

Computation of Control Sets using Subdivision and Continuation Techniques

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Abstract

A key notion for the analysis of the global behavior of control systems are control sets. Control sets are subsets of the state space where approximate controllability holds: from every point in a control set one can steer arbitrarily close to any other point in the control set. In general it is not possible to find explicit formulas for control sets and their domains of attraction. Therefore numerical methods are a natural part of a systematic analysis. We will present a method for the computation of control sets, which is based on subdivision and continuation techniques.

1 Introduction

We consider the following class of control affine systems

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^m u_i(t)X_i(x(t)), \quad t \in \mathbb{R} \quad (1)$$
$$u \in \mathcal{U} = \{u : \mathbb{R} \rightarrow \mathbb{R}^m, u(t) \in U \text{ for almost all } t \in \mathbb{R}, \text{ measurable}\}$$

on a connected Riemannian C^∞ -manifold M of dimension $d < \infty$, where $X_0, \dots, X_m : M \rightarrow \mathbf{T}M$ are C^∞ -vector fields on M and $U \subset \mathbb{R}^m$ is compact and convex. For our purpose it is convenient to set $M = \mathbb{R}^d$. We assume that the right hand side is globally bounded, that is, for all $(x, u) \in \mathbb{R}^d \times U$ we have $\|X_0(x) + \sum_{i=1}^m u_i X_i(x)\| \leq K$ for some constant $K > 0$. Note that this is not a decisive restriction since we exclusively consider compact subsets of the state space. Furthermore, we assume that for each $u \in \mathcal{U}$ and $x \in \mathbb{R}^d$ the equation above has a unique solution $\varphi(t, x, u)$, $t \in \mathbb{R}$, satisfying $\varphi(0, x, u) = x$. We supply \mathcal{U} with the weak* topology of $L_\infty(\mathbb{R}, \mathbb{R}^m)$, then \mathcal{U} becomes a compact, complete, separable metric space (see e.g. Colonius/Kliemann [3, Section 4.2]).

A key notion for the analysis of the global behavior of these systems are control sets, cp. Colonius/Kliemann [3, Chapter 3]. In general it is not possible to find explicit formulas for control sets and their domains of

attraction. Therefore numerical methods are a natural part of a systematic analysis.

We will present a method for the computation of control sets, which is based on subdivision and continuation techniques. These algorithms have been developed and implemented by Michael Dellnitz and Oliver Junge, for the computation of relative global attractors, invariant manifolds and invariant measures of dynamical systems (software package GAIO), see Dellnitz/Hohmann [4], [5] and Dellnitz/Junge [6]. These algorithms can be modified such that it is possible to compute control sets, especially in higher dimensions.

The usage of subdivision techniques directly leads to an algorithm for the computation of viability kernels, similarly as described in Saint-Pierre [8] and Cardaliaguet/Quincampoix/Saint-Pierre [2]. The notion of viability kernels plays an integral role in viability theory. A standard reference is the book by Aubin [1], it deals with differential inclusions, a concept for differential equations with set-valued right hand side. For a given closed subset Q of the state space, the viability kernel is defined as the largest closed viability domain contained in Q . A viability domain is a subset S where for each initial point there exists a solution which remains in S for all positive times. This concept is applied to control system (1), and also introduced for the complete time axes, which yields the definition of the \mathbb{R} -viability kernel of Q .

In section 2 we describe a subdivision algorithm for the computation of viability kernels. In principle, the idea of the method is as follows. First, in a subdivision step, the subset Q under consideration (in the implementation realized as a rectangle of matching dimension) is subdivided into disjoint subsets (boxes) of Q (in the implementation realized as bisection with respect to particular coordinate axes). Then, in a selection step, a selection criterion decides which subsets are kept and which ones are removed. The selection step is repeated until no more subsets are removed. Then, the subdivision step is again applied to the remaining subsets, and so on. In this way, one obtains successively finer coverings of the viability kernel of Q , finally converging

to it.

There is close correspondence between viability kernels and control sets, see Szolnoki [10]. Hence in some cases, this method already allows to compute control sets with a simple geometry. But, in general, subdivision techniques are not sufficient to compute control sets, especially those with a more complicated structure. A crucial restriction originates from the choice of the set Q , since in general there is only a little a priori information on the structure and the position of the control sets in the state space. This problem can be solved by the use of continuation techniques, which are appropriate to approximate reachable sets.

