Sliding-mode Observer For Uncertain Systems
Part I: Linear Systems Case

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Abstract

A new sliding mode observer for linear uncertain systems is proposed. The advantage of the proposed observer is that it works under much less conservative conditions than Wallcot and Zak's observer. In addition, we address the issue of estimating a function of the state as well as unknown inputs or structural uncertainties. Further, the idea is extended to a general class of nonlinear uncertain systems. Numerical examples are used to illustrate the validity of the proposed observer design strategy.

Keywords: Sliding mode observer; Uncertain systems; Functional observer; Unknown input observer.

1 Introduction

In recent years a considerable number of researchers have addressed the design of observer design based on the variable structure systems theory, and sliding mode concept [2, 6, 8, 10, 11]. These existing methods can be classified in two categories: 1) The equivalent control based methods, and 2) sliding mode observer designs based on the method of Lyapunov. Since these techniques are important to the developments in this article, we shall briefly expand on these approaches in the following.

1.1 Equivalent control concepts and Utkin sliding mode observer

Observers based on sliding mode approach first were developed for linear systems [10]. Consider the linear system

$$\begin{aligned} \dot{x} =& \quad Ax + Bu \\ y =& \quad Cx \end{aligned} \tag{1}$$

Without loss of generality it can be assumed that the output distribution matrix can be written as

$$C = [C_1 \quad C_2]$$

where $C_1 \in \mathcal{R}^{p\times(n-p)}, C_2 \in \mathcal{R}^{p\times p}$ and $det(C_2) \neq 0$. Consequently, by using following linear transformation of state variables,

$$T = \begin{bmatrix} I_{n-p} & 0 \\ C_1 & C_2 \end{bmatrix} \tag{2}$$

(1) can be written in the form:

$$\begin{aligned} \dot{x_1} &= A_{11}x_1 + A_{12}y + B_1u \\ \dot{y} &= A_{21}x_1 + A_{22}y + B_2u \end{aligned} \tag{3}$$

The corresponding sliding mode observer for the y subsystem is given by

$$\dot{\hat{y}} = A_{22}\hat{y} + A_{21}\hat{x_1} + B_2u + L_1 sign(y - \hat{y}) \tag{4}$$

where $(\hat{x_1}, \hat{y})$ are the estimates for (x_1, y), and L_1 is a constant nonsingular feedback gain matrix. Define $e_y = y - \hat{y}$ and $e_1 = x_1 - \hat{x_1}$, following error system is obtained

$$\dot{e_y} = A_{22}e_y + A_{21}e_1 - L_1 sign(e_y) \tag{5}$$

It can be shown that for large enough L_1 a sliding mode motion can be induced on the output error state in (5). It follows that, after elapse of some finite time t_s, $e_y = 0$ and $\dot{e_y} = 0$ for all subsequent times.

For the second subsystem, the observer equation is

$$\dot{\hat{x_1}} = A_{11}\hat{x_1} + A_{12}y + B_2u + L_2L_1 sign(e_y) \tag{6}$$

which gives the following estimation error equation,

$$\dot{e_1} = A_{11}e_1 - L_2L_1 sign(e_y) \tag{7}$$

According to the equivalent control method, the system in sliding mode behaves as if $L_1 sign(e_y)$ is replaced by its equivalent value $(L_1 sign(e_y))_{eq}$, which can be calculated from the subsystem (5) assuming $e_y = 0$ and $\dot{e_y} = 0$. Hence

$$(L_1 sign(e_y))_{eq} = A_{21}e_1 \tag{8}$$

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Substituting (8) into (7) we obtain

$$\dot{e_1} = (A_{11} - L_2 A_{21})e_1 \tag{9}$$

Since pair (A_{11}, A_{21}) is observable if (A, C) is observable, $e_1 \to 0$ by appropriate choice of L_2.

