Second Order Sliding Mode Control of Systems with Nonlinear Friction

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Abstract

In this paper a second order sliding mode control algorithm conceived as a solution to the chattering elimination problem is proved to be effective also against discontinuous disturbances such as friction. The stabilization problem for a mechanical system is considered and solved by the proposed control algorithm. The method is effective in avoiding complex stick-slip phenomena, since during the transients no oscillations or overshoot take place.

1 Introduction

This paper deals with the control of mechanical systems subjected to uncertainties and disturbances of various nature, including friction. The proposed solution is based on sliding-mode control theory, which has been shown to be highly effective in counteracting uncertainties and disturbances for some classes of uncertain nonlinear systems. Specific drawback presented by the classical sliding mode techniques is the chattering phenomenon. This problem is addressed in the paper by exploiting the robustness properties of second-order sliding-mode control algorithms. An algorithm of this kind, recently developed by the authors, is proved to be effective to stabilize, with arbitrarily exponentially fast transient, mechanical systems subjected to static and Coulomb friction.

2 Statement of the problem

Consider a mechanical system affected by a bounded and discontinuous disturbance modeling the friction phenomena

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 = f(x_1, x_2) + g(x_1)\, u - C(x_2)\, \mathrm{sign}\,(x_2) \end{cases} \tag{1}$$

$C(x_2)$ is the modulus of the friction term which in general depends on system states and external actions [1], [2], but it is assumed, together with its derivatives, bounded in the bounded domain of interest. For example, according to models proposed in literature [3], [4], $C(x_2)$ could be expressed as $C(x_2) = C_c + (C_s - C_c)\exp^{-\left(\frac{x_2}{v_s}\right)^2}$, where C_c is the Coulomb friction level, C_s is the level of stiction, and v_s is the Stribeck velocity. The viscous friction effect is incorporated in the term $f(x_1, x_2)$, therefore $C(x_2) \leq C_s$. Moreover assume to know a function $F(\|x\|)$ non decreasing with its argument $\|x\|$ and two constants g_1 and g_2 such that $|f(x_1, x_2)| < F(\|x\|)$ and $0 < g_1 \leq g(x_1) \leq g_2$. The control aim is that of steering exponentially to zero both $x_1(t)$ and $x_2(t)$ by means of a continuous control $u(t)$. This is not a trivial task, indeed an undesired sliding motion on $x_2 = 0$ ("trapping point") can occur if

$$|f(x_1, x_2) + g(x_1)\, u(t)| \leq C(x_2) \tag{2}$$

and the point is maintained until the sign of the inequality changed.

Pathological phenomena such as stick-slip limit cycles associated with the occurrence of a sequence of trapping points, can arise when standard control algorithms are used [1]. The solution proposed in this paper avoids any problem of this kind since it is ensured that during the motion from the initial position to the desired final one at most a single trapping point is encountered.

Let the continuous control $u(t)$ be the output of a dynamical system

$$\dot{u} = h(u, v)$$

where $v(t)$ is the actual control signal, in particular $\dot{u} = v$.

System (1) with friction and continuous control can be interpreted as a discontinuous system, due to the presence of friction, in which the control objective is that of avoiding sliding motion on $x_2 = 0$.

If a sliding motion on $x_2 = 0$ occurs, system (1) is in a "trapping point" $x_1 = \overline{x}_1$, $x_2 = 0$, and according to the Filippov's solution concept, the system representation is

$$\begin{cases} x_1 = \overline{x}_1 \neq 0 \\ \dot{x}_1 = x_2 = 0 \\ \dot{x}_2 = x_3 = f(\overline{x}_1, 0) + g(\overline{x}_1)u + (1-2\gamma)C(0) \\ \dot{u} = v \\ \gamma = \frac{f(\overline{x}_1,0)+g(\overline{x}_1)u+C(0)}{2C(0)} \end{cases} \tag{3}$$

where $\gamma \in [0,1]$ according to condition (2).

