Nonnegative linear systems in the behavioral approach: the autonomous case

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Abstract

Nonnegative linear systems, which have been traditionally investigated within the state-space framework, are here introduced and analyzed by means of the behavioral approach. Starting from certain definitions and results presented in [9], we have explored the general autonomous case, by deriving an extended set of necessary and sufficient conditions for an autonomous behavior to be nonnegative.

1 Introduction

Since the early seventies positive linear systems have attracted the interest of several researchers. Even though many issues have been addressed and solved in this context, most of the attention has been devoted to two specific problems: positive controllability/reachability [2, 3, 4, 13] and positive realization [1, 5, 8]. Concrete applications of these apparently abstract problems can be found in various fields, like econometric, bioengineering, chemistry and, generally speaking, in every context where the variables involved are intrinsically nonnegative.

In the last decade, the behavioral approach to dynamic systems [11] has received an increasingly broader acceptance, within the control community, as a natural framework where most of the results of classic (linear time-invariant) system theory can be recasted and further extended. Quite recently, there has been an attempt to develop, within this framework, a general theory of positive linear system. In a very nice paper [10], Nieuwenhuis has first introduced the notion of nonnegative discrete behavior (whose trajectories are defined on the time axis $\mathbb{Z}_+$), based on the notion of most powerful unfalsified behavior [6, 17], and later given some preliminary results, mostly concerned with behaviors which are one dimensional (namely with trajectories in $(\mathbb{R})^{\mathbb{Z}_+}$) and autonomous, or two-dimensional (with trajectories in $(\mathbb{R}^2)^{\mathbb{Z}_+}$) and controllable.

The aim of this contribution is that of extending some of these results to the general autonomous case, and provide an exhaustive characterization of nonnegativity for autonomous behaviors. To this end, we will introduce some new entities, like the "positive part" of a behavior (namely the set of all nonnegative trajectories belonging to the behavior) and the set of initial conditions, to which correspond, by means of a minimal realization, all the nonnegative trajectories in the behavior.

The paper is organized as follows: section 2 summarizes up the basic definitions and results about (linear, left shift-invariant, complete) behaviors whose trajectories are defined on $\mathbb{Z}_+$ and take values in $\mathbb{R}^q$. Also, the basic definitions which are necessary in order to introduce positive behaviors are recalled. In section 3, the nonnegativity property for autonomous systems, in the behavioral approach, is investigated, and necessary and sufficient conditions for an autonomous behavior to be nonnegative are derived.

Throughout the paper we let $\mathbb{R}^n_+$ denote the nonnegative orthant, namely the set of nonnegative vectors in the n-dimensional Euclidean space $\mathbb{R}^n$. A set $\mathcal{K} \subset \mathbb{R}^n$ is said to be a *cone* if all finite nonnegative linear combinations of elements of $\mathcal{K}$ belong to $\mathcal{K}$. A cone $\mathcal{K}$ is *convex* if it contains, with any two points, the line segment between them, namely $\alpha \mathbf{v}_1 + (1-\alpha)\mathbf{v}_2 \in \mathcal{K}$, for every $\alpha \in [0,1]$ and every pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{K}$. A convex cone $\mathcal{K}$ is *solid* if it contains an open set (a ball) of $\mathbb{R}^n$, and it is *pointed* if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$. A closed, pointed, solid convex cone is called a *proper cone*. A cone $\mathcal{K}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This amounts to saying that a positive integer r and r vectors in $\mathbb{R}^n$, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$, can be found, such that $\mathcal{K}$ coincides with the set of nonnegative combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$. In this case, we adopt the notation $\mathcal{K} := \text{Cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r)$. The extension of this notation to the case when the vectors $\mathbf{v}_i$ are replaced by matrices is immediate. Also, the extensions of the previous definitions to an arbitrary real vector space, on which it has been introduced some topology, are straightforward.

If A is an $n \times n$ real matrix, we denote by $\sigma(A)$ its *spectrum* and by $\rho(A)$ its *spectral radius*, i.e. $\rho(A) :=$

$\max\{|\lambda| : \lambda \in \sigma(A)\}$. For every $\lambda \in \sigma(A)$, the *degree* of λ in A, $\deg \lambda$, is the size of the largest diagonal block in the Jordan canonical form of A which contains λ (i.e., the multiplicity of λ as a zero of the minimal polynomial of A).