Section 3 clarifies the relation of control sets and reachable sets. For that purpose we introduce an appropriate notion of chain control sets relative to Q and chain reachable sets relative to Q . Chain control sets are maximal viable subsets of chain controllability. If Q is the whole state space, the relative concept coincides with the standard concept of chain control sets given in Colonius/Kliemann [3, Section 3.4]. There one also finds results on the relation of chain control sets and “usual” control sets. If x lies in a relative chain control set E_Q , then E_Q is given by the viability kernel of the intersection of the positive and the negative relative chain reachable set of x .

The continuation method for the computation of chain reachable sets relative to Q is described in section 4. For a given grid covering Q , the algorithm starts with grid element B containing x . In a continuation step, we add all those grid elements that can be reached from B under a discretized dynamic. The subsequent continuation step is applied to all elements added in the previous step, and so on. The algorithm terminates after a finite number of steps, and, for a grid fine enough, we obtain a suitable covering of the relative chain reachable set.

We now can combine the subdivision and continuation algorithms to compute relative chain control sets E_Q . We use the continuation method to construct an appropriate covering of E_Q , and the subdivision algorithm to approximate the chain control set itself.

In section 5, the method is applied to the so called “full brusselator”. This three dimensional system is a simplified model for the Zhabotinski-Belousov reaction, see Hairer/Nørsett/Wanner [7, Section I.16]. We examine the system with an additional control in each coordinate.

2 The Subdivision Algorithm

First let us recall the notion of viability domains and viability kernels.

Definition 1 A set $S \subset \mathbb{R}^d$ is called an \mathbb{R} -viability domain for control system (1), if for all $x \in S$ there is a control function $u \in \mathcal{U}$, such that the solution $\varphi(t, x, u)$ remains in S for all $t \in \mathbb{R}$.

We now consider viability domains contained in some closed subset Q . More precisely, we are interested in the largest closed viability domain contained in Q .

Definition 2 Let Q be a closed subset of \mathbb{R}^d . The largest closed \mathbb{R} -viability domain in Q is called the \mathbb{R} -viability kernel of Q for control system (1) and is denoted by $\text{Viab}^{\mathbb{R}}(Q)$.

Proposition 3

$$\text{Viab}^{\mathbb{R}}(Q) = \{x \in Q, \text{ there exists } u \in \mathcal{U} \text{ such that } \varphi(t, x, u) \in Q \text{ for all } t \in \mathbb{R}\}.$$

The proof is given in Szolnoki [10]. For control system (1), the corresponding exact time discretization for time step $\tau > 0$ is given by

$$\begin{aligned} x_{k+1} &= \varphi_{\tau}(x_k, u(t_k + \cdot)), \\ t_0 &= 0, t_{k+1} = t_k + \tau, k \in \mathbb{Z} \\ u &\in \mathcal{U} \end{aligned} \quad (2)$$

The unique solution for $x_0 \in \mathbb{R}^d$ and $u \in \mathcal{U}$ is denoted by $(x_k)_{k \in \mathbb{Z}}^u$.

Definition 4 A set $S \subset \mathbb{R}^d$ is called a discrete viability domain for system (2), if for all $x \in S$ there is a control function $u \in \mathcal{U}$ such that $\varphi_{-\tau}(x, u) \in S$ and $\varphi_{\tau}(x, u) \in S$.

Definition 5 The discrete viability kernel for system (2) of a given closed set $Q \subset \mathbb{R}^d$ is defined by

$$\text{Viab}_{\tau}(Q) := \{x_0 \in Q, \text{ there exists } u \in \mathcal{U} \text{ such that } (x_k)_{k \in \mathbb{Z}}^u \subset Q\}.$$

Lemma 6

$$\text{Viab}_{\tau}(Q) = \{x_0 \in Q, \text{ there exists } u \in \mathcal{U} \text{ such that } (x_k)_{k \in \mathbb{Z}}^u \subset \text{Viab}_{\tau}(Q)\}.$$

Proof: For $x_0 \in \text{Viab}_{\tau}(Q)$, there exists $u \in \mathcal{U}$ such that $(x_k)_{k \in \mathbb{Z}}^u \subset Q$. For any arbitrary $j \in \mathbb{Z}$, set $\hat{x}_0 := x_j \in (x_k)_{k \in \mathbb{Z}}^u$ and $v := u(j\tau + \cdot)$. By construction, we obtain $(\hat{x}_i)_{i \in \mathbb{Z}}^v \subset Q$, that is, $\hat{x}_0 = x_j \in \text{Viab}_{\tau}(Q)$, which completes the proof. ■

Remark 7 It can be seen easily that discrete viability kernels are maximal in the sense that every discrete viability domain $S \subset Q$ is contained in $\text{Viab}_\tau(Q)$.

