Semi-Global Stabilization of Linear Systems Subject to Output Saturation[1]

Zongli Lin & Tingshu Hu
Department of Electrical Engineering
University of Virginia
Charlottesville, VA 22903
Email: th7f, zl5y@virginia.edu

Abstract

It is established that a SISO linear system subject to output saturation can be semi-globally stabilized by linear output feedback if all its *invariant zeros* are in the closed left-half plane, no matter where the open loop poles are. This result can be viewed as dual to a well-known result: a linear system subject to input saturation can be semi-globally stabilized by linear output feedback if all its *poles* are in the open left-half plane, no matter where the invariant zeros are.

1 Introduction

Physical limitations on actuators and sensors often cause the control input and/or measurement output to saturate. In control design, it is thus necessary to take the effects of input and/or output saturation into account.

While input saturation has been addressed in much detail in the literature (see, for example, [1] and the references therein), fewer results are available that deal with output saturation. For example, issues related to the observability of a linear system subject to output saturation was discussed in detail in [3]. A discontinuous dead beat controller was recently constructed for single input single output (SISO) linear systems in the presence of output saturation [2] that drives every initial state to the origin in a finite time.

In this paper, we consider the problem of semi-globally stabilizing linear systems using *linear* feedback of the saturated output measurement. Here, by semi-global stabilization we mean the construction of a stabilizing feedback law that yields a domain of attraction that contains any *a priori* given (arbitrarily large) bounded set. This problem was motivated by its counterpart for linear systems subject to input saturation [5]. More specifically, it was established in [5] that a linear system subject to input saturation can be semi-globally stabilized using linear feedback if the system is stabilizable and detectable in the usual linear sense and all its open loop *poles* are in the closed left-half plane, no matter where the invariant zeros are. What we will show in this paper is that a single input single output linear system subject to output saturation can be semi-globally stabilized by linear output feedback if the system is stabilizable and detectable in the usual linear sense and all its *invariant zeros* are in the closed left-half plane, no matter where the open loop poles are. Although this result can be viewed as dual to its input saturation counterpart in [5], the mechanisms behind the stabilizing feedback laws are completely different. In the case of actuator saturation, we construct low gain feedback laws that avoid the saturation of the input signal for all initial states inside the *a priori* given set and the closed-loop system behaves linearly. Here in the case of output saturation, the output matrix is fixed and the output signal is always saturated for large initial states. Once the output is saturated, no information other than its sign is available for feedback. Our linear feedback laws

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are designed in such a way that they use the saturated output to cause the system output to oscillate into the linear region of output saturation function and remain in there in a finite time. The *same* linear feedback laws then stabilize the system at the origin. This is possible since all the invariant zeros are in the closed left-half plane and the feedback gains can be designed such that the overshoot of the output is arbitrarily small.

The precise problem formulation and the main results are presented in Section 2, which also concludes the paper with some simulation results.

2 Main Results

Consider the following single input single output linear system subject to output saturation,

$$\begin{cases} \dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}, \\ y &= \sigma(Cx), \quad y \in \mathbb{R}, \end{cases} \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is the standard saturation function, i.e., $\sigma(u) = \mathrm{sign}(u)\min\{\alpha, |u|\}$. Our main results on semi-global stabilizability of the system (1) is given in the following theorem.

Theorem 1 *The system (1) is semi-globally asymptotically stabilizable by linear feedback of the saturated output if*

- *The pair (A, B) is stabilizable;*
- *The pair (A, C) is detectable; and*
- *All invariant zeros of the triple (A, B, C) are in the closed left-half plane.*

More specifically, for any a priori *given bounded set $\mathcal{X}_0 \subset \mathbb{R}^{2n}$, there exists a linear dynamic output feedback law of the form*

$$\begin{cases} \dot{z} &= Fz + Gy, \quad z \in \mathbb{R}^n, \\ u &= Hz + H_0 y, \end{cases} \tag{2}$$

such that the equilibrium $(x, z) = (0, 0)$ of the closed-loop system is asymptotically stable with $\mathcal{X}_0$ contained in its domain of attraction.

