleq \text{infimal self-bounded } (A, B)\text{-invariant set contained in } \Omega.$$

$\mathcal{C}_\infty(\Omega)$ is the set defined by the intersection of all self-bounded (A, B) -invariant sets in Ω . It should be noticed that $\mathcal{C}_\infty(\Omega)$ cannot be empty as it was assumed that the origin belongs to any self-bounded set.

Theorem 4.1 *Consider the following sequence of sets:*

$$\begin{aligned} \mathcal{C}_0 &= \{0\}, \\ \mathcal{C}_{i+1} &= \{x \in \mathcal{C}^\infty(\Omega) : \exists x_i, u_i : x = Ax_i + Bu_i\}. \end{aligned} \quad (10)$$

The infimal self-bounded (A, B) -invariant set contained in Ω is given by: $\mathcal{C}_\infty(\Omega) = \lim_{i \rightarrow \infty} \mathcal{C}_i$ ($i = 0, 1, 2, \dots$).

Proof: First we prove, by induction, that $\mathcal{C}_i \subset \mathcal{C}_{i+1} \forall i = 0, 1, 2, \dots$. Suppose $\mathcal{C}_{i-1} \subset \mathcal{C}_i$. Then, it is clear that any $x \in \mathcal{C}^\infty(\Omega)$ such that there exist $x_i \in \mathcal{C}_{i-1}$ and u_i for which $x = Ax_i + Bu_i$ belongs to \mathcal{C}_{i+1} . In other words, any $x \in \mathcal{C}_i$ belongs to \mathcal{C}_{i+1} . Therefore, if $\mathcal{C}_{i-1} \subset \mathcal{C}_i$ then $\mathcal{C}_i \subset \mathcal{C}_{i+1}$. Since clearly $\mathcal{C}_0 \subset \mathcal{C}_1$, then, by induction, $\mathcal{C}_i \subset \mathcal{C}_{i+1} \forall i = 0, 1, 2, \dots$

Consider now an admissible trajectory starting from $x(0) \in \mathcal{C}_\infty(\Omega)$. A trajectory will be said to be admissible if it does not leave Ω (hence $\mathcal{C}^\infty(\Omega)$) $\forall k > 0$. Since $\mathcal{C}_i \subset \mathcal{C}_\infty(\Omega) \forall i = 0, 1, 2, \dots$ then $x(0) \in \mathcal{C}_i$ for some i . Hence, $x(1) \in \mathcal{C}_{i+1} \subset \mathcal{C}_\infty(\Omega)$. Therefore, any admissible trajectory is such that $x(k) \in \mathcal{C}_\infty(\Omega) \forall k > 0$. This proves that $\mathcal{C}_\infty(\Omega)$ is self-bounded (A, B) -invariant.

Let now $\mathcal{C} \subset \Omega$ be an arbitrary self-bounded (A, B) -invariant set, and suppose $\mathcal{C}_i \subset \mathcal{C}$ for some i . Then, $Ax_i + Bu_i \in \mathcal{C} \forall x_i \in \mathcal{C}$, u_i such that $Ax_i + Bu_i \in \mathcal{C}^\infty(\Omega)$. Therefore, $x = Ax_i + Bu_i \in \mathcal{C} \forall x_i \in \mathcal{C}_i$, u_i such that $Ax_i + Bu_i \in \mathcal{C}^\infty(\Omega)$. Hence, any $x \in \mathcal{C}_{i+1}$ also belongs to \mathcal{C} . Therefore, $\mathcal{C}_i \subset \mathcal{C} \forall i$ hence $\mathcal{C}_\infty(\Omega) \subset \mathcal{C}$. This proves that $\mathcal{C}_\infty(\Omega)$ is infimal. \square

\mathcal{C}_i is the set of states which can be reached from the origin in i steps by means of a control sequence such that $x(k) \in \mathcal{C}^\infty(\Omega) \forall k \geq 0$. Therefore, one can conclude that $\mathcal{C}_\infty(\Omega)$ is the set of reachable states from the origin inside $\mathcal{C}^\infty(\Omega)$. It can be also noticed that \mathcal{C}_{i+1} is the projection of $\mathcal{I}(\mathcal{C}_i, \Omega)$ onto the state space \mathfrak{R}^n .