1.2 Lyapunov based sliding mode observer synthesis

Consider a system described as

$$\begin{aligned} \dot{x} = & \quad Ax + Bu + Gd(x, u, t) \\ y = & \quad Cx \end{aligned} \tag{10}$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^r$ is the control input, $w \in \mathcal{R}^m$ represents the nonlinear/uncertainty or some faults, and $y \in \mathcal{R}^p$ is the output. The matrices A, B, G and C are of appropriate dimensions and G, C are of full rank, and $d(x, u, t)$ can not only represent nonlinearities, but also time-varying terms and internal/external disturbances. An observer for above class of systems has the form:

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) + \rho G sign(Wy - WC\hat{x}) \tag{11}$$

where the gain matrices K, and W are chosen so that the following conditions are satisfied:

$$P(A - KC) + (A - KC)^T P = -Q \tag{12}$$

$$WC = G^T P \tag{13}$$

where P, and Q are symmetric positive definite matrices. This technique was suggested by Walcott and Zak in [11], and further developed by Misawa in [6]. It is called Walcott-Zak observer in [2] and we shall refer to it by the same name in this article as well. However, a systematic algorithm for finding matrices P, Q, K and W was proposed only recently in [2, 12]. In theory, sliding-mode observer has good robustness to bounded modeling errors. However, there are some differences in terms of the robustness properties of the above two sliding observer design methodologies discussed in the above. The analysis in [8] have shown that there exist bounded estimation error for bounded modeling errors when equivalent control based sliding mode observer is employed. In the other word, the estimation will not be accurate when uncertainties are present. The Lyapunov based sliding-mode observer (11) design makes $\hat{x} \to x$. Because $d(x, u, t)$ may represent both nonlinearities and unknown inputs due to uncertainties (modeling error or disturbance), it means observer (11) results in exact estimation for certain class of nonlinear system under existence of certain class of uncertainties. This is an important difference between the two design techniques.

This paper first explores the reason for the above difference. However, the main contribution of the paper is to extend the design of Walcott-Zak observer to a more general class of linear uncertain systems based on a new explanation about principle of Walcott-Zak observer. Further, the design is extended to a large class of nonlinear uncertain systems. Two examples are given in order to illustrate the validity of the proposed observer.

2 The Principle of Walcott-Zak SMO

The two conditions (12) and (13) in a round about way impose some structural constraints on the system under consideration. It is difficult to design the Wallcot-Zak observer using (12) and (13). As a result, the difficulty in finding the Lyapunov pair $\{P, Q\}$ and gain matrices K, and W has limited the use of this observer. Recently, an explicit equivalent condition for (12) and (13), in terms of original system matrices was derived by Corless and Tu [1], and Xiong and Saif [12]. Corless and Tu work focuses on robust state and input estimator and does not address the connection of their result to sliding mode observer. In the following we shall present an overview of these results.

Lemma 1 *There exists symmetric positive definite matrices P, and Q and gain matrices K, and W that satisfy (12) and (13) if and only if $rank(CG) = rank(G)$ and the triplet $\{A, G, C\}$ is minimum phase.*

Remark 1 Note that the existence conditions for unknown inputs observers (UIO) [5] are exactly the same as those stated in Lemma 1. This is interesting in that it is generally perceived that the sliding mode observers can be designed under less restrictive conditions than say UIOs. Considering the dynamics of UIO is much simpler than that of Walcott-Zak observer, and assumption of boundedness of unknown inputs is unnecessary, Lemma 1 puts the applicability and advantage of SMO in a different light. Another interesting point is that UIO and SMO rely on different operating principles to achieve their robustness to matched uncertainties.

The system satisfying the conditions stated in Lemma 1 has a canonical form, which is described in the following Lemma.

Lemma 2 *For system (10) there exist non-singular transformations Γ_1, and Γ_2 such that*

$$x = \Gamma_1[x'_{ab}, x'_d]', y = \Gamma_2[y'_d, y'_b]',$$

in which case the transformed system can be written as

$$\begin{aligned} \dot{x_{ab}} = & \quad A_{ab}x_{ab} + L_d x_d + B_{ab}u \\ \dot{x_d} = & \quad E_{ab}x_{ab} + A_d x_d + B_d u + G_d d(x, u, t) \\ y_b = & \quad C_{ab}x_{ab} \\ y_d = & \quad C_d x_d \end{aligned} \tag{14}$$

where G_d, and C_d are invertible, and the pair (A_{ab}, C_{ab}) is detectable if and only if $rank(CG) = rank(G)$ and the triplet $\{A, G, C\}$ is minimum phase.

