# Probabilistic Robust Design with Linear Quadratic Regulators

B. T. Polyak* and R. Tempo**

*Institute for Control Science
Russian Academy of Sciences
Profsojuznaja 65, Moscow 117806
Russia
boris@ipu.rssi.ru

**IRITI-CNR
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy
tempo@polito.it

### Abstract

In this paper, we study robust design of uncertain systems in a probabilistic setting by means of Linear Quadratic Regulators. We consider systems affected by random bounded nonlinear uncertainty so that classical optimization methods based on Linear Matrix Inequalities cannot be used without conservatism.

The approach followed here is a blend of *randomization* techniques for the uncertainty together with *convex optimization* for the controller parameters. In particular, we propose an iterative algorithm for designing a controller which is based upon subgradient iterations. At each step of the sequence, we first generate a random sample and then we make a subgradient step for a convex constraint defined by the LQR problem.

The main result of the paper is to prove that this iterative algorithm provides a controller which quadratically stabilizes the uncertain system with probability one in a finite number of steps. In addition, at a fixed step, we compute a lower bound of the probability that a quadratically stabilizing controller is found.

## 1 Introduction

In recent years, we have seen a growing interest in combining robust and probabilistic approaches for analysis and design of control systems. The main idea is to use worst-case uncertainty bounds together with probabilistic information. That is, the uncertainty $\Delta$ entering into the system is bounded within a set $\boldsymbol{\Delta}$ and a certain probability distribution on $\Delta$, for example uniform, is given. In this context, several problems have been already studied and solved, including the computation of bounds on the sample size [8, 12, 21], the connections between randomized algorithms and Learning Theory [13, 23], the generation of samples in various sets [6] and other probabilistic-based problems [3, 18]. In this paper, using the standard setting of Linear Quadratic Regulators, we develop a sequential algorithm, based on subgradient iterations, for designing a controller which quadratically stabilizes the uncertain system with probability one in a finite number of steps. At each step of the sequence, the algorithm is based upon two stages:

- Generation of a random sample $\Delta^k$ of the uncertainty $\Delta \in \boldsymbol{\Delta}$.
- Subgradient step for a convex constraint defined by $\Delta^k$.

The first stage depends on the specific uncertainty structure under attention and can be performed using for example the randomization techniques described in [6]. The outcome of this stage is the generation of a random sample $\Delta^k \in \boldsymbol{\Delta}$. The second stage requires an approximate solution of a convex constraint $V(Q, \Delta^k) \leq 0$ defined by the LQR problem, where $Q$ represents the controller parameters. This can be immediately performed by subgradient step and projection on a cone of nonnegative definite matrices. The outcome of this stage is to obtain a set $Q^k$ of controller parameters.

The main result of this paper is to demonstrate that this sequential algorithm provides a controller which quadratically stabilizes (or guarantees a given cost) the uncertain system with probability one in a finite number of steps. Subsequently, at a fixed step $k$, we also compute a lower bound of the probability that a quadratically stabilizing controller is found. These results are based on two assumptions. Roughly speaking, we first assume that there exists a feasible solution, i.e. a quadratically stabilizing controller. Secondly, we assume that the probability to distinguish if an approximation $Q^k$ is feasible or not is positive, see Section 5 for details on these assumptions and Section 7 for a discussion regarding the case when there is no stabilizing controller.

We notice that methods similar to those developed in this paper have been already used in a deterministic setting in the context of adaptive control, unknown-but-bounded identification and alternating projection techniques, see e.g. [2, 4, 10, 20]. From the technical point of view, one of the novelties of this paper is to study fast iterative methods for solving Linear and Quadratic Matrix Inequalities (LMIs and QMIs) and, more importantly, to formulate the problem in a probabilistic setting and to state probabilistic convergence results. Other recent papers following the same approach developed here are [7] and [9].