Note that

$$\text{Viab}^{\mathbb{R}}(Q) \subseteq \text{Viab}_\tau(Q) \quad \text{for all } \tau > 0.$$

This is immediate from proposition 3. Furthermore, if $\text{Viab}_\tau(Q)$ lies sufficiently “deep” inside Q , that is,

$$\mathcal{B}_{\frac{1}{2}K\tau}(x_0) \subset Q \quad \text{for all } x_0 \in \text{Viab}_\tau(Q), \quad (3)$$

where $\mathcal{B}_{\frac{1}{2}K\tau}(x_0)$ denotes the ball of radius $\frac{1}{2}K\tau$ centered in x_0 , then $\text{Viab}_\tau(Q)$ and $\text{Viab}^{\mathbb{R}}(Q)$ coincide.

Proposition 8 Suppose assumption (3) is fulfilled. Then

$$\text{Viab}_\tau(Q) = \text{Viab}^{\mathbb{R}}(Q).$$

Proof: Let $x_0 \in \text{Viab}_\tau(Q)$. By lemma 6, there exists $u \in \mathcal{U}$ such that $(x_k)_{k \in \mathbb{Z}}^u \subset \text{Viab}_\tau(Q)$. It remains to show that for any $j \in \mathbb{Z}$ we have that $\varphi(t, x_j, u(t_j + \cdot)) \in Q$ for all $t \in [0, \tau]$. By the mean value theorem and the boundedness of the right hand side of control system (1) we obtain that $d(x_j, \varphi(t, x_j, u(t_j + \cdot))) \leq \frac{1}{2}K\tau$ for all $t \in [0, \frac{1}{2}\tau]$ and $d(\varphi(t, x_j, u(t_j + \cdot)), x_{j+1}) \leq \frac{1}{2}K\tau$ for all $t \in [\frac{1}{2}\tau, \tau]$. Hence for all $t \in [0, \tau]$, $\varphi(t, x_j, u(t_j + \cdot)) \in \mathcal{B}_{\frac{1}{2}K\tau}(x_j) \cup \mathcal{B}_{\frac{1}{2}K\tau}(x_{j+1})$, which is a subset of Q by assumption (3), since $x_j, x_{j+1} \in \text{Viab}_\tau(Q)$. ■

For the rest of the paper we assume that assumption (3) is satisfied. Hence we may restrict our attention to discrete viability kernels. For a more detailed analysis of this point we refer to Szolnoki [9]. We now describe an algorithm for the approximation of $\text{Viab}_\tau(Q)$. Consider a subdivision Ψ_δ of the state space \mathbb{R}^d into compact subsets B of diameter at most δ . We denote

$$\begin{aligned} \varphi_\tau(B, u) &:= \{y \in \mathbb{R}^d, \text{ there exists } x \in B \\ &\quad \text{such that } y = \varphi_\tau(x, u)\}, \\ \Phi_\tau(B, u) &:= \{\hat{B} \in \Psi_\delta, \varphi_\tau(B, u) \cap \hat{B} \neq \emptyset\} \end{aligned}$$

and define a dynamic on Ψ_δ , which is in fact an inclusion:

$$\begin{aligned} B_{k+1} &\in \Phi_\tau(B_k, u(t_k + \cdot)), \\ t_0 &= 0, t_{k+1} = t_k + \tau, k \in \mathbb{Z} \\ u &\in \mathcal{U} \end{aligned} \quad (4)$$

For $B_0 \in \Psi_\delta$ and $u \in \mathcal{U}$, the (nonunique) solution of (4) is denoted by $(B_k)_{k \in \mathbb{Z}}^u$.