This Lemma 2 is a direct result of the work on the special coordinate transformation (SCB) theory [9]. Lemmas 1 and 2 imply that Walcott-Zak observer can actually be designed for systems which can be transformed into the form given in (14). For transformed system (14), it should be noted that the nonlinear term $Gsign(WCe)$ actually appears in the subsystem which is affected directly by unknown inputs. Letting $K = \begin{bmatrix} K_{ab} & 0 \\ 0 & K_d \end{bmatrix}$, the observer has following form,

$$\begin{array}{rcl} \dot{\hat{x_{ab}}} & = & A_{ab}\hat{x_{ab}} + L_d\hat{x_d} + B_{ab}u + K_{ab}(y_b - \hat{y_b}) \\ \dot{\hat{x_d}} & = & E_{ab}\hat{x_{ab}} + A_d\hat{x_d} + B_du + K_d(y_d - \hat{y_d}) + \\ & & \rho sign(y_d - \hat{y_d}) \end{array}$$

Obviously, by making a suitable choice of suitable K_d and ρ, it is possible to make $\hat{x_d} - x_d$ to converge exponentially to zero in spite of the presence of the unknown inputs since all states in this subsystem are measurable. So long that $\{A, G, C\}$ is minimum phase, there exists K_{ab} that can make $\hat{x_{ab}} - x_{ab}$ converge to zero. This can be carried out using classical Luenberger observer design. This procedure considerably simplifies the Walcott-Zak SMO design because no constrained Lyapunov equation is involved in the design process. Recalling Utkin SMO (4)-(6) and using the equivalent control method, we can see why under existence of matched uncertainty, the estimation error would remain bounded. Using the transformation (2) for Utkin SMO, the linear uncertain system (10) can be transformed into following canonical form,

$$\begin{array}{rl} \dot{x_1} = & A_{11}x_1 + A_{12}y + B_1u + G_1d(x,u,t) \\ \dot{y} = & A_{21}x_1 + A_{22}y + B_2u + G_2d(x,u,t) \end{array} \quad (15)$$

Under sliding observer (4), the equivalent control signal will be

$$(L_1 sign(e_y))_{eq} = A_{21}e_1 + G_2d(x,u,t) \quad (16)$$

The error dynamics of e_1 in this case will become,

$$\dot{e_1} = (A_{11} - L_2A_{21})e_1 + (G_1 + L_2G_2)d(x,u,t) \quad (17)$$

Clearly, unless $G_1 + L_2G_2 = 0$, the error e_1 will not approach zero if $d(x,u,t)$ is nonzero. Even if x_1 subsystem is unknown input free ($G_1 = 0$), equivalent control signal may introduce unknown input into it. The Utkin SMO never put $G_1 + L_2G_2 = 0$ as a constraint, which is also generally not possible. For example, if $G_1 = 0$ and G_2 is nonsingular, L_2 must be zero. However, the estimation error may be controlled within an acceptable level by suitable choice of the gain L_2 under assumption of $\|d(x,u,t)\|$ being small enough, as discussed in [8]. In such a case however, the performance is not guaranteed because it is difficult to ensure that $\|d(x,u,t)\|$ is always small.

Walcott-Zak observer requires each unknown input element to be compensated by each output directly, and uses remaining outputs to design observer for the unknown input free subsystem. If the number of unknown inputs are equal to outputs, it will require the unknown input free subsystem itself to be stable. In this case no equivalent control information is used. As such, Walcott-Zak observer imposes strong structural constraint on the system and limits its application. On the other hand, we show that the drawback of current equivalent control method is to introduce unknown input into error dynamics for those unmeasurable states. In the next section we propose a novel sliding mode observer design technique by exploiting the structure property of subsystem upon unknown input and using equivalent control signal carefully. Our design significantly reduces the structural constraint.

3 A Novel Sliding Mode Observer For Linear Uncertain systems

First consider some preliminary results from the SCB form of linear systems given below.

