Stability of Digital Control Systems Subject to Jump Linear Random Perturbations

W. Steven Gray **Oscar R. González** **Sudarshan Patilkulkarni**

Department of Electrical and Computer Engineering
Old Dominion University
Norfolk, Virginia 23529-0246 U.S.A.

{gray,gonzalez}@ece.odu.edu spati001@ece.odu.edu

Abstract

In a number of applications involving fault tolerant digital control systems, there naturally arises a class of jump linear discrete-time systems characterized by having random perturbations in their drift terms. In this paper, a necessary and sufficient condition for mean square stability of such systems is developed and then applied to the stability analysis of digital flight control systems operating in electromagnetic (EM) environments. In particular, the stability degradation due to EM induced digital memory errors is examined.

1. Introduction

Consider a finite set of $n \times n$ matrices $A := \{A_0, A_1, \ldots, A_{N-1}\}$ and a Markov chain $\boldsymbol{\theta}(i)$ with states $\{0, 1, \ldots, N-1\}$ and transition probabilities Π. The classic homogeneous jump linear system is

$$\boldsymbol{x}(i+1) = A_{\theta(i)}\boldsymbol{x}(i), \quad \boldsymbol{x}(0) = \boldsymbol{x}_o, \tag{1}$$

where $\boldsymbol{x}_o$ is a vector of random variables defined on an underlying probability space $(\Omega, \mathcal{B}, P)$. (Boldface type will indicate a random variable or process.) Such systems can be viewed as a special case of the stochastic discrete-time system

$$\boldsymbol{x}(i+1) = \boldsymbol{A}(i)\boldsymbol{x}(i), \quad \boldsymbol{x}(0) = \boldsymbol{x}_o, \tag{2}$$

where $\boldsymbol{A}(i)$ denotes an $n \times n$ matrix-valued random process. Stability analysis of system (2) is a classic topic when the matrices $\{\boldsymbol{A}(i) : i \geq 0\}$ are i.i.d. [4, 5]. In the case of system (1), a very complete stability analysis also exists [2, 11]. Primarily motivated by applications, however, the following type of stochastic jump linear system is the focus of this paper:

$$\boldsymbol{x}(i+1) = \boldsymbol{A}_{\theta(i)}\boldsymbol{x}(i), \quad \boldsymbol{x}(0) = \boldsymbol{x}_o, \tag{3}$$

where $\boldsymbol{A} := \{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_{N-1}\}$ is a set of $n \times n$ *random* matrices independent of the Markov chain $\boldsymbol{\theta}(i)$ and the initial condition $\boldsymbol{x}_o$. The existing stability theory for systems (1) and (2) is not directly applicable here. Hence, in the first half of the paper, a necessary and sufficient condition for mean square stability of such systems is developed.

In the second half of the paper, the new stability theory is applied in the context of digital flight control. With the introduction of fly-by-wire systems in civilian aviation, the problem of designing, implementing, testing and certifying highly reliable computer control systems has become a major challenge to the industry and the FAA [3, 14]. Aside from the usual problems of plant uncertainty, sensor and actuator failures, and environmental uncertainties, avionic systems are also subject to electromagnetic (EM) disturbances from both natural and man-made sources [13]. Such EM disturbances can introduce transient signals on analog sensor and actuator lines, change data on digital buses and in memory, or even produce logic changes in the CPU. The result of these so called *computer upsets* is the introduction of some degradation in the quality of the control signal ranging from a perturbation error over a few sample periods to a permanent error mode or computer failure. In [7]-[9], the authors introduced an EM disturbance model which can be used for stability analysis and augmentation of a digitally implemented linear state feedback control law. The model consisted of a Markovian exosystem supplying radiation events to a discrete-time jump linear system, which modeled how the radiation interfered with the nominal operation of the closed-loop system. The statistics of the exosystem were proposed to be equivalent to those of a $(M|M|\infty)$ queue. The interference model mapped each queue state of the exosystem to a corresponding *deterministic* perturbation of the control law, which was introduced with a certain probability dependent on the queue state. It was assumed that the control law perturbations yielded a worst-case effect on the closed-loop system. In this paper the technique is now generalized to allow for stochastic perturbations, a more realistic model. One application of this extension is in modeling EM induced random bit flips in computer memory (see in particular [8, 9]). It is assumed throughout that the upset condition is mild enough to prevent the system from going into a permanent error mode. Hence, the error in the control signal as a result of EM interference will be modeled as a structural perturbation to the ideal control law.