Given $A \in \mathbb{R}^{n\times n}$ and a cone $\mathcal{K} \subseteq \mathbb{R}^n$, we say that A *leaves* $\mathcal{K}$ *invariant* ($\mathcal{K}$ is A-invariant) if $A\mathcal{K} \subseteq \mathcal{K}$. If $A = [a_{ij}]$ is a matrix (in particular, a vector), we write

- $A \geq 0$ (A *nonnegative*), if $a_{ij} \geq 0$ for all i, j;
- $A > 0$ (A *nonzero nonnegative*), if $a_{ij} \geq 0$ for all i, j, and $a_{hk} > 0$ for at least one pair (h, k);
- $A \gg 0$ (A *positive*), if $a_{ij} > 0$ for all i, j.

In the paper, all (discrete) sequences will be defined on the set $\mathbb{Z}_+$ of nonnegative integers. The right (forward) and the left (backward) shift operators on $(\mathbb{R}^q)^{\mathbb{Z}_+}$, the set of sequences defined on $\mathbb{Z}_+$ and taking values in $\mathbb{R}^q$, are defined as

$$\begin{aligned} \tau \ &: \ (\mathbb{R}^q)^{\mathbb{Z}_+} \rightarrow (\mathbb{R}^q)^{\mathbb{Z}_+} \\ &: \ (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \cdots) \mapsto (0, \mathbf{v}_0, \mathbf{v}_1, \cdots) \end{aligned}$$

$$\begin{aligned} \sigma \ &: \ (\mathbb{R}^q)^{\mathbb{Z}_+} \rightarrow (\mathbb{R}^q)^{\mathbb{Z}_+} \\ &: \ (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \cdots) \mapsto (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots). \end{aligned}$$

As we will deal with sets of sequences (our *behaviors*) which are left shift-invariant, we can restrict our attention to the left shift operator σ. For every positive integer i, the ith power of σ is naturally defined by composition as $\sigma^i = \sigma \circ \sigma \circ \ldots \circ \sigma$ (i times).

Also, we can further extend the set of shift operators. Indeed, to every polynomial matrix $R(z) = \sum_{i=0}^{L} R_i z^i \in \mathbb{R}[z]^{p\times q}$ we can associate the polynomial matrix operator $R(\sigma) = \sum_{i=0}^{L} R_i \sigma^i$ (from $(\mathbb{R}^q)^{\mathbb{Z}_+}$ to $(\mathbb{R}^p)^{\mathbb{Z}_+}$), mapping every sequence $\{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ into the sequence $\{R(\sigma)\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$, where $R(\sigma)\mathbf{w}(t) = R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \ldots + R_L\mathbf{w}(t+L)$, for every $t \in \mathbb{Z}_+$. It can be proved that $R(\sigma)$ describes an injective map if and only if R is a right prime matrix, and a surjective map if and only if R is of full row rank.

2 Identifiability issues and nonnegativity property for a complete behavior

Before proceeding, it is convenient to briefly summarize some basic definitions and results about behaviors whose trajectories have support in $\mathbb{Z}_+$. Further details on the subject can be found in [10, 12, 14].

In this paper, by a *dynamic system* we mean a triple $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, \mathfrak{B})$, where $\mathbb{Z}_+$ represents the *time set*, $\mathbb{R}^q$ is the *signal alphabet*, namely the set where the system trajectories take values, and $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is the *behavior*, namely the set of trajectories which are compatible with the system laws. A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be *linear* if it is a vector subspace (over $\mathbb{R}$) of $(\mathbb{R}^q)^{\mathbb{Z}_+}$, and *left shift-invariant* if $\sigma\mathfrak{B} \subseteq \mathfrak{B}$. A linear left shift-invariant behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is *complete* if for every sequence $\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, the condition $\mathbf{w}|_{\mathcal{S}} \in \mathfrak{B}|_{\mathcal{S}}$ for every finite set $\mathcal{S} \subset \mathbb{Z}_+$ implies $\mathbf{w} \in \mathfrak{B}$, where $\mathbf{w}|_{\mathcal{S}}$ denotes the restriction to $\mathcal{S}$ of the trajectory $\mathbf{w}$ and $\mathfrak{B}|_{\mathcal{S}}$ the set of all restrictions to $\mathcal{S}$ of behavior trajectories.