Theorem 1 *[9] For system (10), there exist non-singular transformations Γ_1, Γ_2 and Γ_3, and integer $m_d \leq q, q_i = 1, ..., m_d$, such that*

$$\overline{x} = \Gamma_1^{-1}x, \overline{y} = \Gamma_2^{-1}y, \overline{d} = \Gamma_3^{-1}d;$$

$$\overline{x} = [(x_a)^{'}, x_b^{'}, x_c^{'}, x_d^{'}]^{'}$$

$$\overline{y} = [y_d^{'}, y_b^{'}]^{'}, \overline{d} = [d_d^{'}, d_c^{'}]^{'}$$

$$y_d = [y_1, y_2, ..., y_{m_d}]', d_d = [d_1, d_2, ..., d_{m_d}]'$$

and

$$\begin{array}{rl} \dot{x_a} = & A_ax_a + L_{ad}y_d + L_{ab}y_b + B_au \\ \dot{x_b} = & A_bx_b + L_{bd}y_d + B_bu, \qquad y_b = C_bx_b \\ \dot{x_c} = & A_cx_c + L_{cd}y_d + L_{cb}y_b + G_cE_{ca}x_a + G_cd_c + B_cu \end{array} \quad (18)$$

and $x_d = \begin{bmatrix} x_{1d} & x_{2d} & \dots & x_{m_dd} \end{bmatrix}^T$, for each $x_{id}, i = 1, ..., m_d$,

$$\begin{array}{rcl} \dot{x_{id}} & = & A_{q_i}x_{id} + L_{id}y_d + G_{q_i}[E_{ia}x_a + E_{ib}x_b + E_{ic}x_c \\ & & + \sum_{j=1}^{m_d} E_{ij}x_{jd} + d_i] + B_{id}u \end{array} \quad (19)$$

$$y_i = C_{q_i}x_{id}, \qquad y_d = C_dx_d \quad (20)$$

Here the states x_a, x_b, x_c, x_d are of dimensions n_a, n_b, n_c and $n_d = \sum_{i=1}^{m_d} q_i$ respectively, while x_{id} is of dimension q_i for each $i = 1, ..., m_d$. The matrices A_{q_i}, G_{q_i} and C_{q_i} have following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}; G_{q_i} = \begin{bmatrix} 0_{(q_i-1)\times 1} \\ 1 \end{bmatrix};$$

$$C_{q_i} = \left[\begin{array}{cc} 1 & 0_{1\times(q_i-1)} \end{array} \right]; \tag{21}$$

and last row of L_{id} is identically zero.

To this end, x_d subsystem is further decomposed into m_d subsystems. Let

$$x_{id} = \left[\begin{array}{cccc} x_1^i & x_2^i & ... & x_{q_i}^i \end{array} \right]';$$

The special form of A_{q_i}, G_{q_i} implies that the equations of subsystem x_{id} in (19) can be rewritten as

$$\begin{array}{lcl} \dot{x}_1^i & = & x_2^i + L_1^i y_d + B_1^i u \\ \dot{x}_2^i & = & x_3^i + L_2^i y_d + B_2^i u \\ \dot{\cdots} & \cdot & \cdots \\ \dot{x}_{q_i-1}^i & = & x_{q_i}^i + L_{q_i-1}^i y_d + B_{q_i-1}^i u \\ \dot{x}_{q_i}^i & = & E_{ia}x_a + E_{ib}x_b + E_{ic}x_c + \sum_{j=1}^{m_d} E_{ij}x_{jd} + \\ & & B_{q_i}^i u + d_i \\ y_i & = & x_1^i \end{array} \tag{22}$$