The focus is now turned towards the computation of $\mathcal{C}_\infty(\Omega)$ when Ω is a polyhedral set represented by $R[W, \zeta]$ and $\mathcal{C}^\infty(\Omega) = R[V, \nu]$. Let the set \mathcal{C}_i be given by: $\mathcal{C}_i = \{x : G_i x \leq \rho_i\}$. From (10), the set \mathcal{C}_{i+1} is then given by:

$$\mathcal{C}_{i+1} = \{x : \exists x_i, u_i : x - Ax_i - Bu_i = 0, G_i x_i \leq \rho_i, Vx \leq \nu\}.$$

The relations defining \mathcal{C}_{i+1} can be written in the following matrix form:

$$\begin{bmatrix} I \\ -I \\ 0 \\ V \end{bmatrix} x + \begin{bmatrix} -A & -B \\ A & B \\ G_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ \rho_i \\ \nu \end{bmatrix}. \quad (11)$$

The dependency on x_i and u_i in the definition of \mathcal{C}_{i+1} can be eliminated as follows. Let the rows of matrix $T = [T_1^+ \ T_1^- \ T_2]$ form a minimal generating set of the polyhedral cone defined by the vectors $w = [w_1^+ \ w_1^- \ w_2]$, with $w_1^+, w_1, w_2 \geq 0$, such that:

$$[w_1^+ \ w_1^- \ w_2] \begin{bmatrix} -A & -B \\ A & B \\ G_i & 0 \end{bmatrix} = [0 \ 0]. \quad \text{Then,}$$

from application of Farkas' Lemma [9, 3], $\exists x_i, u_i$ such that the first three inequalities in (11) are verified if

$$\text{and only if } [T_1^+ \ T_1^- \ T_2] \left(\begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ 0 \\ \rho_i \end{bmatrix} \right) \leq 0. \quad \text{Therefore, the set } \mathcal{C}_{i+1} \text{ is given by: } \mathcal{C}_{i+1} = \left\{ x : \begin{bmatrix} T_1^+ - T_1^- \\ V \end{bmatrix} x \leq \begin{bmatrix} T_2 \rho_i \\ \nu \end{bmatrix} \right\}.$$

As shown in [10], it is possible to compute matrix T by means of Fourier-Motzkin elimination technique [9].

After suppression of redundant constraints, the polyhedral set \mathcal{C}_{i+1} can be written in the form: $\mathcal{C}_{i+1} = \{x : G_{i+1} x \leq \rho_{i+1}\}$. Algorithm (10) is then run until $\mathcal{C}_{i+1} = \mathcal{C}_i$ up to a given accuracy. Since all the sets in question are polyhedra, this test can be performed via linear programming [8].

If $\Omega = R[W, \zeta]$ is an unbounded symmetrical polyhedron, then the set $\mathcal{C}_\infty(\Omega)$ can have infinite directions as well. In this case, the preceding procedure is not able to compute $\mathcal{C}_\infty(\Omega)$. One is then led to decompose its computation by "subtracting" from Ω the minimal self-bounded (A, B) -invariant subspace contained in $\ker(W)$ and applying the algorithm above to a polyhedron defined on a reduced order space (see [3] for further details).

5 Extensions

5.1 Persistently disturbed systems

Consider the following linear discrete-time system:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k), \quad (12)$$

where $d \in \mathfrak{R}^q$ is a disturbance vector, supposed constrained to evolve inside a bounded domain $\mathcal{D} \subset \mathfrak{R}^q$:

$$d(k) \in \mathcal{D} \quad \forall k \geq 0. \quad (13)$$

This kind of disturbances acts continuously in time. This is why they are named by some authors *persistent disturbances*.

A closed set $\Omega \subset \mathfrak{R}^n$ is said to be \mathcal{D} - (A, B) -invariant with respect to system (12),(13) if for all initial state $x(0) \in \Omega$ there exists a control sequence $\{u(k)\}$ such that $x(k) \in \Omega \quad \forall d \in \mathcal{D}, \forall k > 0$. This definition assumes that the disturbance vector is not measured.

Consider the following convex sets:

- $\mathcal{I}(\Pi, \Omega, \mathcal{D})$: defined similarly to $\mathcal{I}(\Pi, \Omega)$ (2), but with the second condition changed to $Ax + Bu + Ed \in \mathcal{C}^\infty(\Omega, \mathcal{D}), \forall d \in \mathcal{D}$, where $\mathcal{C}^\infty(\Omega, \mathcal{D})$ is the supremal \mathcal{D} - (A, B) -invariant set contained in Ω .
- $\mathcal{O}(\Pi, \mathcal{D})$: defined similarly to $\mathcal{O}(\Pi)$ (3), but with the condition changed to $Ax + Bu + Ed \in \Pi, \forall d \in \mathcal{D}$.

Then, the set $\Pi \subset \Omega$ is self-bounded \mathcal{D} - (A, B) -invariant with respect to Ω if and only if $\mathcal{I}(\Pi, \Omega, \mathcal{D}) \subset \mathcal{O}(\Pi, \mathcal{D})$.

It should be pointed out that $\mathcal{C}^\infty(\Omega, \mathcal{D})$ may turn out to be empty. The following development assumes non-emptiness of $\mathcal{C}^\infty(\Omega, \mathcal{D})$.