Definition 1:

System (1) represented by (3) during the sliding motion, is leaving the trapping point $(\overline{x}_1, 0)$ *at* $t = t_0$*, if*

$$\gamma[x(t_0), u(t_0)] = \begin{cases} 1 & \frac{d}{dt}\left[\gamma(x,u)\right]_{t=t_0} > 0 \\ 0 & \frac{d}{dt}\left[\gamma(x,u)\right]_{t=t_0} < 0 \end{cases}$$

After $t = t_0$ and until a new trapping point is reached, system (1), under mild regularity conditions on $f(x_1, x_2)$ and $g(x_1)$, can be represented by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 = f(x_1,x_2) + g(x_1)\,u \pm C(x_2) \\ \dot{x}_3 = \ddot{x}_2 = \frac{d}{dt}f(x_1,x_2) + \frac{d}{dt}g(x_1)\,u \pm \frac{d}{dt}C(x_2) + g(x_1)\,v \end{cases}$$

If $s(x_1,x_2) = x_2 + c\,x_1$ is steered to zero in a finite time by a possibly discontinuous signal $v(t)$, the same control objective as in the standard first order sliding mode control case is achieved by means of a continuous control action $u(t) = u(0) + \int_0^t v(\tau)d\tau$, therefore the chattering phenomenon is eliminated.

Differentiate $s(x_1,x_2) = x_2 + c\,x_1$ two times and set $s(x_1,x_2) = y_1$, $\dot{y}_1 = y_2$, then, the whole system can be represented as

$$\begin{cases} \dot{x}_1 = x_2 = y_1 - c\,x_1 \\ \dot{y}_1 = y_2 = f(x_1,x_2) + g(x_1)u + c\,x_2 \\ \dot{y}_2 = m_1(x_1,x_2) + m_2(x_1,x_2)u + g(x_1)v \\ \dot{u} = v \end{cases} \tag{4}$$

$$\begin{aligned} m_1(x_1,x_2) &= \left[\tfrac{\partial}{\partial x_1} f(x_1,x_2)\right] x_2 + \left[\tilde{C}(x_1,x_2) + c\right] f(x_1,x_2) \\ m_2(x_1,x_2) &= \left[\tfrac{\partial}{\partial x_1} g(x_1)\right] x_2 + \left[\tilde{C}(x_1,x_2) + c\right] g(x_1) \\ \tilde{C}(x_1,x_2) &= \left[\tfrac{\partial}{\partial x_2} f(x_1,x_2) \pm \tfrac{\partial}{\partial x_2} C(x_2)\right] \end{aligned}$$

Assume that there exist two functions $\overline{m}_1(\|x\|)$ and $\overline{m}_2(\|x\|)$ non decreasing with their argument $\|x\|$ such that the following inequalities hold

$$|m_i(x_1,x_2)| < \overline{m}_i(\|x\|) \;\; i = 1,2 \tag{5}$$

3 Main Result

The aim of this section is that of identifying a control law $v(t)$ guaranteeing that, starting from any trapping point, the origin is reached according to the following three steps: A trapping point once reached is left in a finite time;
Starting from a trapping point the sliding manifold can be reached in a finite time;
On the sliding manifold the origin of the phase plane is reached exponentially.
The proposed solution relies on the fact that, if a sliding mode occurs on the sliding surface $s(x_1,x_2) = x_2 + c\,x_1 = 0$, the only reachable trapping point is $x_1 = x_2 = 0$, in fact, on $s(x_1,x_2) = 0$, the friction term turns out to be continuous until the state origin is reached and starting from any point of the sliding manifold the state trajectories converge exponentially to the origin according to the zero dynamics of the system constrained to the surface.

Considering equation (4) which describes the relation between the sliding output $y_1 = x_2 + c\,x_1$ and the input $v(t)$, it is evident that the desired sliding motion on $y_1 = 0$ belongs to the class of second order sliding motion since the control $v(t)$ affects $\ddot{y}_1$, the second time derivative of the sliding quantity. Due to the uncertainties in the system dynamics $\dot{y}_1$ is not available, therefore it is apparent that to analyze and solve the considered problem it must be exploited the mathematical tool of the second order differential inequalities.

The behaviour of a system satisfying a second order differential inequality of the type $\ddot{y}_1 y_1 \leq -h^2|y_1|$ is characterized by a focus in the origin of the phase plane y_1, y_2. A sequence of singular points $\{y_1(t_{M_i}), t_{M_i} : y_2(t_{M_i}) = 0\}$, is generated and the related behaviour can range from explosive or persistently oscillating behaviour to the desired stable one.

