

Computation of Almost Invariant Sets for Perturbed Systems

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Abstract

Using the relation between the supports of invariant Markov measures and invariant control sets we discuss the characterization of almost invariant sets for Markov diffusion systems.

1 Introduction

It is well known that Markov diffusion systems can be analyzed via an associated nonlinear control system. This relation, based on the support theorem of Stroock and Varadhan, yields information on the supports of the invariant measures and the multistability behavior. In this paper we propose to extend this connection to the analysis of almost invariant sets. These sets cannot be computed directly. Via their relation to control sets the numerical algorithms developed for nonlinear control systems (compare the contribution of D. Szolnoki in these proceedings) can be used to extract information on almost invariant sets for Markov diffusion systems.

More precisely, we work in the following setup. We start from a perturbed system on a smooth manifold M given by

$$\begin{aligned} d\eta &= Y_0(\eta)dt + \sum_{j=1}^{\ell} Y_j(\eta) \circ dW_j, & \eta_0 &= \eta_0^*, \\ \dot{x} &= X_0(x) + \sum_{i=1}^m f_i(\eta_t) X_i(x) \end{aligned} \quad (1)$$

Here the stochastic perturbation η is given by a stochastic differential equation on a C^∞ -manifold N (of finite dimension) where Y_0, Y_1, \dots, Y_ℓ are C^∞ -vector fields on N and ‘ \circ ’ denotes the symmetric (Stratonovich) stochastic differential. We assume that this equation admits at least one stationary Markov solution, see, e.g., [4] and force this solution to be the unique stationary Markovian one by imposing a Lie algebra rank

condition of the form

$$\dim \mathcal{L}\mathcal{A}\{Y_1, \dots, Y_\ell\}(q) = \dim N \text{ for all } q \in N. \quad (2)$$

The unique stationary solution on N is denoted by η_t^* , and we consider this process as a background noise. It is mapped via a surjective function

$$f : N \rightarrow U \quad (3)$$

onto the perturbation space $U \subset \mathbb{R}^m$. Then $\xi_t = f(\eta_t^*)$ is a stationary stochastic process on U . Thus we arrive at the Markov diffusion process on the state space $N \times M$. The associated control system is given by

$$\dot{x} = X_0(x) + \sum_{i=1}^m u(t) X_i(x), \quad (4)$$

$$u \in \mathcal{U} = \{u : \mathbb{R} \rightarrow \mathbb{R}^m, u(t) \in U \text{ for } t \in \mathbb{R}\}. \quad (5)$$

The trajectories of (4) are denoted by $\varphi(t, x, u)$, $t \in \mathbb{R}$. The behavior of the system (1) can now be studied using imbedding of the stationary process η_t^* into the flow for (4) and then exploiting connections with control theory via the support theorem of Stroock and Varadhan [5]. It turns out that it is sufficient for the global analysis of the Markov diffusion model to understand the invariant control sets of (4) and the corresponding multistability regions. Precise statements are given in [1], [2]. The basic relation is that for every compact invariant control set C there is a unique Markov measure on $N \times M$ such that the projection of its support to M coincides with C .

Of course, supports of invariant Markov measures are not the only invariant sets of the system (1): Adding, e.g., to an invariant control set C its strict domain of attraction

$$\mathbf{A}^{strict}(C) = \left\{ x \in M, \begin{array}{l} \text{for all } u \in \mathcal{U} \text{ there is } t(x, u) \\ \text{with } \varphi(t(x, u), x, u) \in C \end{array} \right\}$$

yields the stochastically invariant set $\mathbf{A}^{strict}(C) \supset C$. But invariant control sets are at the core of any invariant set of the stochastic system (1).

2 Almost Invariant Sets

We now consider almost invariant sets in the sense that the system exits these sets only after a very long time. (Concepts for sets that the process exits with small probability are discussed in [1].) In order to discuss these subsets of the state space M we consider systems with varying range of the perturbation. More precisely, we replace in equation (1) the perturbed system by

$$\dot{x} = X_0(x) + \sum_{i=1}^m \rho f_i(\eta_i) X_i(x)$$

where $\rho \geq 0$, i.e., f is replaced by ρf , and we assume that $0 \in U$. Then the associated control system is given by