The paper is organized as follows: In Section 2, we introduce definitions and notation and in Section 3 we study Linear Quadratic Regulators with random uncertainty. In particular, we study systems in state space form subject to bounded random uncertainty and we formulate a guaranteed cost problem in this context. In Section 4, this problem is shown to be equivalent to the solution of a set of Quadratic Matrix Inequalities. In Section 5, the algorithm is presented and its convergence properties are rigorously stated and proved. In Section 6, we study an application example of quadratic stabilization of an aircraft subject to parametric uncertainty entering multiaffinely into the system. Finally, in Section 7 we provide concluding remarks and, in particular, we briefly discuss the probabilistic solution of unfeasible sets of QMIs.

## 2 Definitions and Notation

First, we recall the notion of projection of a symmetric matrix on a cone of nonnegative definite matrices. Let $E_n$ be a Euclidean space of real symmetric matrices $A \in \mathbf{R}^{n,n}$ equipped with Frobenius norm $||A|| \doteq (\sum_{i,k=1}^{n} a_{ik}^2)^{1/2}$ and scalar product $< A, B > \doteq \mathrm{tr}(AB)$. Let

$$\mathcal{C} \doteq \{A \in E_n : A \geq 0\}$$

be a cone of symmetric nonnegative definite matrices. A projection $A^+$ of a matrix $A \in E_n$ on this cone is the nearest matrix in $\mathcal{C}$ to $A$. This projection can be found explicitly ([16], see also [20]). For a real symmetric matrix $A$, we write

$$A = T \Lambda T^T$$

where $T$ is an orthogonal matrix and $\Lambda \doteq \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then

$$A^+ = T \Lambda^+ T^T \qquad (1)$$

where $\Lambda^+ \doteq \mathrm{diag}\ (\lambda_1^+, \lambda_2^+, \ldots, \lambda_n^+)$ and $\lambda_i^+ \doteq \max\{0, \lambda_i\}$ for $i = 1, 2, \ldots, n$.

Next, we define a *random matrix* $\Delta$ with probability density function (p.d.f.) $f_\Delta(\Delta)$. A real random matrix $\Delta \in \mathbf{R}^{n,m}$ is a matrix of random variables $\Delta_{ik}, i = 1, \ldots, n; k = 1, \ldots, m$. The probability density function $f_\Delta(\Delta)$ is defined as the joint probability density of the elements of $\Delta$. We assume that the random matrix $\Delta$ is bounded in the set $\boldsymbol{\Delta}$. A classical example of p.d.f. is when $\Delta$ is uniformly distributed within $\boldsymbol{\Delta}$. In this case, we have

$$f_\Delta(\Delta) = \mathcal{U}[\Delta] \doteq \begin{cases} \frac{1}{\mathrm{vol}(\boldsymbol{\Delta})} & \text{if } \Delta \in \boldsymbol{\Delta}; \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

where $\mathrm{vol}\,(\boldsymbol{\Delta})$ is the volume of the set $\boldsymbol{\Delta}$.

## 3 LQ Regulators with Random Uncertainty

In this section, we formulate the LQ Regulator problem in the presence of random bounded uncertainty. We consider the state space description

$$\dot{x}(t) = A(\Delta)x(t) + Bu(t), x(0) = x_0, x(t) \in \mathbf{R}^n, u(t) \in \mathbf{R}^m \qquad (3)$$

where the uncertainty $\Delta \in \boldsymbol{\Delta}$ enters into the state matrix. In this paper, we do not make any specific assumption on the dependence of $A(\Delta)$ on $\Delta$, except for boundedness of $A(\Delta)$ for all $\Delta \in \boldsymbol{\Delta}$. For example, $A(\Delta)$ may be affected by parametric (possibly nonlinear) and nonparametric uncertainty. In the former case, the entries of $A(\Delta)$ are functions of uncertain parameters which are bounded within intervals. In the latter case, $\Delta$ is a norm-bounded matrix perturbing the *nominal* state matrix. This matrix is denoted by $A_0$. That is,

$$A_0 = A(0).$$

In the LQ Regulator problem studied in this paper, the performance index is the standard quadratic cost function

$$J(\Delta) \doteq \int_0^\infty (x^T(t) S(\Delta) x(t) + u^T(t) R u(t)) dt \qquad (4)$$

where the weighting factor $S(\Delta) = S^T(\Delta) > 0$ may depend on the uncertainty $\Delta$ while $R = R^T > 0$ is fixed. To simplify the notation, in the following we write $S$ instead of $S(\Delta)$.