The control objective can be stated in terms of the convergence property of the two sequences $\{y_1(t_{M_i}), t_{M_i} : y_2(t_{M_i}) = 0\}$ and $\{\Delta_i = t_{M_i} - t_{M_{i-1}}\}$. Indeed if the sequences are strictly contractive, that is $\frac{|y_1(t_{M_{i+1}})|}{|y_1(t_{M_i})|} \leq \rho < 1$ and $\frac{\Delta_{i+1}}{\Delta_i} \leq q < 1$, and it holds that $\lim_{i\to\infty} y_1(t_{M_i}) = 0$ and $\sum_{i=1}^{\infty}(t_{M_i} - t_{M_{i-1}}) = T < \infty$, then at $t = T$ both y_1 and y_2 are steered to zero.

In order to better explain the second order control strategies, consider the double integrator system $\dot{z}_1 = z_2$, $\dot{z}_2 = h(z) + d(z)w$ perturbed by uncertain terms for which constant bounds are known $|h(z)| < H$, $0 < d_1 < d(z) < d_2$.

The following control algorithm [5] can be successfully applied to the perturbed double integrator.

Algorithm 1:

When $t = 0$, set $z_{1_M} = z_1(0)$, $i = 0$, $t_{M_i} = 0$.

During the control interval, that is $\forall t \in [0, \infty)$, the following steps are performed:

If $z_2(t) = 0$ then set $z_{1_M} = z_1(t)$, $i = i+1$, and $t_{M_i} = t$.

It is applied the control

$$w(t) = -W(t)\mathrm{sign}\left[z_1(t) - \frac{1}{2} z_{1_M}\right] \qquad (6)$$

$$W(t) = \begin{cases} W_M & z_{1_M}\left[z_1(t) - \frac{1}{2} z_{1_M}\right] > 0 \\ \alpha W_M & z_{1_M}\left[z_1(t) - \frac{1}{2} z_{1_M}\right] < 0 \end{cases}$$

$$W_M > \max\left(\frac{H}{d_1}, \frac{4H}{3\alpha d_1 - d_2}\right) \qquad \alpha > 1 \quad \alpha \neq \frac{d_2}{3d_1}$$

It can be proved [5] that, despite the uncertainties, if the control amplitude is sufficiently high, the application of the control strategy (6) generates a sequence of successive singular points $\{z_1(t_{M_i}), t_{M_i} : z_2(t_{M_i}) = 0\}$, and this sequence is strictly contractive, that is $\frac{|z_1(t_{M_{i+1}})|}{|z_1(t_{M_i})|} \leq q < 1$. Moreover, the reaching time is a series of positive elements upper-bounded by a geometric series with ratio strictly less than one. Therefore, $\sum_{i=1}^{\infty}(t_{M_i} - t_{M_{i-1}}) = T < \infty$.

Assume that, at $t = 0$, system (1) is in a trapping point. It must be proved that it is possible to escape from any trapping point in a finite time by means of the continuous control $u(t)$ resulting from a $v(t)$ chosen according to the Algorithm 1 with suitably chosen design constants. To this end the following proposition is proved.

Proposition 1:

Assume that at $t = 0$ system (1) is in the following trapping point $x_1(0) = X_1$, $x_2(0) = 0$. After a finite time t_0, under the action of a control $u(t) = u(0) + \int_0^t v(\tau)d\tau$, with $v(t)$

$$v(t) = -V(t)\mathrm{sign}\left[y_1(t) - \frac{1}{2} y_{1_M}\right] \qquad (7)$$

$$V(t) = \begin{cases} V_M & y_{1_M}\left[y_1(t) - \frac{1}{2} y_{1_M}\right] > 0 \\ \alpha V_M & y_{1_M}\left[y_1(t) - \frac{1}{2} y_{1_M}\right] \leq 0 \end{cases}$$

the trapping point is left and the the phase trajectories satisfy the following conditions $\forall t > t_0$ $x_2(t) \neq 0$ and $\mathrm{sign}(x_2) = -\mathrm{sign}(x_1)$, *that is after a finite time the even quadrants of the phase plane are reached.*

Proof:

At $t = 0$ system (1) is assumed to be in a trapping point then its equivalent representation is (3). It follows that $y_2(0) = 0$ and that, according to the given definition $y_1(0) = cX_1 = y_{1M}$ is a singular point. While the system remains in the trapping point, i.e. $\forall t : 0 \leq t \leq t_0$, the applied control, according to Algorithm 1, is $v(t) = -V_M \mathrm{sign}\left[y_1(t) - \frac{1}{2}y_1(0)\right] = -V_M \mathrm{sign}\left[\frac{1}{2}y_1(0)\right]$, therefore $v(t) = \dot{u}(t) = -V_M \mathrm{sign}(X_1)$.