Lemma 9 Let $B_1, B_2 \in \Psi_\delta$ and $u \in \mathcal{U}$. Then

$$B_2 \in \Phi_\tau(B_1, u) \iff B_1 \in \Phi_{-\tau}(B_2, u).$$

Proof: Let $B_2 \in \Phi_\tau(B_1, u)$. Thus $\varphi_\tau(B_1, u) \cap B_2 \neq \emptyset$, that is, there exist $x_1 \in B_1$ and $x_2 \in B_2$ such that $x_2 = \varphi_\tau(x_1, u)$ or $x_1 = \varphi_{-\tau}(x_2, u)$. Hence $x_1 \in \varphi_{-\tau}(B_2, u)$ and it immediately follows $B_1 \in \Phi_{-\tau}(B_2, u)$. The other implication follows analogously. ■

Definition 10 The finite viability kernel for system (4) of a given finite subset Q of Ψ_δ is given by

$$\begin{aligned} \text{Viab}_{\tau, \delta}(Q) &:= \{B_0 \in Q, \text{ there exists } u \in \mathcal{U} \\ &\quad \text{such that } (B_k)_{k \in \mathbb{Z}}^u \subset Q\}. \end{aligned}$$

Lemma 11

$$\begin{aligned} \text{Viab}_{\tau, \delta}(Q) &= \{B_0 \in Q, \text{ there exists } u \in \mathcal{U} \\ &\quad \text{such that } (B_k)_{k \in \mathbb{Z}}^u \subset \text{Viab}_{\tau, \delta}(Q)\}. \end{aligned}$$

Proof: The proof is exactly the same as the proof of lemma 6, with B instead of x and $\text{Viab}_{\tau, \delta}(Q)$ instead of $\text{Viab}_\tau(Q)$. ■

The finite viability kernel $\text{Viab}_{\tau, \delta}(Q)$ can be computed in a finite number of steps by the following algorithm. Set $Q_0 = Q$ and define an iteration of selection steps by

$$\begin{aligned} Q_{\ell+1} &= \{B \in Q_\ell, \text{ there exist } u_1, u_2 \in \mathcal{U} \\ &\quad \text{such that } \Phi_\tau(B, u_1) \cap Q_\ell \neq \emptyset \\ &\quad \text{and } \Phi_{-\tau}(B, u_2) \cap Q_\ell \neq \emptyset\}. \end{aligned} \quad (5)$$

The algorithm terminates if $Q_{\ell+1} = Q_\ell$, where $\ell \leq |Q|$ and $|Q|$ denotes the number of elements in Q .

Proposition 12 Let Q_ℓ be the result of the above described algorithm. Then

$$Q_\ell = \text{Viab}_{\tau, \delta}(Q).$$

Proof: “ \subseteq ”: Let $B_0 \in Q_\ell$, hence there exist $u_{-1}, u_0 \in \mathcal{U}$ and $B_{-1}, B_1 \in Q_\ell$ such that $B_1 \in \Phi_\tau(B_0, u_0)$ and $B_{-1} \in \Phi_{-\tau}(B_0, u_{-1})$ or, by lemma 9, $B_0 \in \Phi_\tau(B_{-1}, u_{-1})$. For B_1 there exist $u_1 \in \mathcal{U}, B_2 \in Q_\ell$ with $B_2 \in \Phi_\tau(B_1, u_1)$ and for B_{-1} there exist $u_{-2} \in \mathcal{U}, B_{-2} \in Q_\ell$ with $B_{-1} \in \Phi_\tau(B_{-2}, u_{-2})$, and so on. Defining $u \in \mathcal{U}$ by $u|_{[k\tau, (k+1)\tau]}(t) := u_k|_{[0, \tau]}(t - k\tau), k \in \mathbb{Z}$, we have constructed a solution $(B_k)_{k \in \mathbb{Z}}^u$ of (4), which is contained in $Q_\ell \subset Q$, that is, $B_0 \in \text{Viab}_{\tau, \delta}(Q)$.

“ \supseteq ”: We have $\text{Viab}_{\tau, \delta}(Q) \subseteq Q_0 = Q$. Suppose there exists $B_0 \in \text{Viab}_{\tau, \delta}(Q) \subseteq Q_i$ such that $B_0 \notin Q_{i+1}$ for some $i < \ell$. This means that B_0 has been removed in the i^{th} selection step, and for all $u \in \mathcal{U}$ we obtain

$$\Phi_\tau(B_0, u) \cap Q_i = \emptyset \text{ or } \Phi_{-\tau}(B_0, u) \cap Q_i = \emptyset$$

On the other hand, $B_0 \in \text{Viab}_{\tau, \delta}(Q)$, and lemma 11 yields that there exists $u \in \mathcal{U}$ and $B_{-1}, B_1 \in \text{Viab}_{\tau, \delta}(Q) \subseteq Q_i$ such that $B_1 \in \Phi_{\tau}(B_0, u)$ and $B_{-1} \in \Phi_{-\tau}(B_0, u)$, which yields the desired contradiction. ■

For a given strictly decreasing sequence $\delta_i \rightarrow 0^+$, $i \rightarrow \infty$, we consider a corresponding sequence of grids (Ψ_{δ_i}) , where $\Psi_{\delta_{i+1}}$ is constructed from Ψ_{δ_i} by subdividing each element $B_{\delta_i} \in \Psi_{\delta_i}$ into subsets $B_{\delta_{i+1}}$ of diameter at most δ_{i+1} .