Next, we consider a state feedback law which coincides with the optimal feedback when the state matrix is fixed and there is no uncertainty. That is, we take

$$u(t) = -R^{-1}B^TQ^{-1}x(t) \tag{5}$$

where $Q = Q^T > 0$ is chosen so that quadratic stability is guaranteed. It is well-known [5, 14] that this is equivalent to the solution of the LMI

$$A(\Delta)Q + QA^T(\Delta) - 2BR^{-1}B^T < 0, \quad Q = Q^T > 0 \tag{6}$$

for all $\Delta \in \mathbf{\Delta}$. That is, if a common solution $Q$ of (6) exists, then the control law (5) quadratically stabilizes the system. In many important cases, which include interval matrices, this problem can be reduced to the solution of a finite number of LMIs. Let $A(\Delta)$ denote an $n \times n$ interval matrix with $2^\ell$, $\ell = n^2$, vertex matrices

$$A^1, A^2, \ldots, A^{2^\ell}.$$

In this case, if

$$A^kQ + Q(A^k)^T - 2BR^{-1}B^T < 0, \quad Q = Q^T > 0$$

for all $k = 1, 2, \ldots, 2^\ell$, then (6) holds for all $\Delta \in \mathbf{\Delta}$, see e.g. [11] for details. A major computational issue, however, is that the number of LMIs which should be simultaneously solved may be too large. For example, if a $5 \times 5$ interval matrix is considered, we need to solve $2^{25}$ LMIs and this is beyond the present capabilities of LMIs solvers. To overcome this drawback, a probabilistic measure of violation has been introduced in [3]. That is, taking some probability density $f_\Delta(\Delta)$, the goal is to find a common solution which "approximately solves" the system of LMIs in a probabilistic sense.

## 4 Guaranteed Cost with Random Uncertainty

In this section, we generalize the problem formulation of the previous section and we study the so-called guaranteed cost problem for uncertain systems, see e.g. [15]. That is, given a parameter $\gamma > 0$ and initial conditions $x_0$, the objective is to guarantee that

$$J(\Delta) \leq \gamma^{-1}x_0^TQ^{-1}x_0 \tag{7}$$

for all $\Delta \in \mathbf{\Delta}$. It is well-known that this problem can be formulated in terms of block structured LMIs, see e.g. [5]. However, we use an alternative formulation which contains no block structure and requires the solution of QMIs. The reason why we introduce QMIs can be easily explained as follows: The subgradient iteration for a convex constraint can be immediately performed by means of the projection on a cone of nonnegative definite matrices in the case of LMIs and QMIs but not in the case of block structured LMIs.

The solution of the guaranteed cost problem in terms of QMIs is formally stated in the next lemma, see e.g. [15] for similar results and [19] for a proof.

**Lemma 1** *Let $Q = Q^T > 0$ be a solution of the QMIs*

$$A(\Delta)Q + QA^T(\Delta) - 2BR^{-1}B^T + \gamma(QSQ + BR^{-1}B^T) \leq 0 \tag{8}$$

*for all $\Delta \in \mathbf{\Delta}$. Then, the control*

$$u(t) = -R^{-1}B^TQ^{-1}x(t)$$

*quadratically stabilizes the system (3) and the cost*

$$J(\Delta) \leq \gamma^{-1}x_0^TQ^{-1}x_0$$

*is guaranteed for all $\Delta \in \mathbf{\Delta}$.*

We remark that guaranteed cost is an extension of the quadratic stability problem studied in the previous section. That is, taking $\gamma$ sufficiently small, we immediately recover equation (6). Therefore, the special case of quadratic stability can be easily handled in this more general setting by a suitable selection of the parameter $\gamma$. Thus, $\gamma$ may be viewed as a design parameter which can be adequately chosen.