While in a trapping point $\gamma[X_1, u(t)] = \frac{f(X_1,0)+g(X_1)\,u+C(0)}{2C(0)} = K_1 + K_2\,u$ where K_1 and K_2 are constants the values of which depend on X_1. It follows that, since $\dot{\gamma} = K_2\,\dot{u}$, if $X_1 > 0$ $\exists t_0 : \gamma[X_1, u(t_0)] = 0$ and $\dot{\gamma}(t_0) < 0$ while if $X_1 < 0$ $\exists t_0 : \gamma[X_1, u(t_0)] = 1$ and $\dot{\gamma}(t_0) > 0$. This fact, according to Definition 1 guarantees that at $t = t_0$ the system leaves the trapping point. Q.E.D.

At $t = t_0$ the system is leaving a trapping point; what must be assured is that, applying the control Algorithm 1, a sliding motion on $y_1 = x_2 + c\,x_1 = 0$ occurs in a finite time. If during the transient induced by the application of Algorithm 1, the state trajectory cross the $x_2 = 0$ axis, the contraction of the sequence of successive singular points $\{y_1(t_{M_i}), t_{M_i} : y_2(t_{M_i}) = 0\}$ can be still guaranteed, but the corresponding sequence $\{t_{M_i} : y_2(t_{M_i}) = 0\}$ cannot be proved to be contractive, and therefore no finite time convergence can be ensured. On the contrary, if during the reaching phase the system trajectory never crosses the $x_2 = 0$ axis, the conditions guaranteeing the finite time convergence are still satisfied. Guaranteeing this last condition allows to completely avoid the analysis of complex stick-slip phenomena, including limit cycles.

Algorithm 1 requires the knowledge of constant upperbounds of the relevant uncertain term in order to choose the parameter V_M and α guaranteeing the finite time convergence. The use of constant upperbound simplifies the analysis of the system behaviour which can be carried on relying on the use of limiting curves.

To this end an upperbound of the modulus of the drift term $m_1(x_1, x_2) + m_2(x_1, x_2)\,u$ in (4) must be available at $t = t_0$. Unfortunately the choice of the control parameters V_M and α affects the future behaviour of the system and therefore the uncertain drift term. In the sequel what could appear to be a logic loop is proved to be solved by the adopted control strategy on the basis of assumptions (5).

Proposition 2:

If in any either finite or infinite time interval starting from $t = t_0$, the motion of system (4) is within the boundary layer $\{y_1(t) : |y_1(t)| < Y_1, Y_1 > 0\}$, the norm of the state vector $x(t)$ can be upperbounded by Y_1, the maximal value possibly assumed by $y_1(t)$, suitably scaled by a constant $k > 0$ chosen on the basis of the initial condition $x_1(t_0)$.

Proof:

Consider the first equation in (4), since it describes the dynamics of a perfectly known first order stable system with input $y_1(t)$, it follows

$$x_1(t) = x_1(t_0)\exp^{-c\,t} + \int_{t_0}^{t} \exp^{-c\,(t-\tau)}\, y_1(\tau)\mathrm{d}\tau$$

therefore

$$|x_1(t)| \leq |x_1(t_0)| + Y_1$$

and

$$|x_2(t)| \leq c\,|x_1(t)| + Y_1 \leq c\,|x_1(t_0)| + (c+1)Y_1$$

The state component $x_1(t)$ is available $\forall t \in [0, \infty)$ therefore it is always possible to compute the constant k such that

$$\|x(t)\| < kY_1$$

Q.E.D.

System (1) at $t = t_0$ is leaving a trapping point, that is $x_1(t_0^+) = X_1$, $x_2(t_0^+) = 0$, $\dot{x}_2(t_0^+) = 0$, $|\ddot{x}_2(t_0^+)| \neq 0$, and $\mathrm{sign}\left[\ddot{x}_2(t_0^+)\right] = -\mathrm{sign}\,(X_1)$; since $y_2(t_0) = 0$ it can be posed $t_0 = t_{M_1}$ and $|y_1(t_{M_1})| = |c\,X_1| = Y_1$.