Remark 13 Fix $Q_{\delta_0} \subset \Psi_{\delta_0}$. Note that for each $i \in \mathbb{N}$, Q_{δ_0} can also be represented as a subset $Q_{\delta_i} \subset \Psi_{\delta_i}$, such that $\bigcup_{B \in Q_{\delta_0}} B = \bigcup_{B \in Q_{\delta_i}} B$. For abbreviation, we will write Q instead of Q_{δ_i} , if the corresponding grid is obvious. Furthermore, for any $B_{\delta_{i+1}} \in Q$ there exists $B_{\delta_i} \in Q$ such that $B_{\delta_{i+1}} \subset B_{\delta_i}$.

Under these assumptions, the corresponding viability kernels $\text{Viab}_{\tau, \delta_i}(Q)$ of a given subset $Q \subset \Psi_{\delta_0}$ define a decreasing sequence in the following sense.

Lemma 14 For each $i \in \mathbb{N}_0$,

$$\text{Viab}_{\tau, \delta_{i+1}}(Q) \subset \text{Viab}_{\tau, \delta_i}(Q),$$

where $\text{Viab}_{\tau, \delta_i}(Q)$ represents a subset in $\Psi_{\delta_{i+1}}$.

Proof: Let $B_{\delta_{i+1}}^0 \in \text{Viab}_{\tau, \delta_{i+1}}(Q)$. By definition, there exists $u \in \mathcal{U}$ such that $(B_{\delta_{i+1}}^k)_k^u \subset Q$. By Remark 13, for each $k \in \mathbb{Z}$ there exists $B_{\delta_i}^k \in Q$ such that $B_{\delta_{i+1}}^k \subset B_{\delta_i}^k$, hence, $(B_{\delta_i}^k)_k^u \subset Q$ and it immediately follows $B_{\delta_{i+1}}^0 \subset B_{\delta_i}^0 \in \text{Viab}_{\tau, \delta_i}(Q)$. ■

Considering $\text{Viab}_{\tau, \delta_i}(Q)$ as a subset of \mathbb{R}^d , we obtain for each $i \in \mathbb{N}_0$

$$\text{Viab}_{\tau, \delta_i}(Q) = \bigcap_{j=0}^i \text{Viab}_{\tau, \delta_j}(Q)$$

and we denote by

$$\text{Viab}_{\tau, 0}(Q) := \bigcap_{j=0}^{\infty} \text{Viab}_{\tau, \delta_j}(Q)$$

the corresponding limit set.

Lemma 15 For each $i \in \mathbb{N}_0$,

$$\text{Viab}_{\tau}(Q) \subset \text{Viab}_{\tau, \delta_i}(Q).$$

Proof: Let $x_0 \in \text{Viab}_{\tau}(Q)$, that is, there exists $u \in \mathcal{U}$ such that $(x_k)_k^u \subset Q$. For each $k \in \mathbb{Z}$, there is $B_{\delta_i}^k \subset Q$ such that $x_k \in B_{\delta_i}^k$. The sequence $(B_{\delta_i}^k)_k^u \subset Q$ defines a solution of system (4). We obtain $B_{\delta_i}^0 \in \text{Viab}_{\tau, \delta_i}(Q)$ and $x_0 \in B_{\delta_i}^0$ completes the proof. ■

Lemma 16 $\text{Viab}_{\tau, 0}(Q)$ is a discrete viability domain.

For the proof we refer to Szolnoki [9]. Combining lemmas 15, 16 and Remark 7, we obtain that the discrete viability kernel can be approximated by finite viability kernels.

Theorem 17

$$\text{Viab}_{\tau, 0}(Q) = \text{Viab}_{\tau}(Q).$$

Note that

$$\text{Viab}_{\tau, \delta_{i+1}}(Q) = \text{Viab}_{\tau, \delta_{i+1}}(\text{Viab}_{\tau, \delta_i}(Q))$$

for each $i \in \mathbb{N}$, where $\text{Viab}_{\tau, \delta_i}(Q)$ represents a subset in $\Psi_{\delta_{i+1}}$. This is an immediate consequence of lemma 14.