Linear left shift-invariant complete behaviors are kernels of polynomial matrices in the left shift operator σ, which amounts to saying that the trajectories $\mathbf{w} = \{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ of $\mathfrak{B}$ can be identified with the set of solutions in $(\mathbb{R}^q)^{\mathbb{Z}_+}$ of a system of difference equations

$$R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \cdots + R_L\mathbf{w}(t+L) = 0, \ t \in \mathbb{Z}_+,$$

with $R_i \in \mathbb{R}^{p\times q}$, and hence described by the equation

$$R(\sigma)\mathbf{w} = 0, \tag{1}$$

where $R(z) := \sum_{i=0}^{L} R_i z^i$ belongs to $\mathbb{R}[z]^{p\times q}$. In the sequel, a behavior $\mathfrak{B}$ described as in (1) will be denoted, for short, as $\mathfrak{B} = \ker(R(\sigma))$. Also, we will restrict our attention to linear, left shift-invariant and complete behaviors $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$, and refer to them simply as *behaviors*.

Definition 1 *A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be* autonomous *if there exists $m \in \mathbb{Z}_+$ such that if $\mathbf{w}^1, \mathbf{w}^2 \in \mathfrak{B}$ and $\mathbf{w}^1|_{[0,m]} = \mathbf{w}^2|_{[0,m]}$, then $\mathbf{w}^1 = \mathbf{w}^2$.*

As is well-known, a behavior $\mathfrak{B} = \ker(R(\sigma))$, with $R \in \mathbb{R}[z]^{p\times q}$, is autonomous if and only if it is a finite dimensional vector subspace of $(\mathbb{R}^q)^{\mathbb{Z}_+}$, or, equivalently, if and only if R has full column rank q [14].

We now address certain identifiability issues which are fundamental in order to introduce the notion of positive behavior. Such concepts are only marginally touched upon here. For further details we refer the interested reader to [6, 10].

Definition 2 *Let $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$ be m trajectories in $(\mathbb{R}^q)^{\mathbb{Z}_+}$. A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be* the most powerful unfalsified behavior (MPUB) explaining *$\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$, if*

- *$\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$ belong to $\mathfrak{B}$, and*
- *for any other behavior $\bar{\mathfrak{B}}$ having $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$ among its trajectories, we have $\mathfrak{B} \subseteq \bar{\mathfrak{B}}$.*

For every choice of the trajectories $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$, the most powerful unfalsified behavior explaining $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$, denoted by $\mathfrak{B}(\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m)$, exists and represents the smallest (linear left shift-invariant and complete) behavior including $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$.

A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be *identifiable* if there exists a finite number of its trajectories, say $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$, such that $\mathfrak{B} \equiv \mathfrak{B}(\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m)$. Under the linearity, left shift-invariance and completeness assumptions we steadily adopt, every behavior is, indeed, identifiable.

By resorting to the notion of identifiability, Nieuwenhuis proposed in [10] the following definition of nonnegative behavior.

Definition 3 *A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be* nonnegative *if there exist $m \in \mathbb{N}$ and nonnegative trajectories $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$ such that $\mathfrak{B} \equiv \mathfrak{B}(\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m)$.*

3 Nonnegative autonomous behaviors

Among the basic properties an autonomous behavior is endowed with, a major role will be played in this paper by the fact that every autonomous behavior can be "realized" by means of a state-space model [16]. Indeed, if $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is an autonomous behavior, then there exist $n \in \mathbb{N}$ and real matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{q\times n}$ such that

$$\mathfrak{B} = \{\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}_+} : \exists\, \mathbf{x}(0) \text{ s.t. } \mathbf{x}(t+1) = A\mathbf{x}(t), \mathbf{w}(t) = C\mathbf{x}(t),\ t \in \mathbb{Z}_+\}.$$

The pair (A, C) is an *n-dimensional realization of* $\mathfrak{B}$. Those realizations of $\mathfrak{B}$ for which n is minimal are called *minimal*.

By referring to a minimal realization of $\mathfrak{B}$, we can provide an efficient characterizations of those finite sets of trajectories, say $\{\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m\}$, which allow to uniquely identify the behavior $\mathfrak{B}$, by this meaning that $\mathfrak{B}$ is the MPUB explaining such sets.