Considering the second equation in (4), at $t = t_0 = t_{M_1}$, it is possible to calculate a known constant upperbound of the modulus of $u(t_{M_1})$

$$|u(t_{M_1})| < \frac{F(kY_1) + c\,kY_1}{g_1} = U_{M_1}$$

furthermore, within the boundary layer $\{y_1(t) : |y_1(t)| \leq Y_1, |y_1(t_{M_1})| = Y_1\}$, the drift term in the third equation in (4), by virtue of Proposition 2, is overestimated by

$$|m_1(x_1,x_2) + m_2(x_1,x_2)u| < \overline{m}_1(kY_1) + \overline{m}_2(kY_1)|u|$$

In the actual case what must be proved is the existence of a constant U_M^* such that for any $t \in [t_{M_1}, \infty)$ the application of the Algorithm 1 with $z_1 = y_1$, $z_2 = y_2$, $w = v$, $d_1 = g_1$, $d_2 = g_2$, and $H = \overline{m}_1(kY_1) + \overline{m}_2(kY_1)U_M^*$ guarantees that $|u(t)| < U_M^*$ and that $y_1(t)$ tends to zero in a finite time without leaving the boundary layer $\{y_1(t) : |y_1(t)| \leq Y_1, |y_1(t_{M_1})| = Y_1\}$. To obtain the control objective the control law $v(t)$, chosen in accordance with the second order Algorithm 1, must guarantee that its integral $u(t)$ never leaves a boundary layer $\{u(t) : |u(t)| \leq U_M^*\}$. The constant U_M^* must be available at $t = t_{M_1}$ and remain valid $\forall t \in [t_{M_1}, \infty)$.

The following Theorem guarantees that: the constant U_M^* can be found, the convergence of state trajectories to the sliding manifold takes place in a finite time, and no trapping point except the origin is encountered.

Theorem 1

Consider $\overline{u}_1$ the maximal positive solution of the equation

$$a\left[\overline{m}_1(kY_1) + \overline{m}_2(kY_1)u\right] = (u - U_{M_1})^2 \tag{8}$$

$$a = \max\left(\frac{\nu^2}{\nu - 1}, \frac{\alpha^2\,\nu^2}{\alpha\,\nu - 1}\right)\frac{Y_1}{g_1^2}$$

with $\nu > 1$, $\alpha = \frac{2}{\nu} + \frac{g_2}{g_1}$, and $U_{M_1} = \frac{F(kY_1)+c\,kY_1}{g_1}$; let $U_M^* > \overline{u}_1$ and $H^* = \overline{m}_1(kY_1) + \overline{m}_2(kY_1)U_M^*$.

Consider system (4): the application of the control law (7), designed according to the control Algorithm 1, with $V_M = \nu\frac{H^*}{g_1}$ and $\alpha = \frac{2}{\nu} + \frac{g_2}{g_1}$, ensures that $\forall t \in [t_{M_1}, \infty)$ the system motion is within the boundary layers $\{y_1(t) : |y_1(t)| \leq Y_1, |y_1(t_{M_1})| = Y_1\}$ and $\{u(t) : |u(t)| \leq U_M^*\}$. The sequence $\{|y_1(t_{M_i})|, t_{M_i} : y_2(t_{M_i}) = 0\}$ is strictly contractive and the contraction $\frac{|y_1(t_{M_{i+1}})|}{|y_1(t_{M_i})|} \leq \rho < 1$ occurs with no sign commutation of $x_2(t)$, furthermore $\sum_{k=1}^{\infty}(t_{M_{i+1}} - t_{M_i}) = T < \infty$.

Proof:

The Theorem is proved, without loss of generality, in the case $X_1 > 0$.

Assume that an upperbound U_M^* exists and it is available at $t = t_{M_1}$. It must be proved that it is possible to choose $\nu > 1$ and α such that the sequence of singular points of system (4) has the desired properties, i.e. it is contractive and all the $y_1(t_{M_i})$ have the same sign.