We propose the following sliding mode observer for each x_{id} subsystem based on equivalent control method,

$$\begin{array}{lcl} \dot{\hat{x}}_1^i & = & \hat{x}_2^i + L_1^i y_d + B_1^i u + \lambda_1^i sign(y_i - \hat{x}_1^i) \\ \dot{\hat{x}}_2^i & = & \hat{x}_3^i + L_2^i y_d + B_2^i u + \lambda_2^i sign(\overline{e_2^i}) \\ \cdots & & \cdots \\ \dot{\hat{x}}_{q_i-1}^i & = & \hat{x}_{q_i}^i + L_{q_i-1}^i y_d + B_{q_i-1}^i u + \lambda_{q_i-1}^i sign(\overline{e_{q_i-1}^i}) \\ \dot{\hat{x}}_{q_i}^i & = & E_{ia}\hat{x_a} + E_{ib}\hat{x_b} + E_{ic}\hat{x_c} + \sum_{j=1}^{m_d} E_{ij}\hat{x_{jd}} + \\ & & B_{q_i}^i u + \lambda_{q_i}^i sign(\overline{e_{q_i}^i}) \end{array} \tag{23}$$

where $\hat{x_a}, \hat{x_b}, \hat{x_c}$ are estimation for states x_a, x_b, x_c respectively, coming from sub observers given later. And

$$\overline{e_j^i} = (\lambda_{j-1}^i sign(\overline{e_{j-1}^i}))_{eq} \tag{24}$$

for $j = 2, ..., q_i$, and $\overline{e_1^i} = y_1 - \hat{x}_1^i = e_1$ and can be obtained directly. The equivalent control signal $(v)_{eq}$ for signal v is calculated by low pass filtering signal v with anti-peaking structure [4]. This anti-peaking structure stems from the fact that we do not inject the observation error information before reaching the sliding manifold linked with this information. Moreover, we reach the manifold one by one. Doing this we obtain a subdynamics of dimension one and consequently we do not have peaking phenomena. More precisely,

$$(\lambda_{j-1}^i sign(\overline{e_{j-1}^i}))_{eq} = 0$$

before $\overline{e_{j-1}^i}$ reaching its sliding manifold.

Theorem 2 *Considering system (22) and observer (23), for any initial state and any bounded unknown inputs $d_i(x, u, t)$, there exists a choice of*

$$\lambda_j^i, i = 1, ..., m_d, j = 1, ..., q_i$$

such that the state estimation converges in finite time to its real value.

A classical Luenberger observer is applied to subsystem x_b described by,

$$\dot{x_b} = A_b x_b + L_{bd} y_d + B_b u, \quad y_b = C_b x_b$$

because it is unknown input free, and (A_b, C_b) forms an observable pair. Of course, the Utkin SMO can be used too. Generally, subsystem x_a can be further decomposed into two subsystems,

$$\begin{array}{rcl} \dot{x}_a^- & = & A_a^- x_a^- + L_a^- y + B_a^- u \\ \dot{x}_a^+ & = & A_a^+ x_a^+ + L_a^+ y + B_a^+ u \end{array}$$

A_a^- is stable, a Luenberger observer exists for subsystem x_a^- even if non of its state variables are measurable. However, A_a^+ is unstable, an alternative approach has to be found in order to estimate x_a^+. Next, the possibility to estimate x_a^+ is discussed together with estimation of x_c. After $e_{q_i}^i$ reaches sliding mode, we may have the following equivalent control signal,

$$\begin{array}{rcl} (\lambda_{q_i}^i sign(e_{q_i}^i))_{eq} & = & E_{ia}^- e_a^- + E_{ia}^+ e_a^+ + E_{ib} e_b + E_{ic} e_c + \\ & & \sum_{j=1}^{m_d} E_{ij} e_{jd} + d_i, \end{array}$$

where $i = 1, ..., m_d$. After all states of x_a^-, x_b and x_d have been estimated, it will equal to

$$(\Lambda sign(e_q))_{eq} = E_{dc} e_c + E_{da}^+ e_a^+ + d_d \tag{25}$$

where all m_d equivalent control signals are written together as a vector. Considering the above equivalent control signal as output of x_c, x_a^+ subsystem, this subsystem can be rewritten as

$$\begin{array}{rl} \dot{x_{ac}} = & A_{ac} x_{ac} + L_{ac} y + B_{ac} u + G_{ac} d \\ y_{ac} = & E_{ac} x_{ac} - E_{ac} \hat{x_{ac}} + G_o d \end{array} \tag{26}$$

where

$$x_{ac} = \left[\begin{array}{c} x_a^+ \\ x_c \end{array} \right], A_{ac} = \left[\begin{array}{cc} A_a^+ & 0 \\ G_c E_{ca}^+ & A_c \end{array} \right], G_{ac} = \left[\begin{array}{cc} 0 & 0 \\ G_c & 0 \end{array} \right],$$

$$E_{ac} = \left[\begin{array}{cc} E_{da}^+ & E_{dc} \end{array} \right], G_o = \left[\begin{array}{cc} 0 & I \end{array} \right],$$