Proof. We will establish this result in two steps. In the first step, we will construct a family of feedback laws of the form (2), parameterized in $\varepsilon \in (0, 1]$. In the second step, we will show that, for any *a priori* given bounded set $\mathcal{X}_0 \subset \mathbb{R}^{2n}$, there exists an $\varepsilon^* \in (0, 1]$ such that, for each $\varepsilon \in (0, \varepsilon^*]$, the equilibrium $(x, z) = (0, 0)$ of the closed-loop system is asymptotically stable with $\mathcal{X}_0$ contained in its domain of attraction.

The construction of the feedback laws follows the following algorithm.

STEP 1. Find a state transformation [6],

$$x = T\bar{x},$$
$$\bar{x} = \begin{bmatrix} x_0^{\mathrm{T}} & x_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$
$$x_1 = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1r} \end{bmatrix}^{\mathrm{T}},$$

such that the system can be written in the following form,

$$\begin{cases} \dot{x}_0 &= A_0 x_0 + B_0 x_{11}, \quad x_0 \in \mathbb{R}^{n_0}, \\ \dot{x}_{11} &= x_{12}, \\ \dot{x}_{12} &= x_{13}, \\ &\vdots \\ \dot{x}_{1r} &= C_0 x_0 + a_1 x_{11} + a_2 x_{12} + \cdots \\ &\quad + a_r x_{1r} + u, \\ y &= \sigma(x_{11}), \end{cases} \tag{3}$$

where (A_0, B_0) is stabilizable and the eigenvalues of A_0 are the invariant zeros of the triple (A, B, C) and hence are all in the closed left-half plane.

We note that the multiple input multiple output counterpart of the above canonical form will in general also require a transformation on the input and the output. The later cannot be performed due to the presence of output saturation.

STEP 2. For $\varepsilon \in (0, 1]$, let $F_0(\varepsilon)$ be such that

$$\lambda(A_0 + B_0 F_0(\varepsilon)) = \{-\varepsilon + \lambda_0(A_0)\} \cup \lambda_-(A_0),$$

where $\lambda_0(A_0)$ and $\lambda_-(A_0)$ denote respectively the sets of eigenvalues of A_0 that are on the imaginary axis and in the open left-half plane. It is clear that $A_0 + B_0 F_0(\varepsilon)$ is Hurwitz for any $\varepsilon \in (0, 1]$ and

$$\|F_0(\varepsilon)\| \le \alpha_0 \varepsilon, \quad \forall \varepsilon \in (0, 1], \tag{4}$$

for some α_0 independent of ε.

Such an $F_0(\varepsilon)$ exists since (A_0, B_0) is stabilizable. We summarize some properties for the triple $(A_0, B_0, F_0(\varepsilon))$ from [4, Lemmas 2.2.3 and 2.2.4 and Theorem 3.3.1].

Lemma 1 *For the given triple $(A_0, B_0, F_0(\varepsilon))$, there exists a nonsingular matrix $T_0(\varepsilon) \in \mathbb{R}^{n_0\times n_0}$ such that*

$$\begin{aligned} \|T_0(\varepsilon)\| &\le \tau_0, & (5)\\ \|F_0(\varepsilon)T_0^{-1}(\varepsilon)\| &\le \beta_0\varepsilon, & (6)\\ \|F_0(\varepsilon)A_0T_0^{-1}(\varepsilon)\| &\le \beta_1\varepsilon, & (7)\\ T_0(\varepsilon)(A_0+B_0F_0(\varepsilon))T_0^{-1}(\varepsilon) &= J_0(\varepsilon), & (8) \end{aligned}$$

where τ_0, β_0 and β_1 are some constants independent of ε and $J_0(\varepsilon) \in \mathbb{R}^{n_0\times n_0}$ is a real matrix. Moreover, there exists a $P_0 > 0$, independent of ε, such that,

$$J_0^{\mathrm{T}}(\varepsilon)P_0 + P_0J_0(\varepsilon) \le -\frac{\varepsilon}{2}I. \tag{9}$$

STEP 3. Let L be such that $A+LC$ is Hurwitz. Such an L exists since the pair (A, C) is detectable.