In the interval $[t_{M_1}, t_{M_2}]$ the actual trajectory of system (4) with the control law (7) belongs to the region of the phase plane bounded by the two curves:

$$\left\{\begin{array}{l} \dot{y}_1^{'} = y_2^{'} \\ \dot{y}_2^{'} = \left\langle \begin{array}{ll} +H^* - g_1V_M & \qquad \frac{Y_1}{2} < y_1^{'} \leq Y_1 \\ +H^* + g_2\alpha V_M & \qquad y_1^{'} \leq \frac{Y_1}{2} \end{array}\right. \end{array}\right. \tag{9}$$

$$\left\{\begin{array}{l} \dot{y}_1^{''} = y_2^{''} \\ \dot{y}_2^{''} = \left\langle \begin{array}{ll} -H^* - g_2V_M & \qquad \frac{Y_1}{2} < y_1^{''} \leq Y_1 \\ -H^* + g_1\alpha V_M & \qquad y_1^{''} \leq \frac{Y_1}{2} \end{array}\right. \end{array}\right. \tag{10}$$

with $y_2^{'}(t_{M_1}) = y_2^{''}(t_{M_1}) = 0$ and $y_1^{'}(t_{M_1}) = y_1^{''}(t_{M_1}) = Y_1$.

Consider the two limiting systems (9) and (10), it is easy to verify that

$$\begin{array}{ll} \exists t_{c_1}^{'} > t_{M_1} : & \qquad y_1^{'}(t_{c_1}^{'}) = \frac{Y_1}{2} \\ \exists t_{c_1}^{''} > t_{M_1} : & \qquad y_1^{''}(t_{c_1}^{''}) = \frac{Y_1}{2} \end{array}$$

and the following holds

$$\begin{array}{ll} t_{c_1}^{'} - t_{M_1} = \sqrt{\frac{Y_1}{g_1V_M - H^*}} & \qquad y_2^{'}(t_{c_1}^{'}) = -\sqrt{(g_1V_M - H^*)Y_1} \\ t_{c_1}^{''} - t_{M_1} = \sqrt{\frac{Y_1}{g_2V_M + H^*}} & \qquad y_2^{''}(t_{c_1}^{''}) = -\sqrt{(g_2V_M + H^*)Y_1} \\ t_{c_1}^{'} - t_{M_1} > t_{c_1}^{''} - t_{M_1} & \qquad |y_2^{''}(t_{c_1}^{''})| > |y_2^{'}(t_{c_1}^{'})| \end{array}$$

furthermore

$$\begin{array}{ll} \exists t^{'}_{M_2} > t^{'}_{c_1} : & y^{'}_2(t^{'}_{M_2}) = 0 \\ \exists t^{''}_{M_2} > t^{''}_{c_1} : & y^{''}_2(t^{''}_{M_2}) = 0 \end{array}$$

and

$$\begin{array}{ll} t^{'}_{M_2} - t^{'}_{c_1} = \frac{\sqrt{(g_1 V_M - H^*) Y_1}}{g_2 \alpha V_M + H^*} & \\ t^{''}_{M_2} - t^{''}_{c_1} = \frac{\sqrt{(g_2 V_M + H^*) Y_1}}{g_1 \alpha V_M - H^*} & \\ t^{''}_{M_2} - t^{''}_{c_1} > t^{'}_{M_2} - t^{'}_{c_1} & y^{'}_1(t^{'}_{M_2}) > y^{''}_1(t^{''}_{M_2}) \end{array}$$

If α and V_M are chosen in order to guarantee that for system (10) the condition $y^{''}_1(t^{''}_{M_2}) = y^{''}_2(t^{''}_{M_2}) = 0$ holds, then $y_1(t_{M_2})$, the second singular point of system (4), has the same sign of $y_1(t_{M_1})$, positive in this case. i The condition is satisfied, with $V_M = \nu \frac{H^*}{g_1}$ and $\nu > 1$, if

$$\alpha = \frac{2}{\nu} + \frac{g_2}{g_1}$$

Now we solve the problem of finding values of U^*_M, possibly in a semi infinite positive real interval, such that, with the parameters V_M and α chosen in order to ensure the desired property of the solution, it is also guaranteed that at least $\forall t \in [t_{M_1}, t_{M_2}]$ under the switching logic (7)

$$|u(t)| = \left| u(t_{M_1}) + \int_{t_{M_1}}^{t} v(\tau) d\tau \right| < U^*_M$$