We conclude with the following algorithm for the approximation of $\text{Viab}_{\tau}(Q)$: Given a sequence $\delta_i \rightarrow 0$ and an initial set $Q \subset \Psi_{\delta_0}$, we compute $\text{Viab}_{\tau, \delta_0}(Q)$ as described in (5). We obtain $\text{Viab}_{\tau, \delta_{i+1}}(Q)$ from $\text{Viab}_{\tau, \delta_i}(Q)$ in two steps:

1. change to grid $\Psi_{\delta_{i+1}}$ by subdividing each element in $\text{Viab}_{\tau, \delta_i}(Q)$ into subsets of diameter at most δ_{i+1} ;
2. compute $\text{Viab}_{\tau, \delta_{i+1}}(\text{Viab}_{\tau, \delta_i}(Q))$ as described in (5).

3 Control Sets and Reachable Sets

We now introduce the notion of relative chain control sets and relative chain reachable sets.

Definition 18 Fix $x, y \in \mathbb{R}^d$ and let $\varepsilon, T > 0$. A controlled (ε, T) -chain from x to y is given by $n \in \mathbb{N}$, $x_0, \dots, x_n \in \mathbb{R}^d$, $u_0, \dots, u_{n-1} \in \mathcal{U}$ and $t_0, \dots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and

$$d(\varphi(t_j, x_j, u_j), x_{j+1}) \leq \varepsilon \quad \text{for all } j = 0, \dots, n-1.$$

Definition 19 A set $E_Q \subset Q$ is called a chain control set relative to Q for control system (1), if

- (i) E_Q is a \mathbb{R} -viability domain,
- (ii) for all $x, y \in E_Q$ and $\varepsilon, T > 0$ there is a controlled (ε, T) -chain from x to y , which is completely contained in Q , that means $\varphi(t, x_j, u_j) \in Q$ for all $t \in [0, t_j]$, $j = 0, \dots, n-1$,

(iii) E_Q is maximal (with respect to set inclusion) with these properties.

Definition 20 The positive and the negative τ -chain reachable set of $x \in Q$ relative to Q are defined by

$$\begin{aligned} \mathcal{O}_{Q,\tau}^+(x) &:= \{y \in Q, \text{ for all } \varepsilon > 0 \\ &\quad \text{there exists a controlled } (\varepsilon, \tau)\text{-chain} \\ &\quad \text{from } x \text{ to } y \text{ in } Q \text{ with jump times} \\ &\quad t_j = \tau \text{ for all } j = 0, \dots, n-1\}, \end{aligned}$$

$$\begin{aligned} \mathcal{O}_{Q,\tau}^-(x) &:= \{y \in Q, \text{ for all } \varepsilon > 0 \\ &\quad \text{there exists a controlled } (\varepsilon, \tau)\text{-chain} \\ &\quad \text{from } y \text{ to } x \text{ in } Q \text{ with jump times} \\ &\quad t_j = \tau \text{ for all } j = 0, \dots, n-1\}. \end{aligned}$$

The proof of the following theorem on the relation of chain control sets and τ -chain reachable sets is given in Szolnoki [9].

Theorem 21 Let $E_Q \subset Q$ be a chain control set relative to Q , and $x \in E_Q$. Then

$$E_Q = \text{Viab}^{\mathbb{R}}(\mathcal{O}_{Q,\tau}^+(x) \cap \mathcal{O}_{Q,\tau}^-(x)).$$

4 The Continuation Algorithm

The τ -chain reachable sets relative to Q described in the previous section are the objects we can approximate by a continuation algorithm. In the following we describe the method for positive τ -chain reachable sets, the corresponding ‘‘negative object’’ can be computed analogously. Set

$$\begin{aligned} \mathcal{O}_0^+(x) &:= \{B \in Q, x \in B\}, \\ \Delta_0 &:= \mathcal{O}_0^+(x). \end{aligned}$$

Then we obtain $\mathcal{O}_{\ell+1}^+(x)$ from $\mathcal{O}_\ell^+(x)$ by

$$\begin{aligned} \Delta_{\ell+1} &= \{B \in Q, \text{ there exist } \hat{B} \in \Delta_\ell, u \in \mathcal{U} \\ &\quad \text{such that } B \in \Phi_\tau(\hat{B}, u)\}, \end{aligned} \quad (6)$$

$$\mathcal{O}_{\ell+1}^+(x) = \mathcal{O}_\ell^+(x) \cup \Delta_{\ell+1}.$$

The algorithm terminates if $\Delta_\ell = \emptyset$, $\ell \leq |Q|$, and we set

$$\mathcal{O}_{Q,\tau,\delta}^+(x) := \mathcal{O}_\ell^+(x).$$

For a given sequence of grids Ψ_{δ_i} , $\delta_i \rightarrow 0^+$, the corresponding finite reachable sets define a decreasing sequence.