Consider the polyhedral case: $\Omega = R[W, \zeta]$, $\Pi = R[G, \rho]$, $\mathcal{C}^\infty(\Omega, \mathcal{D}) = R[V, \nu]$ and $\mathcal{D} = R[D, \omega] = \{d \in \mathfrak{R}^q : Dd \leq \omega\}$. Define the components δ_i and ξ_i of vectors δ and ξ respectively as follows:

$$\delta_i = \max_{d \in R[D, \omega]} G_i Ed, \quad \xi_i = \max_{d \in R[D, \omega]} V_i Ed.$$

Then:

- $\mathcal{I}(\Pi, \Omega, \mathcal{D}) = R \left[\begin{bmatrix} G & 0 \\ VA & VB \end{bmatrix}, \begin{bmatrix} \rho \\ \nu - \xi \end{bmatrix} \right],$
- $\mathcal{O}(\Pi, \mathcal{D}) = R \left[\begin{bmatrix} GA & GB \end{bmatrix}, \rho - \delta \right].$

One can notice that the role of vectors δ and ξ is to absorb the effect of the disturbances. The following result can then be easily derived:

Theorem 5.1 *The convex polyhedron $R[G, \rho] \subset R[W, \zeta]$ is self-bounded \mathcal{D} - (A, B) -invariant with respect to $R[W, \zeta]$ if and only if there exist matrices L and M such that:*

$$LG + MVA = GA, \quad MVB = GB, \\ L\rho + M(\nu - \xi) \leq \rho - \delta, \quad L \geq 0, M \geq 0.$$

The existence of the infimal self-bounded \mathcal{D} - (A, B) -invariant set contained in a given set Ω , denoted $\mathcal{C}_\infty(\Omega, \mathcal{D})$, can be shown by a similar proof as for the undisturbed case. $\mathcal{C}_\infty(\Omega, \mathcal{D})$ can be computed as in (10), but with $\mathcal{C}_{i+1} = \{x \in \mathcal{C}^\infty(\Omega, \mathcal{D}) : \exists x_i \in \mathcal{C}_i, u_i, d_i \in \mathcal{D} : x = Ax_i + Bu_i + Ed_i\}$. An algorithm for computing $\mathcal{C}_\infty(\Omega, \mathcal{D})$ with $\Omega = R[G, \rho]$, $\mathcal{C}^\infty(\Omega, \mathcal{D}) = R[V, \nu]$, $\mathcal{D} = R[D, \omega]$ can be easily derived from the algorithm of section 4.

5.2 Control constrained systems

Consider the system (1) and assume that the values of the control vector elements are constrained to a convex set

$$u(k) \in \mathcal{U} \quad \forall k \geq 0. \quad (14)$$

A closed set $\Omega \subset \mathfrak{R}^n$ is said to be \mathcal{U} - (A, B) -invariant with respect to system (1), subject to constraints (14), if for all initial state $x(0) \in \Omega$ there exists a control sequence $\{u(k)\}$, with $u(k) \in \mathcal{U} \quad \forall k \geq 0$, such that $x(k) \in \Omega \quad \forall k > 0$.

Consider the set $\mathcal{I}(\Pi, \Omega, \mathcal{U})$, defined similarly to $\mathcal{I}(\Pi, \Omega)$ (2), but with the addition of the condition $u \in \mathcal{U}$ and with the second condition changed to $Ax + Bu \in \mathcal{C}^\infty(\Omega, \mathcal{U})$, where $\mathcal{C}^\infty(\Omega, \mathcal{U})$ is the supremal \mathcal{U} - (A, B) -invariant set contained in Ω . Consider also the set $\mathcal{O}(\Pi)$ (3). Then, the set $\Pi \subset \Omega$ is self-bounded \mathcal{U} - (A, B) -invariant with respect to Ω if and only if $\mathcal{I}(\Pi, \Omega, \mathcal{U}) \subset \mathcal{O}(\Pi)$.

Consider now the polyhedral case: $\Omega = R[W, \zeta]$, $\Pi = R[G, \rho]$, $\mathcal{C}^\infty(\Omega, \mathcal{U}) = R[V, \nu]$ and $\mathcal{U} = R[U, \psi] = \{u \in \mathfrak{R}^m : Uu \leq \psi\}$, $\psi \geq 0$. Then, the set $\mathcal{O}(\Pi)$ is given by

$$(7) \text{ and } \mathcal{I}(\Pi, \Omega, \mathcal{U}) = R \left[\begin{bmatrix} G & 0 \\ VA & VB \\ 0 & U \end{bmatrix}, \begin{bmatrix} \rho \\ \nu \\ \psi \end{bmatrix} \right].$$