We are now ready to state the probabilistic solution of this problem.

## 5 Probabilistic Design with Guaranteed Cost

In this section, we state the main assumptions and results of the paper.

### 5.1 Uncertainty Randomization and Assumptions

First, we generate a random matrix $\Delta^k$ in the set $\mathbf{\Delta}$ according to the p.d.f. $f_\Delta(\Delta)$. Thus, we immediately obtain the QMI

$$A(\Delta^k)Q + QA^T(\Delta^k) - 2BR^{-1}B^T + \gamma(BR^{-1}B^T + QSQ) \leq 0. \tag{9}$$

Then, we define a matrix-valued function

$$V(Q, \Delta^k) \doteq A(\Delta^k)Q + QA^T(\Delta^k) - 2BR^{-1}B^T + \gamma(BR^{-1}B^T + QSQ) \tag{10}$$

and introduce the scalar function

$$v(Q, \Delta^k) \doteq ||[V(Q, \Delta^k)]^+|| \tag{11}$$

where $[\cdot]^+$ is the projection defined in (1) and $||\cdot||$ denotes Frobenius norm. An interpretation of this function as a measure of violation is discussed in Section 7.

We now formally state a feasibility assumption regarding the set of solutions $Q$.

**Assumption 1** *The solution set*

$$\mathcal{S}_{\mathbf{\Delta}} \doteq \{Q = Q^T > 0 : V(Q, \Delta) \leq 0 \ \ \textit{for all} \ \ \Delta \in \mathbf{\Delta}\}$$

*has a non-empty interior.*

Next, we introduce a probabilistic assumption on the set $\mathbf{\Delta}$.

**Assumption 2** *For any $Q \notin \mathcal{S}_{\mathbf{\Delta}}$, we assume that*

$$Prob\{v(Q,\Delta) > 0\} > 0.$$

This assumption can be explained as follows: For any $Q \notin \mathcal{S}_{\mathbf{\Delta}}$, there exists $\Delta \in \mathbf{\Delta}$ such that the event

$$\{v(Q,\Delta) > 0\}$$

has a non-zero probability of occurrence. For example, if the p.d.f. $f_{\Delta}(\Delta)$ is uniform, for any $Q$ which is not in the solution set, this assumption implies that the volume of the "bad set"

$$\{\Delta \in \mathbf{\Delta} : v(Q,\Delta) > 0\}$$

is non-zero.

### 5.2 Projection on a Cone and Subgradient Computation

With given initial condition $Q^0 > 0$, we consider iterations of the form

$$Q^{k+1} = \begin{cases} [Q^k - \mu^k \partial_Q\{v(Q^k,\Delta^k)\}]^+ & \text{if } v(Q^k,\Delta^k) > 0; \\ Q^k & \text{otherwise} \end{cases} \tag{12}$$

where $\partial_Q$ denotes the subgradient.

The stepsize $\mu^k$ is given by

$$\mu^k = \frac{v(Q^k,\Delta^k) + r||\partial_Q\{v(Q^k,\Delta^k)\}||}{||\partial_Q\{v(Q^k,\Delta^k)\}||^2} \tag{13}$$

where $r > 0$ is the radius of a ball $\mathcal{B}_r \subseteq \mathcal{S}_{\mathbf{\Delta}}$ with center $Q^*$. Then, for initial condition $Q^0$, we define

$$M = \frac{||Q^0 - Q^*||^2}{r^2}.$$

In the next lemma, we explicitly compute the subgradient of $v(Q^k,\Delta^k)$. See [19] for a proof of this result.