STEP 4. Construct the family of output feedback laws as follows.

$$\begin{cases} \dot z = Az+Bu+L(Cz-y), \\ u = -C_0z_0-\sum_{i=1}^r a_iz_{1i}-\frac{\alpha_1}{\varepsilon^r}(y-F_0(\varepsilon)z_0) \\ \qquad -\frac{\alpha_2}{\varepsilon^{r-1}}z_{12}-\cdots-\frac{\alpha_r}{\varepsilon}z_{1r}, \end{cases} \tag{10}$$

where z_0 and z_{1i}, $i = 1, 2, \cdots, r$, are defined as follows,

$$z = T\bar z,$$

$$\bar z = \begin{bmatrix} z_0^{\mathrm{T}} & z_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$z_1 = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1r} \end{bmatrix}^{\mathrm{T}},$$

and α_i's are chosen such that

$$s^r+\alpha_rs^{r-1}+\alpha_{r-1}s^{r-2}+\cdots+\alpha_2s+\alpha_1 = (s+1)^r,$$

i.e.,

$$\alpha_i = C_r^{i-1} = \frac{r!}{(i-1)!(r-i+1)!}, \quad i = 1, \cdots, r.$$

We now proceed with the second step of the proof: to show that, for any *a priori* given bounded set $\mathcal{X}_0 \subset \mathbb{R}^{2n}$, there exists an $\varepsilon^* \in (0, 1]$ such that, for each $\varepsilon \in (0, \varepsilon^*]$, the equilibrium $(x, z) = (0, 0)$ of the closed-loop system is asymptotically stable with $\mathcal{X}_0$ contained in its domain of attraction. Without loss of generality, let us assume that the system is already in the form of (3), i.e., $T = I$. Letting $e = x - z$, we can write the closed-loop system as follows,

$$\begin{cases} \dot x_0 = A_0x_0+B_0x_{11}, \\ \dot x_{11} = x_{12}, \\ \dot x_{12} = x_{13}, \\ \quad\vdots \\ \dot x_{1r} = C_0(x_0-z_0)+a_1(x_{11}-z_{11}) \\ \qquad +a_2(x_{12}-z_{12})+\cdots+a_r(x_{1r}-z_{1r}) \\ \qquad -\frac{\alpha_1}{\varepsilon^r}(y-F_0(\varepsilon)z_0)-\frac{\alpha_2}{\varepsilon^{r-1}}z_{12}-\cdots-\frac{\alpha_r}{\varepsilon}z_{1r}, \\ \dot z = Az+L(Cz-y) \\ \qquad +B\left[-C_0z_0-a_1z_{11}-\cdots-a_rz_{1r}\right. \\ \qquad \left.-\frac{\alpha_1}{\varepsilon^r}(y-F_0(\varepsilon)z_0)-\frac{\alpha_2}{\varepsilon^{r-1}}z_{12}-\cdots-\frac{\alpha_r}{\varepsilon}z_{1r}\right], \\ y = \sigma(x_{11}). \end{cases} \tag{11}$$

We next define a new set of state variables as follows,

$$\begin{aligned} \tilde x_0 &= T_0(\varepsilon)x_0, \\ \tilde x_{11} &= x_{11}-F_0(\varepsilon)x_0, \\ \tilde x_{1i} &= \varepsilon^{i-1}x_{1i}+C_{i-1}^1\varepsilon^{i-2}x_{1i-1} \\ &\quad +C_{i-1}^2\varepsilon^{i-3}x_{1i-2}+\cdots+C_{i-1}^{i-2}\varepsilon x_{12} \\ &\quad +C_{i-1}^{i-1}(x_{11}-F_0(\varepsilon)x_0), \quad i = 2, 3, \cdots, r, \\ e_0 &= x_0-z_0, \\ e_{1i} &= x_{1i}-z_{1i}, \; i = 1, 2, \cdots, r, \end{aligned} \tag{12}$$

and denote

$$e = \begin{bmatrix} e_0^{\mathrm{T}} & e_{11} & e_{12} & \cdots & e_{1r} \end{bmatrix}^{\mathrm{T}}.$$