Lemma 22 For each $i \in \mathbb{N}$,

$$\mathcal{O}_{Q,\tau,\delta_{i+1}}^+(x) \subset \mathcal{O}_{Q,\tau,\delta_i}^+(x),$$

where $\mathcal{O}_{Q,\tau,\delta_i}^+(x)$ represents a subset in $\Psi_{\delta_{i+1}}$.

Proof: Let $B_{\delta_{i+1}} \in \mathcal{O}_{Q,\tau,\delta_{i+1}}^+(x)$. By construction, there exist $B_{\delta_{i+1}}^j \in \Delta_j$, $j = 0, \dots, k$, $k < \ell$, with $x \in B_{\delta_{i+1}}^0$ and $B_{\delta_{i+1}} = B_{\delta_{i+1}}^k$. By remark 13, for each $B_{\delta_{i+1}}^j$ there exists $B_{\delta_i}^j \in Q_{\delta_i}$ with $B_{\delta_{i+1}}^j \subset B_{\delta_i}^j$. It is easy to see that this yields $B_{\delta_i}^k \in \mathcal{O}_{Q,\tau,\delta_i}^+(x)$ and we finally obtain $B_{\delta_{i+1}} = B_{\delta_{i+1}}^k \subset B_{\delta_i}^k \in \mathcal{O}_{Q,\tau,\delta_i}^+(x)$. ■

Considering $\mathcal{O}_{Q,\tau,\delta_i}^+(x)$ as a subset of \mathbb{R}^d , we obtain for each $i \in \mathbb{N}_0$

$$\mathcal{O}_{Q,\tau,\delta_i}^+(x) = \bigcap_{j=0}^i \mathcal{O}_{Q,\tau,\delta_j}^+(x)$$

and we denote by

$$\mathcal{O}_{Q,\tau,0}^+(x) := \bigcap_{j=0}^{\infty} \mathcal{O}_{Q,\tau,\delta_j}^+(x)$$

the corresponding limit set.

Lemma 23 For each $i \in \mathbb{N}_0$,

$$\mathcal{O}_Q^+(x) \subset \mathcal{O}_{Q,\tau,\delta_i}^+(x).$$

Proof: Let $y \in \mathcal{O}_{Q,\tau}^+(x)$, that is, for all $\varepsilon > 0$ there exist $n \in \mathbb{N}$, $x_0, \dots, x_n \in Q$ and $u_0, \dots, u_{n-1} \in \mathcal{U}$ with $x_0 = x$, $x_n = y$, and $d(\varphi_\tau(x_j, u_j), x_{j+1}) \leq \varepsilon$ for all $j = 0, \dots, n-1$. We may assume that for ε small enough, $\mathcal{B}_\varepsilon(x_j) \subset B_{\delta_i}(x_j)$ for all $j = 0, \dots, n$, where $\mathcal{B}_\varepsilon(x_j)$ denotes the ball of radius ε centered in x_j and $B_{\delta_i}(x_j)$ denotes the element of Q_{δ_i} containing x_j . By construction of the continuation algorithm it immediately follows $B_{\delta_i}(x_j) \in \Delta_j$ for all $j = 0, \dots, n$, hence $y \in B_{\delta_i}(x_n) \in \Delta_n \subset \mathcal{O}_n^+(x) \subset \mathcal{O}_{Q,\tau,\delta_i}^+(x)$. ■

We conclude with the following convergence result.