**Lemma 2** *The function $v(Q,\Delta^k)$ is convex in $Q$ and its subgradient $\partial_Q\{v(Q,\Delta^k)\}$ is given by*

$$\partial_Q\{v(Q,\Delta^k)\} =$$

$$\frac{[V(Q,\Delta^k)]^+(A(\Delta^k)+\gamma QS) + (A(\Delta^k)+\gamma QS)^T[V(Q,\Delta^k)]^+}{v(Q,\Delta^k)} \tag{14}$$

*if $v(Q,\Delta^k) > 0$ or*

$$\partial_Q\{v(Q,\Delta^k)\} = 0$$

*otherwise.*

### 5.3 Probabilistic Convergence Results

We are now ready to state the main result of the paper. This is a probabilistic convergence result when $\mu^k$ is given by (13) and $r$ is known. The proof is given in [19].

**Theorem 1** *Let Assumptions 1 and 2 be satisfied. Then, the algorithm (12) with stepsize (13) converges with probability one in a finite number of iterations. That is,*

$$Prob\{\exists\ k_0 < \infty : Q^k \in \mathcal{S}_{\mathbf{\Delta}} \ \text{for all}\ k \geq k_0\} = 1. \tag{15}$$

*Moreover, if*

$$Prob\{v(Q,\Delta) > 0\} \geq p > 0 \tag{16}$$

*holds, then for $k > M/p$*

$$Prob\{Q^k \in \mathcal{S}_{\mathbf{\Delta}}\} \geq 1 - 2\exp(-2(pk-M)^2/k). \tag{17}$$

The method proposed in Subsection 5.2 can be traced back to the Kaczmarz projection method for solving linear equations which was developed in 1937 and to the Agmon-Motzkin-Shoenberg method for solving linear inequalities which has been studied in 1954. A finite-convergent version of the latter method was proposed by Yakubovich in the middle of the sixties, see [4] and references therein. A similar approach for solving convex inequalities can be also found in [10, 17]. However, in contrast with this literature, the method proposed here is random. Therefore, the probabilistic results given in Theorem 1 are new not only for the QMI case but also for the case of linear constraints. Finally, if compared with the method of alternating projections, see [20], we observe that the iteration (12) given in this section does not require to find projections on QMIs. From the computational point of view, this is a crucial advantage since this operation may be hard to perform.

## 6 Numerical Example

In this section, we consider a multivariable example given in [1] (see also the original paper [22] for a slightly different model and set of data) which studies the design of a controller for the lateral motion of an aircraft. The model consists of four states and two inputs. The state space equation analyzed in this paper is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_p & L_\beta & L_r \\ g/V & 0 & Y_\beta & -1 \\ N_{\dot\beta}(g/V) & N_p & N_\beta + N_{\dot\beta}Y_\beta & N_r - N_{\dot\beta} \end{bmatrix} x(t) +$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t)$$

where $x_1$ is the bank angle, $x_2$ the derivative of the bank angle, $x_3$ is the sideslip angle, $x_4$ the yaw rate, $u_1$ the rudder deflection and $u_2$ the aileron deflection.

We consider the following nominal values for the aircraft parameters entering into the state matrix: $L_p = -2.93$, $L_\beta = -4.75$, $L_r = 0.78$, $g/V = 0.086$, $Y_\beta = -0.11$, $N_{\dot\beta} = 0.1$, $N_p = -0.042$, $N_\beta = 2.601$ and $N_r = -0.29$. These values coincide with those given in [22] with the difference that here we take $N_{\dot\beta} = 0.1$ while in [22] the value of this parameter is set to zero. The perturbed matrix $A(\Delta)$ is constructed as follows: Each nominal aircraft parameter is perturbed by a relative uncertainty which is taken to be equal to 15%; e.g., the parameter $L_\beta$ is bounded in the interval $[-5.4625, -4.0375]$. Then, we assume that the p.d.f. is uniform in these intervals. Finally, taking $\gamma = 0$ we study quadratic stability and we also select weights $R = I$ and $S = 0.01I$.