With these new state variables, the closed-loop system can be written as follows,

$$\begin{cases} \dot{\tilde x}_0 = J_0(\varepsilon)\tilde x_0+T_0(\varepsilon)B_0\tilde x_{11}, \\ \dot{\tilde x}_{11} = -\frac{1}{\varepsilon}\tilde x_{11}+\frac{1}{\varepsilon}\tilde x_{12}-[F_0(\varepsilon)A_0T_0^{-1}(\varepsilon) \\ \qquad +F_0(\varepsilon)B_0F_0(\varepsilon)T_0^{-1}(\varepsilon)]\tilde x_0 \\ \qquad -F_0(\varepsilon)B_0\tilde x_{11}, \\ \dot{\tilde x}_{12} = -\frac{1}{\varepsilon}\tilde x_{12}+\frac{1}{\varepsilon}\tilde x_{13}-[F_0(\varepsilon)A_0T_0^{-1}(\varepsilon) \\ \qquad +F_0(\varepsilon)B_0F_0(\varepsilon)T_0^{-1}(\varepsilon)]\tilde x_0 \\ \qquad -F_0(\varepsilon)B_0\tilde x_{11}, \\ \quad\vdots \\ \dot{\tilde x}_{1r} = -\frac{1}{\varepsilon}\tilde x_{1r}+\frac{1}{\varepsilon}[x_{11}-\sigma(x_{11})]-\frac{1}{\varepsilon}F_0(\varepsilon)e_0 \\ \qquad +\varepsilon^{r-1}[C_0e_0+a_1e_{11}+\cdots+a_re_{1r}] \\ \qquad +\alpha_2e_{12}+\alpha_3\varepsilon e_{13}+\cdots+\alpha_r\varepsilon^{r-2}e_{1r} \\ \qquad -[F_0(\varepsilon)A_0T_0^{-1}(\varepsilon) \\ \qquad +F_0(\varepsilon)B_0F_0(\varepsilon)T_0^{-1}(\varepsilon)]\tilde x_0 \\ \qquad -F_0(\varepsilon)B_0\tilde x_{11}, \\ \dot e = (A+LC)e-L[x_{11}-\sigma(x_{11})]. \end{cases} \tag{13}$$

Choose a Lyapunov function candidate as follows,

$$V(\tilde{x}_0, \tilde{x}_{11}, \cdots, \tilde{x}_{1r}, e) = \nu \tilde{x}_0^{\mathrm{T}} P_0 \tilde{x}_0 + \sum_{i=1}^{r} \tilde{x}_{1i}^2 + \sqrt{\varepsilon} e^{\mathrm{T}} P e, \quad (14)$$

where $\nu \in (0,1]$, independent of ε, is a constant whose value is to be determined later, P_0 is as defined in Lemma 1, and $P > 0$ is such that

$$(A+LC)^{\mathrm{T}} P + P(A+LC) = -I. \quad (15)$$

Let $c > 0$, independent of ε, be such that,

$$c \geq \sup_{(x,z)\in\mathcal{X}_0, \varepsilon\in(0,1], \nu\in(0,1]} V(\tilde{x}_0, \tilde{x}_{11}, \cdots, \tilde{x}_{1r}, e). \quad (16)$$

Such a c exists due to the boundedness of $\mathcal{X}_0$ and the definition of the state variables as given by (12). With this choice of c, it is obvious that $(x,z) \in \mathcal{X}_0$ implies that $(\tilde{x}_0, \tilde{x}_{11}, \cdots, \tilde{x}_{1r}) \in L_V(c) := \{(\tilde{x}_0, \tilde{x}_{11}, \cdots, \tilde{x}_{1r}, e) \in \mathbb{R}^{2n} : V \leq c\}$.

Using Lemma 1, we can calculate the derivative of V inside the level set $L_V(c)$ along the trajectories of the closed-loop system (13) as follows,