Lemma 1 [15] *Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ be an autonomous behavior, and let (A, C) be an n-dimensional and minimal realization of $\mathfrak{B}$. Let $\mathbf{w}^i$, $i = 1, 2, \ldots, m$, denote the trajectory of $\mathfrak{B}$ obtained corresponding to the initial condition $\mathbf{x}(0) = \mathbf{x}_0^i$, and set $X_0 := [\,\mathbf{x}_0^1 \mid \mathbf{x}_0^2 \mid \ldots \mid \mathbf{x}_0^m\,]$. The following facts are equivalent:*

i) $\mathfrak{B} = \mathfrak{B}(\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m)$;

ii) (A, X_0) is a reachable pair.

The technical result of Lemma 1 provides the main building block in the construction of the following set of characterizations of nonnegative autonomous behaviors.

Theorem 2 *Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ be an autonomous behavior, and let (A, C) be an n-dimensional and minimal realization of $\mathfrak{B}$. The following facts are equivalent:*

1) $\mathfrak{B}$ is a nonnegative behavior;

2) there exists a positive integer m and some matrix $X_0 \in \mathbb{R}^{n\times m}$ such that

2a) (A, X_0) is a reachable pair, and

2b) $CA^tX_0 \geq 0$ for every $t \geq 0$;

3) there exists a positive integer m and some matrix $B \in \mathbb{R}^{n\times m}$ such that

3a) (A, B, C) is a minimal realization of its transfer matrix $W(z) := C(zI_n - A)^{-1}B$, and

3b) the Markov coefficients of $W(z)$, i.e. the coefficients W_t of the power series expansion $\sum_{t\geq 0} W_t\, z^{-t}$ of $W(z)$, are all nonnegative matrices;

4) there exists a proper A-invariant cone $\mathcal{K} \subset \mathbb{R}^n$ such that

$$\mathcal{K} \subseteq \{\mathbf{x} \in \mathbb{R}^n : CA^t\mathbf{x} \geq 0,\ \forall\, t \geq 0\}. \qquad (2)$$

Proof 1) $\Leftrightarrow$ 2) $\mathfrak{B}$ is a nonnegative behavior if and only if there exist trajectories, say $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m$, with nonnegative coefficients, such that $\mathfrak{B} = \mathfrak{B}(\mathbf{w}^1, \ldots, \mathbf{w}^m)$. By the previous lemma, this is equivalent to saying that m vectors $\mathbf{x}_0^i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$, can be found, such that $\mathbf{w}^i(t) = CA^t\mathbf{x}_0^i \geq 0$, for every $t \in \mathbb{Z}_+$ and every $i \in \{1, 2, \ldots, m\}$, and (A, X_0) is a reachable pair, where $X_0 := [\,\mathbf{x}_0^1 \mid \mathbf{x}_0^2 \mid \ldots \mid \mathbf{x}_0^m\,]$. This last statement is, in turn, equivalent to 2a)÷2b).

2) $\Leftrightarrow$ 3) As (A, C) is a minimal realization for the autonomous behavior $\mathfrak{B}$, and hence is an observable pair, the equivalence of 2) and 3) is straightforward.

2) $\Rightarrow$ 4) Set $\mathcal{K} := \overline{\text{Cone}}(X_0, AX_0, A^2X_0, \ldots)$, which is the topological closure (w.r.t. the topology of pointwise convergence) of the cone generated by the columns of $X_0, AX_0, A^2X_0, \ldots$. Of course, $\mathcal{K}$ is, by definition, a closed, convex, A-invariant cone. Moreover, 2a) ensures that $\mathcal{K}$ is solid.

To conclude the proof it will be enough to prove (2). Indeed, a cone $\mathcal{K}$ that satisfies (2) is necessarily pointed and hence, putting together this property with those previously remarked, proper. Observe that if $\mathbf{v}$ is a finite (nonnegative) combination of the columns of $\mathcal{K}$ then $\mathbf{v} = \sum_{i\in I} A^iX_0\mathbf{c}_i$, for some finite set I and some nonnegative vectors $\mathbf{c}_i \in \mathbb{R}_+^m$. As a consequence, by the assumption 2b),

$$CA^t\mathbf{v} = CA^t\left(\sum_{i\in I} A^i X_0 \mathbf{c}_i\right) = \sum_{i\in I} CA^{t+i} X_0 \mathbf{c}_i \geq 0.$$