The uncertain parameters enter into the state space equation in a multiaffine fashion. Therefore, classical quadratic stability can be ascertained solving simultaneously $2^9 = 512$ LMIs corresponding to the vertex set, see e.g. [11]. From the computational point of view, we remark that this may be not a very easy task.

We now describe the results of the simulations. The initial condition $Q^0 > 0$ was randomly generated. Then, we sequentially generated 800 random matrices $\Delta^k$ obtaining a sequence of solutions $Q^k$ according to the iteration given in (12). The final solution $Q$ and the corresponding controller $K = B^T Q^{-1}$ are given by

$$Q = \begin{bmatrix} 0.7560 & -0.0843 & 0.1645 & 0.7338 \\ -0.0843 & 1.0927 & 0.7020 & 0.4452 \\ 0.1645 & 0.7020 & 0.7798 & 0.7382 \\ 0.7338 & 0.4452 & 0.7382 & 1.2162 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 38.6191 & -4.3731 & 43.1284 & -49.9587 \\ -2.8814 & -10.1758 & 10.2370 & -0.4954 \end{bmatrix}.$$

This controller satisfies all 512 LMIs given by the vertex set and therefore it is also a quadratic stabilizing controller in a classical worst-case sense. Subsequently, we performed a few experiments for different values of relative uncertainty. The conclusion was that 15% is approximately the largest value of uncertainty for which there exists a probabilistic robust quadratic stabilizing controller.

For the sake of comparison, we then checked the performance of the standard optimal controller $K_0 = B^T P_0$, where $P_0$ is obtained solving the Riccati equation for the nominal system

$$A_0^T P_0 + P_0 A_0 - P_0 B B^T P_0 + S = 0.$$

In this case, the optimal controller $K_0$ quadratically stabilizes only a subset of 240 vertex systems.

## 7 Concluding Remarks

In this paper, we studied probabilistic robust design of Linear Quadratic Regulators via convex optimization and gradient-based methods. The algorithm proposed here enjoys some interesting convergence properties whenever a quadratic stabilizer exists or, in other words, the problem is strictly feasible. This assumption is standard in interior point methods but it may be restrictive in some case. Therefore, for problems for which a solution to the system of QMIs does not necessarily exist, we define a *measure of violation*. This concept is not new and it has been introduced in [3]. The novelty here is to discuss a different and more effective measure of violation. In particular, as a violation function, we consider the function defined in (11)

$$v(Q, \Delta^k) = ||[V(Q, \Delta^k)]^+||.$$

An interpretation in terms of violation can be immediately given when $V(Q, \Delta^k)$ is a scalar function. In this case $[V(Q, \Delta^k)]^+$ simply means to set $v(Q, \Delta^k)$ to zero if (9) is satisfied or, otherwise, to $V(Q, \Delta^k)$ if (9) is violated. For the case when $V(Q, \Delta^k)$ is a matrix, this threshold function is replaced with a projection.

Subsequently, if

$$\Delta^1, \Delta^2, \ldots, \Delta^N$$

are $N$ randomly generated matrices, we study the empirical mean of the violation function $v(Q, \Delta^k)$

$$v(Q) = \frac{1}{N} \sum_{k=1}^{N} v(Q, \Delta^k).$$

The final objective is then to minimize the empirical mean

$$v_{inf} = \inf_{Q>0} v(Q) \tag{18}$$

which is equivalent to minimize the average violation. It is straightforward to show that this problem is convex. In addition, the set of QMIs (9) is feasible and a common solution exists if and only if

$$v_{inf} = 0.$$

On the contrary of the solution proposed for the feasible case in Section 5, in this section the randomization process is not sequential and the number $N$ of QMIs is fixed a priori. Convergence properties of a sequential algorithm of the form (12) with a different stepsize are studied in [7].

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