Note that no mention of how to calculate $\hat{x_c}, \hat{x}_a^+$ has been made up to this point. However, so long it is bounded, its value does not effect the convergence of the sliding mode observer (23) with a large enough gain. And it can be considered as known input for x_{ac} subsystem (26). For simplicity, we can put $\hat{x_c} = \hat{x}_a^+ = 0$ into sliding mode observer (23). An interesting fact is that (A_{ac}, E_{ac}) is detectable if (A, C) is detectable. Unfortunately, there is no unknown input observer for system (26) according to following lemma.

Lemma 3 *[3] There exists an unknown input observer for system (26) only if*

$$rank \left[\begin{array}{cc} G_o & E_c G_s \\ 0 & G_o \end{array} \right] = rank G_o + rank \left[\begin{array}{c} G_s \\ G_o \end{array} \right] \tag{27}$$

Note that based upon the particular form of G_s, G_o, condition (27) will never be satisfied. In this case, we may use H_2 or H_∞ optimal observer design techniques proposed in [9] to make the estimation error as small as possible. At this point, based on the above analysis, we shall summarize our new sliding mode observer design procedure for linear uncertain system as follows,

SMO Design Algorithm

Step 1: Transform system (10) it into SCB form (18)-(19) by non-singular transformations $\Gamma_1, \Gamma_2, \Gamma_3$.

Step 2: Estimate x_d using sliding mode observer (23), and estimate x_a^-, x_b using a regular Luenberger observer. The measurement variable for transformed system y_d, y_b is derived from original output by $[y_d, y_b]' = \Gamma_2^{-1} y$, and the known input distribution matrix is transformed by $\Gamma_1^{-1} B$.

Step 3: If $\{A, G, C\}$ is left invertible(or equivalently $n_c = 0$) and is minimum phase (or $n_a^+ = 0$), the original state can be derived by

$$x = \Gamma_1 [x_a^- \quad x_b \quad x_d]^T$$

Remark 2 Walcott-Zak SMO requires $rank(CG) = rank(G)$, which immediately implies that $n_c = 0$ and $q_i = 1, i = 1, ..., m_d$, namely the number of infinite zero of order i is one. The restriction on system $\{A, G, C\}$ infinite zero structure is removed in our algorithm. Unfortunately, the requirement of $n_c = 0$ is still necessary for estimating all states, and it implies $rank(C) \geq rank(G)$.

Remark 3 In the observer structure, the equivalent control signal with anti-peaking structure allows e_j^i to converges to zero if all the e_k^i with $k < j$ have converged to zero before. This is a basic disadvantage of equivalent control method. However, it does allow us to achieve unknown input decoupled estimation under less conservative conditions than those of any other existing method.

Remark 4 It is not always necessary to design the proposed sliding mode observer based on the transformed SCB model. The basic design rule is to make sure unknown inputs will not appear in the equivalent control signal. Our example illustrates this intuitive design procedure for a simple low-order system. However, SCB-based design is systematic and can be done easily for high-order systems with aid of SCB software [9].

3.1 State Function and Unknown Input Estimation Using SMO

If the system does not satisfy the condition for estimating all the unavailable state variables under the presence of unknown inputs, we can seek to estimate linear function of state, Tx. In this case it is desired for T to have as many linearly independent rows as possible. Based on design algorithm of SMO, we will immediately have the following algorithm for sliding mode functional observer (SMFO):

SMFO Design Algorithm

Step 1: Follow SMO algorithm to get estimation of sub system x_a^-, x_b and x_d.