This proves that $\mathbf{v} \in \left\{\mathbf{x} \in \mathbb{R}^n : CA^t\mathbf{x} \geq 0,\ \forall\, t \geq 0\right\}$. On the other hand, if $\mathbf{v}$ is in $\mathcal{K}$ and it cannot be expressed as a finite (nonnegative) combination of the columns of $\mathcal{K}$, then $\mathbf{v} = \lim_{n\to+\infty} \mathbf{v}_n$, for some sequence of vectors $\{\mathbf{v}_n\}_{n\geq 0}$, each of them being a finite (nonnegative) combination of the columns of $\mathcal{K}$. So, for any $t \in \mathbb{Z}_+$, the fact that condition $CA^t\mathbf{v}_n \geq 0$ holds for every n ensures that $CA^t\mathbf{x} \geq 0$. Therefore $\mathcal{K} \subseteq \left\{\mathbf{x} \in \mathbb{R}^n : CA^t\mathbf{x} \geq 0,\ \forall\, t \geq 0\right\}$.

4) $\Rightarrow$ 2) As $\mathcal{K}$ is a solid cone, n linearly independent vectors, say $\mathbf{x}_0^1, \mathbf{x}_0^2, \dots, \mathbf{x}_0^n$, can be found in $\mathcal{K}$. Of course, $X_0 := [\,\mathbf{x}_0^1 \mid \mathbf{x}_0^2 \mid \dots \mid \mathbf{x}_0^n\,]$ is nonsingular square, and hence, of course, (A, X_0) is a reachable pair. Finally, (2) ensures that $CA^tX_0 \geq 0$ for every $t \geq 0$, and hence also 2b) holds true. ■

The above characterization leads the way to further insights into the meaning of nonnegativity for autonomous behaviors. It is well-known that an autonomous behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is a finite-dimensional vector subspace of $(\mathbb{R}^q)^{\mathbb{Z}_+}$, whose dimension coincides with the dimension of a minimal realization of $\mathfrak{B}$. In fact, it is immediately seen that, under the minimality (namely observability) assumption on the pair (A, C), there exists a bijective correspondence between the set of behavior trajectories and the (vector) space $\mathbb{R}^n$ of initial conditions $\mathbf{x}(0)$. We aim, now, to introduce the *positive part* of a behavior $\mathfrak{B}$, namely the set $\mathfrak{B}_+$ of nonnegative trajectories of the behavior, and relate the nonnegativity of $\mathfrak{B}$ to the algebraic properties of its positive part and to the set $\mathcal{X}_+$ of initial conditions corresponding (by means of (A, C)) to the trajectories of $\mathfrak{B}_+$.

Definition 4 *Given a behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$, we call the* positive part *of $\mathfrak{B}$, and denote it by $\mathfrak{B}_+$, the set of all nonnegative trajectories in $\mathfrak{B}$, namely*

$$\mathfrak{B}_+ := \mathfrak{B} \cap (\mathbb{R}_+^q)^{\mathbb{Z}_+}. \tag{3}$$

If, in addition, $\mathfrak{B}$ is autonomous and (A, C) is an n-dimensional realization of $\mathfrak{B}$, we denote by $\mathcal{X}_+$ the set of all initial conditions that generate, by means of the realization (A, C), all the trajectories in $\mathfrak{B}_+$, i.e.

$$\mathcal{X}_+ := \left\{\mathbf{x}(0) \in \mathbb{R}^n : CA^t\mathbf{x}(0) \geq 0,\ \forall\, t \in \mathbb{Z}_+\right\}.$$

The positive part of a behavior, $\mathfrak{B}_+$ and the set $\mathcal{X}_+$ admit interesting algebraic characterizations, whose proofs can be obtained by directly verifying the definitions of the properties involved.

Proposition 3 *Given a behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$, its positive part, $\mathfrak{B}_+$, is a convex and pointed cone in $(\mathbb{R}^q)^{\mathbb{Z}_+}$, and it is closed (w.r.t. to the topology of the pointwise convergence). Furthermore, if $\mathfrak{B}$ is autonomous and (A, C) is an n-dimensional realization of $\mathfrak{B}$, $\mathcal{X}_+$ is, in turn, a convex, pointed and closed cone in $\mathbb{R}^n$.*

We have the following characterization.