$$\begin{aligned}
\dot{V} &= -\nu \tilde{x}_0^{\mathrm{T}} \tilde{x}_0 + 2\nu \tilde{x}_0^{\mathrm{T}} P_0 T_0(\varepsilon) B_0 \tilde{x}_{11} + \sum_{i=1}^{r} \Big[-\frac{2}{\varepsilon} \tilde{x}_{1i}^2 \\
&\quad + \frac{2}{\varepsilon} \tilde{x}_{1i} \tilde{x}_{1i+1} - 2\tilde{x}_{1i} [F_0(\varepsilon) A_0 T_0^{-1}(\varepsilon) + \\
&\quad F_0(\varepsilon) B_0 F_0(\varepsilon) T_0^{-1}(\varepsilon)] \tilde{x}_0 - 2\tilde{x}_{1i} F_0(\varepsilon) B_0 \tilde{x}_{11} \Big] \\
&\quad + 2x_{1r} \Big[\frac{1}{\varepsilon} [x_{11} - \sigma(x_{11})] - \frac{1}{\varepsilon} F_0(\varepsilon) e_0 \\
&\quad + \varepsilon^{r-1} [C_0 e_0 + a_1 e_{11} + \cdots + a_r e_{1r}] \\
&\quad + \alpha_2 e_{12} + \alpha_3 \varepsilon e_{13} + \cdots + \alpha_r \varepsilon^{r-2} e_{1r} \Big] \\
&\quad - \sqrt{\varepsilon} e^{\mathrm{T}} e - 2\sqrt{\varepsilon} e^{\mathrm{T}} P L [x_{11} - \sigma(x_{11})] \\
&\leq -\nu \tilde{x}_0^{\mathrm{T}} \tilde{x}_0 + 2\delta_{01} \nu \|\tilde{x}_0\| |\tilde{x}_{11}| \qquad (17) \\
&\quad + \sum_{i=1}^{r} \Big[-\frac{2}{\varepsilon} \tilde{x}_{1i}^2 + \frac{2}{\varepsilon} \tilde{x}_{1i} \tilde{x}_{1i+1} \\
&\quad + 2\delta_{i0} \varepsilon |\tilde{x}_{1i}| \|\tilde{x}_0\| + 2\delta_{i1} \varepsilon |\tilde{x}_{1i}| |\tilde{x}_{11}| \Big] \\
&\quad + \frac{2}{\varepsilon} |x_{1r}| |x_{11} - \sigma(x_{11})| + 2\eta_1 |\tilde{x}_{1r}| \|e\| \\
&\quad - \sqrt{\varepsilon} e^{\mathrm{T}} e + 2\eta_2 \sqrt{\varepsilon} \|e\| |x_{11} - \sigma(x_{11})|,
\end{aligned}$$

where δ_{ij}'s and η_i's are some constants, independent of ε.

We will continue our evaluation of $\dot{V}$ by considering two separated cases, $|x_{11}| \leq 1$ and $|x_{11}| > 1$.

Case 1: $|x_{11}| \leq 1$. In this case, we have,

$$\begin{aligned}
\dot{V} &\leq -\nu \tilde{x}_0^{\mathrm{T}} \tilde{x}_0 + 2\delta_{01} \nu \|\tilde{x}_0\| |\tilde{x}_{11}| \\
&\quad - \frac{1}{\varepsilon} \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1r-1} & \tilde{x}_{1r} \end{bmatrix} \\
&\quad \begin{bmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{12} \\ \vdots \\ \tilde{x}_{1r-1} \\ \tilde{x}_{1r} \end{bmatrix} \\
&\quad + \sum_{i=1}^{r} [2\delta_{i0} \varepsilon |\tilde{x}_{1i}| \|\tilde{x}_0\| + 2\delta_{i1} \varepsilon |\tilde{x}_{1i}| |\tilde{x}_{11}|] \\
&\quad + 2\eta_1 |\tilde{x}_{1r}| \|e\| - \sqrt{\varepsilon} e^{\mathrm{T}} e \\
&\leq -\left[\nu - \left(\sum_{i=1}^{r} \delta_{i0} \right) \varepsilon^2 - \delta_{01} \nu^2 \right] \|\tilde{x}_0\|^2 \qquad (18) \\
&\quad - \left[\frac{\delta_1}{\varepsilon} - \delta_{01} - \delta_{10} - 2\delta_{11} \varepsilon - \left(\sum_{i=2}^{r} \delta_{i1} \right) \varepsilon \right] \tilde{x}_{11}^2 \\
&\quad - \sum_{i=2}^{r-1} \left[\frac{\delta_1}{\varepsilon} - \delta_{i0} - \delta_{i1} \varepsilon \right] |\tilde{x}_{1i}|^2 \\
&\quad - \left[\frac{\delta_1}{\varepsilon} - \delta_{r1} \varepsilon - \frac{2\eta_1^2}{\sqrt{\varepsilon}} \right] \tilde{x}_{1r}^2 - \frac{\sqrt{\varepsilon}}{2} \|e\|^2,
\end{aligned}$$

where we have used the fact that the matrix

$$\begin{bmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

is positive definite with its maximum eigenvalue denoted as $\delta_1 > 0$.