Theorem 5.2 *The convex polyhedron $R[G, \rho] \subset R[W, \zeta]$ is self-bounded \mathcal{U} - (A, B) -invariant with respect to $R[W, \zeta]$ if and only if there exist matrices L , M and N such that:*

$$LG + MVA = GA, \quad MVB + NU = GB, \\ L\rho + M\nu + N\psi \leq \rho, \quad L \geq 0, M \geq 0, N \geq 0.$$

The existence of the infimal self-bounded \mathcal{U} - (A, B) -invariant set contained in a given set Ω , denoted $\mathcal{C}_\infty(\Omega, \mathcal{U})$, can be shown. $\mathcal{C}_\infty(\Omega, \mathcal{U})$ can be computed as in (10), but with $\mathcal{C}_{i+1} = \{x \in \mathcal{C}^\infty(\Omega, \mathcal{U}) : \exists x_i \in \mathcal{C}_i, u_i \in \mathcal{U} : x = Ax_i + Bu_i\}$. An algorithm for computation of $\mathcal{C}_\infty(\Omega, \mathcal{U})$ with $\Omega = R[G, \rho]$, $\mathcal{C}^\infty(\Omega, \mathcal{U}) = R[V, \nu]$, $\mathcal{U} = R[U, \psi]$ can also be easily derived from the algorithm of section 4.

6 Numerical Example

Consider the system (1) for which:

$$A = \begin{bmatrix} -0.8 & 0.2 \\ 0.5 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the polyhedron $R[W, \zeta]$ with

$$W = \begin{bmatrix} 0.2 & 0.2 \\ -1 & -1 \\ -1 & 0.35 \\ 0.25 & -0.5 \\ 0.6 & 0.1 \end{bmatrix}, \quad \zeta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The computation of the supremal (A, B) -invariant set contained in $\Omega = R[W, \zeta]$ results in $\mathcal{C}^\infty(\Omega) = R[V, \nu]$, with:

$$V = \begin{bmatrix} W \\ \hline 3.0857 & -0.7714 \end{bmatrix}, \quad \nu = \begin{bmatrix} \zeta \\ \hline -3.8571 \end{bmatrix}.$$

The application of the algorithm proposed for computation of the infimal self-bounded (A, B) -invariant set contained in Ω yields $\mathcal{C}_\infty(\Omega) = R[G, \rho]$ with:

$$G = \begin{bmatrix} 0.2 & 0.2 \\ -1 & -1 \\ -1 & 0.35 \\ 0.25 & -0.5 \\ 17.5 & 0 \\ -5.7143 & 0 \end{bmatrix}, \quad \rho = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 13.8095 \\ 5.4422 \end{bmatrix}.$$

The sets Ω , $\mathcal{C}^\infty(\Omega)$, $\mathcal{C}_\infty(\Omega)$ are shown in Figure 1.

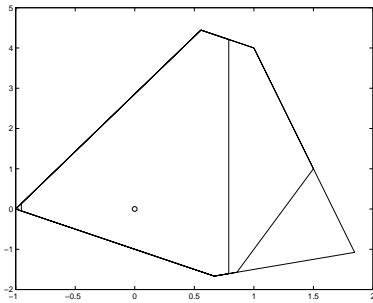


Figure 1: $\{0\} \subset \mathcal{C}_\infty(\Omega) \subset \mathcal{C}^\infty(\Omega) \subset \Omega$.

For $R[G, \rho]$, matrices which verify the conditions of self-boundedness of Theorem (3.2) are given by:

$$L = \begin{bmatrix} 0 & 0.7 & 0 & 0 & 10 & 0 \\ 0.2286 & 0 & 0 & 0 & 0 & 2.2857 \\ \hline \mathbf{0}_{4 \times 5} \end{bmatrix},$$

$$M = \begin{bmatrix} \hline \mathbf{0}_{2 \times 5} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

where $\mathbf{0}_{s \times t}$ represents a null matrix of dimension $s \times t$.

The polyhedron $R[V, \nu]$ can be made positively invariant through a simple state feedback control law $u = Fx$ with: $F = [-0.5 \quad 0.9]$. In agreement with Corollary 3.1, positive invariance and self-boundedness of $R[G, \rho]$ are also achieved using the same state feedback control law.

7 Conclusion

This work studied the concept of self-bounded (A, B) -invariant sets of the state space for discrete-time systems. The existence of such sets implies limitations

in the controllability of trajectories confined in a given set. Self-bounded polyhedral sets were geometrically and analytically characterized.

Given a convex region, there exists a self-bounded set which is contained in any other self-bounded set in this region. If a trajectory completely contained in the region starts from this infimal set, then it will not exit it. Such an infimal set was characterized and a numerical method was proposed for its computation in the polyhedral case.

Some important extensions were considered as well, namely systems subject to persistent disturbances and to control constraints. A promising application of the results on self-bounded sets presented here, besides constrained controllability, lies in the solution of the ℓ^1 control problem. This issue is now under investigation.

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