Theorem 4 *Given an autonomous behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$, let $\mathfrak{B}_+$ be its positive part and let (A, C) be an n-dimensional and minimal realization of $\mathfrak{B}$. The following facts are equivalent:*

i) $\mathfrak{B}$ is a nonnegative behavior;

ii) $\mathfrak{B}$ is the smallest (linear, left shift-invariant and complete) behavior having $\mathfrak{B}_+$ as its positive part;

iii) $\mathfrak{B}_+$ generates an n-dimensional real vector space in $(\mathbb{R}^q)^{\mathbb{Z}_+}$;

iv) the set $\mathcal{X}_+$ of initial conditions to which correspond, by means of the realization (A, C), the nonnegative behavior trajectories is a proper cone.

Proof i) $\Rightarrow$ ii) Suppose, by contradiction, that there exists a behavior $\mathfrak{B}_1$, properly included in $\mathfrak{B}$, having $\mathfrak{B}_+$ as positive part. Then, of course, for every m and every choice of m nonnegative trajectories $\mathbf{w}^i$, $i = 1, 2, \dots, m$, the behavior $\mathfrak{B}(\mathbf{w}^1, \dots, \mathbf{w}^m)$ would be included in the smallest behavior having $\mathfrak{B}_+$ as its positive part, and hence $\mathfrak{B}(\mathbf{w}^1, \dots, \mathbf{w}^m) \subseteq \mathfrak{B}_1 \subset \mathfrak{B}$. Thus, $\mathfrak{B}$ could not be nonnegative.

ii) $\Rightarrow$ iii) Assume, by contradiction, that the vector space, $\mathrm{span}(\mathfrak{B}_+)$, generated (over $\mathbb{R}$) by the trajectories of $\mathfrak{B}_+$ (equivalently, the smallest vector space including $\mathfrak{B}_+$) has dimension $d < n$. It is not hard to see that $\mathrm{span}(\mathfrak{B}_+)$ is, in turn, an autonomous behavior having $\mathfrak{B}_+$ as positive part. Since its dimension is d, it cannot coincide with $\mathfrak{B}$, and hence $\mathfrak{B}$ is not the smallest behavior having $\mathfrak{B}_+$ as its positive part.

iii) $\Rightarrow$ i) If $\mathfrak{B}_+$ generates an n-dimensional vector space, it follows that there exist n linearly independent trajectories in $\mathfrak{B}_+$. If we denote by $\mathbf{x}_0^i$, $i = 1, 2, \dots, n$, the corresponding initial conditions, the n-dimensional square matrix $X_0 := [\mathbf{x}_0^1\ \mathbf{x}_0^2\ \dots\ \mathbf{x}_0^n]$ makes 2a) and 2b) in Theorem 2 satisfied. Consequently, $\mathfrak{B}$ is nonnegative.

iii) $\Leftrightarrow$ iv) It is easily seen that $\mathrm{span}(\mathfrak{B}_+)$ coincides with the set of behavior trajectories generated by the state-space model

$$\mathbf{x}(t+1) = A\mathbf{x}(t) \qquad \mathbf{w}(t) = C\mathbf{x}(t), \qquad t \in \mathbb{Z}_+,$$

corresponding to initial conditions, $\mathbf{x}(0)$, belonging to the vector space span$(\mathcal{X}_+)$. Therefore, $\mathfrak{B}_+$ generates an n-dimensional vector space if and only if span$(\mathcal{X}_+) = \mathbb{R}^n$, namely $\mathcal{X}_+$ is a solid, and hence proper, cone. ■

REMARKS Given an autonomous behavior $\mathfrak{B}$, the trajectories of $\mathfrak{B}_+$ are biuniquely related (by means of a nonnegative realization) to the convex, pointed and closed cone $\mathcal{X}_+$ in $\mathbb{R}^n$. Thus the nonnegativity of $\mathfrak{B}$ corresponds to the fact that such a cone $\mathcal{X}_+$ is a solid one, or, in a sense, is "large" in $\mathbb{R}^n$, just as the nonnegativity of $\mathfrak{B}$ means that the set $\mathfrak{B}_+$ is rich enough to carry on all the information about $\mathfrak{B}$. We want to better understand this fact by means of a pair of examples.