Let ν be such that $\nu \leq \frac{1}{2\delta_{01}}$ and let $\varepsilon_1^* \in (0,1]$ be such that the following hold for all $\varepsilon \in (0, \varepsilon_1^*]$,

$$\begin{aligned}
\nu - \left(\textstyle\sum_{i=1}^{r} \delta_{i0} \right) \varepsilon^2 - \delta_{01} \nu^2 &\geq \tfrac{\nu}{4}, \\
\tfrac{\delta_1}{\varepsilon} - \delta_{01} - \delta_{10} - 2\delta_{11} \varepsilon - \left(\textstyle\sum_{i=2}^{2} \delta_{i1} \right) \varepsilon &\geq \tfrac{\delta_1}{2\varepsilon}, \\
\tfrac{\delta_1}{\varepsilon} - \delta_{i0} - \delta_{i1} \varepsilon &\geq \tfrac{\delta_1}{2\varepsilon}, \\
i = 2, 3, \ \cdots, \ r-1, & \\
\tfrac{\delta_1}{\varepsilon} - \delta_{r1} \varepsilon - \tfrac{2\eta_1^2}{\sqrt{\varepsilon}} &\geq \tfrac{\delta_1}{2\varepsilon}.
\end{aligned}$$

With these choices of ν and ε_1^*, we conclude that, for any $|x_{11}| \leq 1$ and any $\varepsilon \in (0, \varepsilon_1^*]$,

$$\dot{V} \leq -\frac{\nu}{4} \|\tilde{x}_0\|^2 - \frac{\delta_1}{2\varepsilon} \sum_{i=1}^{r} |\tilde{x}_{1i}|^2 - \frac{\sqrt{\varepsilon}}{2} \|e\|^2. \quad (19)$$

Case 2: $|x_{11}| > 1$. In this case, we have,

$$\begin{aligned}\dot V &\le -\nu\tilde x_0^{\mathrm T}\tilde x_0 + 2\delta_{01}\nu\|\tilde x_0\||\tilde x_{11}| \\ &\quad -\frac{1}{\varepsilon}\left[\tilde x_{11}^2 - (|x_{11}|-1)^2\right] - \sqrt{\varepsilon}e^{\mathrm T}e \\ &\quad +\sum_{i=1}^{r}\left[(\delta_{i0}+\delta_{i1})\varepsilon\tilde x_{1i}^2 + \delta_{i0}\varepsilon\|\tilde x_0\|^2 + \delta_{i1}\varepsilon\tilde x_{11}^2\right] \\ &\quad +\eta_1\tilde x_{1r}^2 + \left(\eta_1+\eta_2\sqrt{\varepsilon}\right)\|e\|^2 \\ &\quad +\eta_2\sqrt{\varepsilon}(|x_{11}|-1)^2. \end{aligned} \tag{20}$$

Now let $\varepsilon_2^* \in (0,1]$ be such that, for all $\varepsilon \in (0,\varepsilon_2^*]$, $(\tilde x_0, \tilde x_{11}, \cdots, \tilde x_{1r}, e) \in L_V(c)$ implies that,

$$|F_0(\varepsilon)x_0| \le \frac{1}{2}, \tag{21}$$

and

$$\begin{aligned} &2\delta_{01}\nu\|\tilde x_0\||\tilde x_{11}| \\ &+\sum_{i=1}^{r}\left[(\delta_{i0}+\delta_{i1})\varepsilon\tilde x_{1i}^2 + \delta_{i0}\varepsilon\|\tilde x_0\|^2 + \delta_{i1}\varepsilon\tilde x_{11}^2\right] \\ &+\eta_1\tilde x_{1r}^2 + \left(\eta_1+\eta_2\sqrt{\varepsilon}\right)\|e\|^2 + \eta_2\sqrt{\varepsilon}(|x_{11}|-1)^2 \\ &\le \frac{1}{8\varepsilon}. \end{aligned} \tag{22}$$

The inequality (21) is due to (4) and implies that,

$$\tilde x_{11}^2 - (|x_{11}|-1)^2 \ge \frac{1}{4}.$$

With this choice of ε_2^*, we have that, for any $|x_{11}| > 1$,

$$\dot V \le -\nu\tilde x_0^{\mathrm T}\tilde x_0 - \sqrt{\varepsilon}e^{\mathrm T}e - \frac{1}{8\varepsilon}, \quad \varepsilon \in (0,\varepsilon_2^*]. \tag{23}$$