$$\dot{x} = X_0(x) + \sum_{i=1}^m u(t) X_i(x), \quad u(t) \in \rho U.$$

Suppose that for $\rho = \rho_0$ one has an invariant control set $C(\rho_0)$. There exists a unique invariant density h with support equal to $C(\rho_0)$ of the associated Markov measure. Numerical simulations show that the values of h are extremely small in a wide region near the boundary of C . If we increase ρ then there are still control sets for the associated control system. However, at some ρ -value, say for $\rho > \rho_1 > \rho_0$, the invariant control set may lose its invariance. Then the associated stochastic system has no invariant measure with support on $C(\rho)$. However, the time to leave $C(\rho)$ will be extremely large for $\rho > \rho_1$ with $\rho - \rho_1 > 0$ small. Thus the largest ρ -value where $C(\rho)$ is invariant determines a subset of the state space which is 'almost invariant' for $\rho > \rho_1$. Hence in order to understand the behavior of the stochastic system we propose to study the loss of invariance for control sets. This should be done not only for the simple parameter dependence given by the factor ρ which increases the range of the perturbations but for arbitrary parameters α occurring in the system. One arrives at the following parameter dependent system

$$\dot{x} = X_0(\alpha, x) + \sum_{i=1}^m u(t) X_i(\alpha, x), \quad u(t) \in U. \quad (6)$$

where $\alpha \in A \subset \mathbb{R}^k$.

First we characterize the situation where the invariance is preserved under small perturbations. The associated control flow is given by

$$\Phi_t : \mathcal{U} \times M \rightarrow \mathcal{U} \times M, \quad \Phi_t(u, x) = (u(t + \cdot), \varphi(t, x, u))$$

and its chain recurrent components are relevant (cp. [3]). The projections of the chain recurrent components onto M are chain control sets, which are ordered via their approximate controllability properties.

Theorem 1 *Let C^{α_0} be a compact invariant control set of $(6)^{\alpha_0}$ with $\alpha_0 \in A$, and assume that the system $(6)^{\alpha_0}$ satisfies a Lie algebra rank condition (the accessibility rank condition) on C^{α_0} . Assume that C^{α_0} is a maximal chain control set, $E^{\alpha_0} = C^{\alpha_0}$. Then for all α in a neighborhood of α_0 there are unique control sets C^α depending continuously in the Hausdorff metric on α at $\alpha = \alpha^0$ with $C^\alpha \cap \text{int } C^{\alpha_0} \neq \emptyset$ and*

$$\lim_{\alpha \rightarrow \alpha_0} C^\alpha = C^{\alpha_0} = E^{\alpha_0}.$$

Furthermore, suppose that every chain recurrent component of $\dot{x} = X_0(\alpha_0, x)$ has void intersection with ∂C^{α_0} . Then the control sets C^α of $(6)^\alpha$ are invariant for α close to α_0 .

Theorem 1 says that, under standard accessibility and compactness assumptions, invariant control sets which are maximal chain control sets maintain their invariance under small perturbations of the system vector fields. Hence control sets can lose their invariance only when they become part of a larger or of a non-maximal chain control set. While a general characterization of these scenarios seems out of reach at this moment, there are typical scenarios which we observe in a number of numerical examples. In the situation where the parameter is the range ρ of the perturbations, the following holds: Define the invariant domain of attraction $\mathbf{A}^{inv}(C)$ of an invariant control set C as the set of points x for which C is the only invariant control set that can be reached from x .

Theorem 2 *Let an inner pair condition be satisfied and consider an invariant control set $C(\rho_0)$ for ρ_0 . Suppose that for an increasing control set family $C(\rho)$ $\rho > \rho_0$, containing $C(\rho_0)$ one has for some $\rho_1 > \rho_0$ that for all $\rho \in [\rho_0, \rho_1]$ the set $C(\rho)$ is an invariant control set and*

$$C(\rho_1) \cap \partial \mathbf{A}^{inv}(C(\rho_1)) \neq \emptyset.$$

Assume that the chain control set $E(\rho_1)$ containing $C(\rho_1)$ does not contain another invariant control set. Then for $\rho > \rho_1$ with $\rho - \rho_1$ small, the control set $C(\rho)$ is not invariant, the map $\rho \mapsto C(\rho)$ is discontinuous (in the Hausdorff topology) at $\rho = \rho_1$ and $C(\rho_1) \neq E(\rho_1)$. Furthermore, there is a control set $D \neq C(\rho_1)$ for the control range $\rho_1 U$ such that for $\rho > \rho_1$ the control sets $C(\rho)$ satisfy

$$C(\rho) \supset C(\rho_1) \cup D.$$

Thus the loss of invariance of a control set C is related to the collision of C with another control set. This has as an immediate corollary a result on associated Markov diffusion processes: At ρ_1 the invariant measure supported by $C(\rho_1)$ vanishes. Instead a subset survives which can be left only after long time (tending to infinity as ρ approaches ρ_1). Computation of the resulting control sets with continuation/subdivision techniques as presented in the paper by D. Szolnoki at the same time computes sets that are almost invariant in this sense. This scenario plays an important role in dynamic reliability theory where one is interested in the statistics of failure times, i.e., of exit times from safety regions.

References

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