EXAMPLE 1 Consider the scalar autonomous behavior $\mathfrak{B} = \ker\ (r(\sigma))$, where $r(z) = (z-2)(z-1) = z^2 - 3z + 2$. The trajectories of $\mathfrak{B}$ are, of course, the (real-valued) sequences $w = \{w(t)\}_{t\in\mathbb{Z}_+}$ satisfying $w(t+2) = 3w(t+1) - 2w(t)$ for every $t \in \mathbb{Z}_+$. It is easily seen that every trajectory in $\mathfrak{B}$ is uniquely determined by its values for $t = 0$ and $t = 1$, and hence there is a bijective correspondence between trajectories of $\mathfrak{B}$ and points of $\mathbb{R}^2$. It is also easy to verify that the trajectory sample at time t can be expressed as $w(t) = n_t(w(1) - w(0)) + w(1)$, where $\{n_t\}_{t\in\mathbb{Z}_+}$ is a diverging sequence, as t goes to $+\infty$. This ensures that w is a nonnegative behavior trajectory if and only if $w(0) \geq 0$, $w(1) \geq 0$ and $w(1) \geq w(0)$. So, the set of initial values of $w(t)$, for $t = 0, 1$, to which corresponds a nonnegative behavior trajectory is the set

$$\left\{(w(0), w(1)) \in \mathbb{R}\times\mathbb{R} : w(1) \geq w(0) \geq 0\right\},$$

and the set $\mathfrak{B}_+$ generates a two-dimensional vector space. A minimal realization of $\mathfrak{B}$ is given, for instance, by the pair

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \qquad C = [\,1 \quad 0\,].$$

The set $\{\mathbf{x} : CA^t\mathbf{x} \geq 0,\ \forall t \in \mathbb{Z}_+\}$ coincides with $\mathbb{R}^2_+$, and hence a proper A-invariant cone $\mathcal{K} \subseteq \mathbb{R}^2_+$ is the (polyhedral) cone

$$\mathcal{K} := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_2 \geq x_1 \geq 0 \right\},$$

generated by the two vectors $[1\ 1]^T$ and $[0\ 1]^T$.

Indeed, $\mathfrak{B}$ is a nonnegative behavior as $\mathfrak{B} = \mathfrak{B}(w^1)$, where w^1 is the nonnegative sequence defined as $w^1(t) = 2^{t+1} - 1$, for $t \in \mathbb{Z}_+$.

EXAMPLE 2 Consider the scalar autonomous behavior $\mathfrak{B} = \ker\ (r(\sigma))$, where $r(z) = (z+2)(z-1) = z^2 + z - 2$. The trajectories of $\mathfrak{B}$ are the (real-valued) sequences $w = \{w(t)\}_{t\in\mathbb{Z}_+}$ satisfying $w(t+2) = -w(t+1) + 2w(t)$ for every $t \in \mathbb{Z}_+$. Again, the behavior trajectories are uniquely determined by their values for $t = 0$ and $t = 1$, and the trajectory sample at time t can be expressed

- for t even, as $w(t) = n_t(w(1) - w(0)) + w(1)$, where $\{n_t\}_{t\in\mathbb{Z}_+}$ is a diverging sequence, as t goes to $+\infty$;
- for t odd, as $w(t) = m_t(w(0) - w(1)) + w(0)$, where $\{m_t\}_{t\in\mathbb{Z}_+}$ is a diverging sequence, as t goes to $+\infty$.

This, therefore, implies that w is a nonnegative behavior trajectory if and only if $w(0) = w(1) \geq 0$. So, the set of initial conditions to which corresponds a nonnegative behavior trajectory is the set

$$\left\{(w(0), w(1)) \in \mathbb{R}\times\mathbb{R} : w(1) = w(0) \geq 0\right\},$$

and the set $\mathfrak{B}_+$ generates a one-dimensional vector space. A minimal realization of $\mathfrak{B}$ is given by

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \qquad C = [\,1 \quad 0\,].$$

Again, we have that $\{\mathbf{x} : CA^t\mathbf{x} \geq 0,\ \forall t \in \mathbb{Z}_+\} = \mathbb{R}^2_+$, and it is not hard to prove that no proper A-invariant cone $\mathcal{K} \subseteq \mathbb{R}^2_+$ exists. Consequently, $\mathfrak{B}$ is not a nonnegative behavior.

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