At $t = t_{M_1}$ system (4) is in an extremal value $y_2(t_{M_1}) = 0$ and $y_1(t_{M_1}) = Y_1$, then the applied control is $v(t) = -V_M$ until the time instant $t = t_{c_1}$ when $y_1(t_{c_1}) = \frac{Y_1}{2}$. The time instant $t = t_{c_1}$ is so that

$$t^{''}_{c_1} - t_{M_1} \leq t_{c_1} - t_{M_1} \leq t^{'}_{c_1} - t_{M_1}$$

It follows that $\forall t \in [t_{M_1}, t_{c_1}]$

$$u(t) = u(t_{M_1}) - V_M (t - t_{M_1})$$

It must be guaranteed that

$$|u(t)| \leq U_{M_1} + V_M (t_{c_1} - t_{M_1})_{max} \leq U^*_M$$

From the previous inequality, considering that

$$(t_{c_1} - t_{M_1})_{max} = t^{'}_{c_1} - t_{M_1} = \sqrt{\frac{Y_1}{g_1 V_M - H^*}}$$

and that by definition

$$V_M = \nu \frac{\overline{m}_1(kY_1) + \overline{m}_2(kY_1) U^*_M}{g_1}$$

it is obtained that U^*_M must satisfy the following condition

$$\frac{\nu^2}{\nu - 1} \frac{Y_1}{g_1^2} \left[\overline{m}_1(kY_1) + \overline{m}_2(kY_1) U^*_M \right] \leq (U^*_M - U_{M_1})^2 \tag{11}$$

At $t = t_{c_1}$ according to the control switching logic (7) $v(t)$ commutes and the applied control is $v(t) = +\alpha V_M$.

During the time interval $[t_{M_1}, t_{c_1}]$ the signal $u(t)$ linearly decreases with time, therefore $u(t_{c_1}) < u(t_{M_1}) < U_{M_1}$. After the commutation of $v(t)$, in the interval $[t_{c_1}, t_{M_2}]$, the actual control $u(t)$ is given by

$$u(t) = u(t_{c_1}) + \alpha V_M (t - t_{c_1})$$

The condition to be verified is

$$|u(t)| < U_{M_1} + \alpha V_M (t_{M_2} - t_{c_1})_{max} < U^*_M$$

Considering that $t^{'}_{M_2} - t^{'}_{c_1} < t_{M_2} - t_{c_1} < t^{''}_{M_2} - t^{''}_{c_1}$ the following holds

$$(t_{M_2} - t_{c_1})_{max} = t^{''}_{M_2} - t^{''}_{c_1} = \sqrt{\frac{Y_1}{g_1 \alpha V_M - H^*}}$$

$$\frac{\alpha^2 \nu^2}{\alpha \nu - 1} \frac{Y_1}{g_1^2} \left[\overline{m}_1(kY_1) + \overline{m}_2(kY_1) U^*_M \right] \leq (U^*_M - U_{M_1})^2 \tag{12}$$

The two inequalities (11) and (12) have the same form, and they are both satisfied if U^*_M is chosen to be greater than $\overline{u}_1$ the maximal positive solution of the algebraic equation (8).

At $t = t_{M_2}$, since the validity of the contraction properties of the control Algorithm 1 have not been violated during the interval $[t_{M_1}, t_{M_2}]$, system (4) is in an extremal point $y_2(t_{M_2}) = 0$ and $Y_2 = y_1(t_{M_2}) < y_1(t_{M_1}) = Y_1$. It follows that

$$|u(t_{M_2})| < \frac{F(k Y_2) + c\, k\, Y_2}{g_1}$$

$$U_{M_2} = \frac{F(k Y_2) + c\, k\, Y_2}{g_1} < \frac{F(k Y_1) + c\, k\, Y_1}{g_1} = U_{M_1} < U^*_M$$

If U_{M_2} replaces U_{M_1} in the equation (8), the relevant solution $\overline{u}_2$ is such that $\overline{u}_2 < \overline{u}_1 < U^*_M$. Repeating this reasoning it can be concluded that the upperbound U^*_M of $|u(t)|$, calculated at $t = t_{M_1} = t_0$ on the basis of the assumptions made and of the knowledge of Y_1, is never violated in any successive interval.

As a result the drift term in (4) is proved to be overestimated by a constant $H^* = \overline{m}_1(k\, Y_1) + \overline{m}_2(k\, Y_1) U^*_M$ and all the results of previously obtained [5] can be applied.

This is sufficient to conclude that if no further trapping points are encountered, there exists a $t = T < \infty$ such that $\forall t > T$ the motion is constrained on the manifold $y_1 = 0$.

It remains to be proved that no further trapping point can occur.