Step 2: If $n_c \neq 0$ or $n_a^+ \neq 0$, any linear function of states, $T_{k \times n} x$ can be estimated, where T must satisfy following condition,

$$T\Gamma_1 = [T_a^- \quad 0 \quad T^b \quad 0 \quad T^d]_{k \times n} = \hat{T}_{k \times n}$$

where T_a^-, T_b, T_d are any matrices of dimension $k \times n_a^-, k \times n_b$ and $k \times n_d$ respectively. Obviously, the maximum linear independent rows number $k = n_a^- + n_b + n_d$.

Checking the equivalent control signal (25), we have following corollary about unknown inputs estimation,

Corollary 1 *If system $\{A, G, C\}$ is left invertible (i.e. $n_c = 0$) and all unstable transmission zeros are unobservable modes (i.e. in SCB form, $E_{da}^+ = 0$), all unknown inputs can be estimated exactly using sliding mode observer (23). If system $\{A, G, C\}$ is not left invertible (i.e. $n_c > 0$) but all eigenvalues of corresponding n_c subsystem are unobservable modes (i.e. $E_{dc} = 0$), and all unstable transmission zeros are unobservable modes (i.e. $E_{da}^+ = 0$), at least m_d unknown inputs can be estimated exactly using sliding mode observer (23).*

Remark 5 In [13], linear state function Tx is estimated using unknown input functional observer (UIFO), and the maximum rank of T is $n_a^- + n_b$. The maximum rank of T has been increased significantly using our new SMFO design. Compared with input estimator based on UIFO, which was discussed in [1, 14], our new SMFO has better capability in estimating the unknown inputs as well.

4 Illustrative Example

Example 1 The system under consideration is a one-link manipulator with revolute joints actuated by a DC motor. The system dynamics are nonlinear and of the form

$$\dot{x} = Ax + \Phi(x) + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix},$$

$$\Phi(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33 sin(x_3) \end{bmatrix}$$

The above parameters for the system are typical and have been taken from [7]. Under condition of motor position and velocity are measured, [7] proposed an observer by considering this system as a Lipschitz nonlinear system. Obviously, this systems can be considered as a linear system subject to bounded unknown input, namely the nonlinear term $-3.33 sin(x_3)$. Further it is easy to verify that even if only motor position is measured, or

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

a SMO can be designed using the proposed algorithm. This SMO design will relieve the need for measurement of motor velocity. It should be stressed that no Walcott-Zak SMO or linear UIO for the above robotic system exists because $rank(CG) = 0 < rank(G) = 1$. The design using SCB transform is omitted here due to limited space. This system has a simple format so that direct design is possible. First, build the observer for x_1 as

$$\dot{\hat{x}}_1 = \hat{x}_2 + \lambda_1 sign(y_1 - \hat{x}_1)$$

After e_1 has approximately reached zero, we know $e_2 = (\lambda_1 sign(e_1))_{eq}$. Next, build the observer for x_2 as

$$\dot{\hat{x}}_2 = -48.6 y_1 - 1.25 \hat{x}_2 + 48.6 \hat{x}_3 + \lambda_2 sign(e_2)$$

After e_2 approximates to zero, we know that

$$e_3 = (\lambda_1 sign(e_2))_{eq}/48.6$$

and the observer for x_3 is built as

$$\dot{\hat{x}}_3 = \hat{x}_4 + \lambda_3 sign(e_3)$$

Finally, $e_4 = (\lambda_3 sign(e_3))_{eq}$ and the observer for x_4 is

$$\dot{\hat{x}}_4 = -19.5 y_1 + 19.5 \hat{x}_4 + \lambda_4 sign(e_4)$$

e_4 will go zero in spite of the nonlinear term $-3.33 sin(x_3)$. Actually, it does not matter if the nonlinear term is more complicated, Lipschitz or not.

5 Conclusions

This first part of a two part paper explored some underlying similarity and connections between seemingly two different methodologies to designing observers, namely that of the sliding mode, and unknown input observer. Based on some results on the special coordinate basis (SCB) form for linear systems, the paper extended the Walcott-Zak observer to general class of linear uncertain systems. In addition, the extension of the state observer to function of state observer was discussed.

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