Combining Cases 1 and 2, we conclude that, for any $\varepsilon \in (0,\varepsilon^*]$ with $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*\}$,

$$\dot V < 0, \ \ \forall (\tilde x_0, \tilde x_{11}, \tilde x_{12}, \cdots, \tilde x_{1e}, e) \in L_V(c)\setminus\{0\}, \tag{24}$$

which, in turn, shows that the equilibrium $(x,z) = (0,0)$ of the closed-loop system is asymptotically stable with $\mathcal{X}_0$ contained in its domain of attraction. □

In what follows, we will use a simple example to demonstrate the closed-loop system behavior. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^{\mathrm T}.$$

It can be easily verified that this system is controllable and observable with an invariant zero at $s = 0$. The open loop poles are located at $\{-1, \pm j, 1\}$. Following the design algorithm we proposed above, we construct a family of parameterized output feedback laws as follows,

$$\left\{\begin{array}{rcl} \dot z_1 &=& z_2 - 2(z_2 - y), \\ \dot z_2 &=& z_3 - 4(z_2 - y), \\ \dot z_3 &=& z_4 - 6(z_2 - y), \\ \dot z_4 &=& z_1 - 4(z_2 - y) + u, \\ u &=& -z_1 - \frac{1}{\varepsilon^3}(y + \varepsilon z_1) - \frac{3}{\varepsilon^2}z_3 - \frac{3}{\varepsilon}z_4. \end{array}\right. \tag{25}$$

Some simulation results are shown in Figs. 1 and 2. In the simulation, initial conditions are taken randomly as $[-4.5625, -18.8880, 8.2065, -4.0685, 34.5569, 15.4136, -25.4760, 15.8338]^{\mathrm T}$. In Fig. 1, ε is chosen to be $\varepsilon = 0.1$. It is clear that with this choice of ε, the initial conditions are not inside the domain of attraction. In Fig. 2, ε is chosen to be $\varepsilon = 0.001$. It is clear that, the output is out of saturation after some time and the closed-loop system become linear and all its states converge to zero. This demonstrates that as ε decreases, the domain of attraction is enlarged.

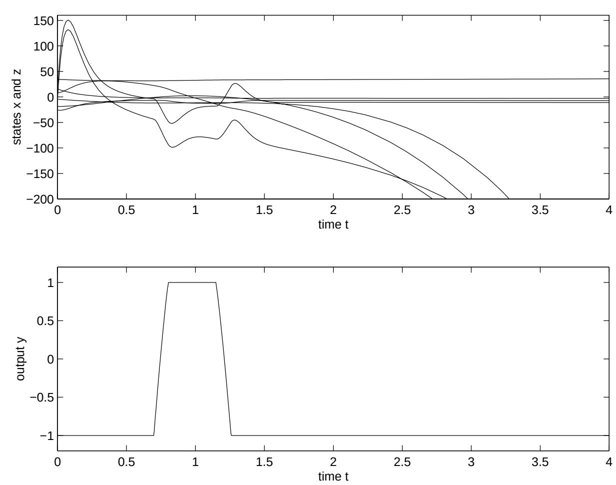

Figure 1: $\varepsilon = 0.1$

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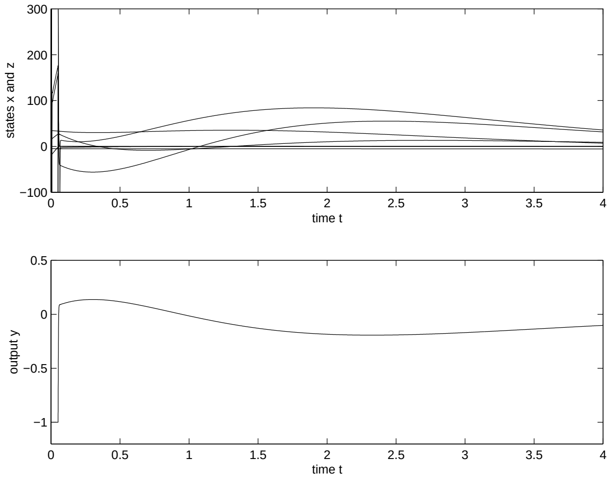

Figure 2: $\varepsilon = 0.001$

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