After $t = t_{M_1}$, the system is no longer in a trapping point, therefore it is correctly represented by the following ODE

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = m_1(x_1, x_2) + m_2(x_1, x_2)u + g(x_1)v \\ \dot{x}_3 = m_1(x_1, x_2) - c\,x_3 + m_2(x_1, x_2)u + g(x_1)v \\ \dot{u} = v \end{cases} \tag{13}$$

Consider the trajectory of the system in the phase plane x_2, x_3. During the interval $[t_{M_1}, t_{c_1}]$ the trajectories lie in the third quadrant since $x_2(t^+_{M_1}) = x_3(t^+_{M_1}) = 0$ and $\dot{x}_3(t^+_{M_1})$ is negative. At $t = t_{c_1}$ $\dot{x}_3(t_{c_1})$ is negative, since $v(t)$ dominates the drift term of both the equations in (13), until at $t = t_{M_2}$, a new singular point is reached. The only possibility for $x_2(t)$ to change its sign is along trajectories crossing the second quadrant and reaching the first one. This is impossible not only in $[t_{M_1}, t_{M_2}]$, but even in any successive interval $[t_{M_i}, t_{M_{i+1}}]$. Indeed the constraint $y_2(t) \leq 0$ means that $x_3 + c\,x_2 \leq 0$ and therefore no phase trajectory can cross the line $x_3 + c\,x_2 = 0$. Being $x_2(t)$ always of the same sign it can be proved in a standard way that exist a time instant T such that for $\forall t \geq T$ $y_1(t) = 0$. Q.E.D.

It has been considered a mechanical system which, at $t = 0$, is in a “trapping” point. As for the regulation problem it is applied the proposed control algorithm and it is obtained the asymptotic convergence of x_1 and x_2 to zero. This result is obtained by means of a continuous control $u(t)$, Figure 1, which is the integral of a discontinuous signal $v(t)$designed following the second order sliding mode Algorithm 1 with the parameters chosen according to the conditions found in the paper. It is interesting to note the system phase trajectories. In Figure 2 it is clearly shown how the system behaviour is confined in the fourth quadrant of the plane y_1, y_2, while the trajectory in the plane $x_2, \dot{x}_2$ is presented in Figure 3.

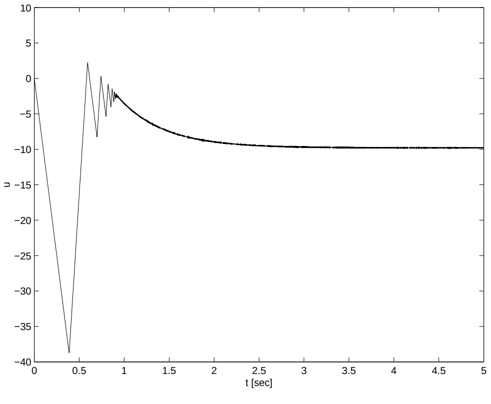

Figure 1: The continuous control signal u

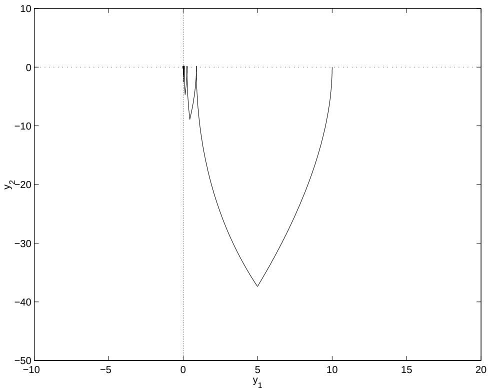

Figure 2: The trajectory of the controlled system in the phase plane y_1,y_2

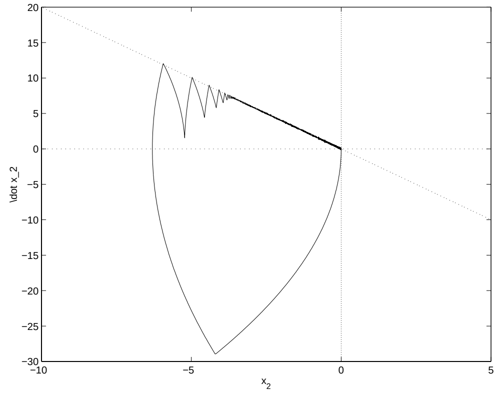

Figure 3: The trajectory of the controlled system in the phase plane x_2,$\dot{x}_2$

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