Theorem 24

$$\mathcal{O}_{Q,\tau,0}^+(x) = \mathcal{O}_{Q,\tau}^+(x).$$

Proof: With lemma 23, it suffices to show $\mathcal{O}_{Q,\tau,0}^+(x) \subset \mathcal{O}_{Q,\tau}^+(x)$. Let $y \in \mathcal{O}_{Q,\tau,0}^+(x)$. For contradiction we assume that there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $x_0, \dots, x_n \in Q$ and $u_0, \dots, u_{n-1} \in \mathcal{U}$ with $x_0 = x$, $x_n = y$, there exists $j \in \{0, \dots, n-1\}$ with $d(\varphi_\tau(x_j, u_j), x_{j+1}) > \varepsilon$. Hence we say ‘‘every chain from x to y has a jump bigger than ε ’’. On the other hand, $y \in \mathcal{O}_{Q,\tau,0}^+(x)$, and lemma 22 yields that $B_{\delta_i}(y) \in \mathcal{O}_{Q,\tau,\delta_i}^+(x)$ for all $i \in \mathbb{N}_0$. Choose $i \in \mathbb{N}_0$ such that $\delta_i < \varepsilon$. By construction of the continuation algorithm, there exist $n \in \mathbb{N}$, $B_{\delta_i}^0, \dots, B_{\delta_i}^n \in Q_{\delta_i}$, $x_j \in B_{\delta_i}^j$, $j = 0, \dots, n$ with $x_0 = x$, $x_n = y$ and $u_0, \dots, u_{n-1} \in \mathcal{U}$

such that $\varphi_\tau(x_j, u_j) \in B_{\delta_i}^{j+1}$, $j = 0, \dots, n-1$ and from $\delta_i < \varepsilon$ we obtain $d(\varphi_\tau(x_j, u_j), x_{j+1}) < \varepsilon$ for all $j = 0, \dots, n-1$. This contradicts that every chain from x to y has a jump bigger than ε , hence $y \in \mathcal{O}_{Q,\tau}^+(x)$. ■

We now can combine the subdivision and continuation algorithms to compute relative chain control sets E_Q . We use the continuation method to construct an appropriate covering of E_Q , and the subdivision technique to approximate the chain control set itself. We end up with the following computation method:

1. Choose $x \in E_Q$ and $\tau, \delta > 0$ small enough,
2. compute $\mathcal{O}_{Q,\tau,\delta}^+(x) \cap \mathcal{O}_{Q,\tau,\delta}^-(x)$ with the continuation algorithm,
3. approximate $E_Q = \text{Viab}^{\mathbb{R}}(\mathcal{O}_{Q,\tau,\delta}^+(x) \cap \mathcal{O}_{Q,\tau,\delta}^-(x))$ with the subdivision algorithm.

5 Example

The controlled full brusselator is given by

$$\begin{aligned}\dot{x} &= 1 + x^2 y - (z + 1)x + u_1(t) \\ \dot{y} &= xz - x^2 y + u_2(t) \\ \dot{z} &= -xz + \alpha + u_3(t).\end{aligned}$$

The uncontrolled system possesses a critical point at $x = 1$, $y = z = \alpha$. For $\alpha > \frac{1}{4}(9 - \sqrt{17}) = 1.21922$ there arises a limit cycle which exists for all values of α up to approximately 1.5, see Hairer/Nørsett/Wanner [7, Section I.16]. Figure 1 shows the control sets of the controlled brusselator for $\alpha = 1.42$. There is a variant control set containing the critical point, and an invariant control set containing the limit cycle. Figure 2 additionally shows the positive domain of attraction of the variant control set.

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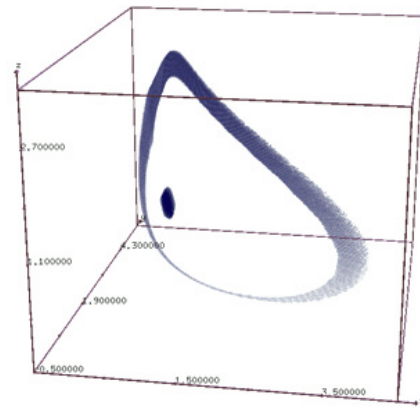


Figure 1: Control sets of the full brusselator

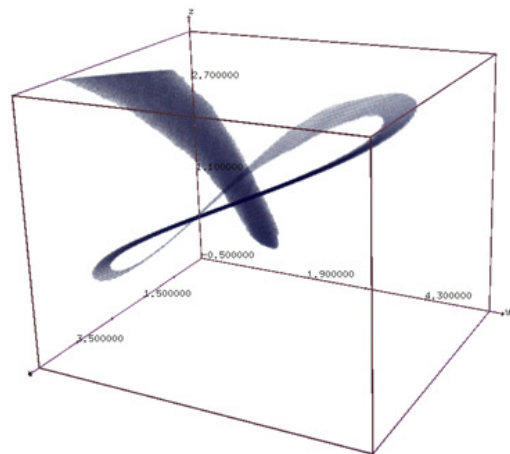


Figure 2: Domain of attraction of the variant control set

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