The paper is organized as follows. In the next section, the new stability theory associated with system (3) is presented. Then in Section 3, this theory is applied to EM disturbance modeling and analysis. In the final section, some simulation results are presented.

2. Stochastic Stability Analysis

Consider a homogeneous stochastic difference equation of the form (2). It assumed throughout that the distributions for each $\boldsymbol{A}(i)$, $i \geq 0$, and $\boldsymbol{x}_o$ are such that the system has at least finite second order moments, that is, $Q(i) := E\{\mathbf{x}(i)\mathbf{x}^T(i)\}$ has finite components for each $i \geq 0$. The main notion of stochastic stability considered here is given below.

Definition 2.1 *The system (2) is* **mean square stable** *if for any initial condition* $\mathbf{x}_o$ *there exists a matrix* $Q \in \mathbb{R}^{n\times n}$ *not depending on* $\mathbf{x}_o$ *such that*

$$\|Q(i) - Q\| \to 0 \text{ as } i \to \infty,$$

where $\|\cdot\|$ *is any induced matrix norm.*

The main stability result is a necessary and sufficient condition for mean-square stability of a homogeneous jump-linear system with random A matrices. The method of proof actually builds on the deterministic A matrix case in [2].

Theorem 2.1 *Let* $\boldsymbol{\theta}(i)$ *be a finite state Markov chain with states* $\{0, 1, \ldots, N-1\}$ *and transition probabilities* Π. *Assume the random variables comprising* $\boldsymbol{x}_o$ *have finite joint second-order moments and are independent of* $\{\boldsymbol{\theta}(i) : i \in \mathcal{Z}^+\}$. *Let* $\boldsymbol{A} := \{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_{N-1}\}$ *be a set of* $n \times n$ *matrices, where each matrix has random components with finite joint moments and independent of* $\{\boldsymbol{\theta}(i) : i \in \mathcal{Z}^+\}$ *and* $\boldsymbol{x}_o$. *Then the dynamical system*

$$\boldsymbol{x}(i+1) = \boldsymbol{A}_{\theta(i)}\boldsymbol{x}(i), \quad \boldsymbol{x}(0) = \boldsymbol{x}_o$$

is mean-square stable if and only if

$$\rho(\mathcal{A}_1(\boldsymbol{A})) < 1 \ \text{a.e.},$$

where $\rho(\cdot)$ *denotes the spectral radius and*

$$\mathcal{A}_1(\boldsymbol{A}) = (\Pi^T \otimes I_{n^2})\, diag(\boldsymbol{A}_0 \otimes \boldsymbol{A}_0, \ldots, \boldsymbol{A}_{N-1} \otimes \boldsymbol{A}_{N-1}).$$

Proof: First observe the basic decomposition

$$\begin{aligned} Q(i) &:= E\left\{\boldsymbol{x}(i)\boldsymbol{x}^T(i)\right\} \\ &= \sum_{k=0}^{N-1} E\{\boldsymbol{x}(i)\boldsymbol{x}^T(i)\mathbf{1}_{\{\theta(i)=k\}}\} \\ &= \sum_{k=0}^{N-1} Q_k(i), \end{aligned}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the Dirac measure. For fixed $\boldsymbol{A} = A$ and any $k = 0, \ldots, N-1$ it follows directly that

$$\begin{aligned} Q_k(i+1|A) &= E\{\boldsymbol{x}(i+1)\boldsymbol{x}^T(i+1)\mathbf{1}_{\{\theta(i+1)=k\}}|A\} \\ &= E\left\{\left[\sum_{j=0}^{N-1} A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}}\right]\mathbf{1}_{\{\theta(i+1)=k\}}\,\middle|\, A\right\} \\ &= \sum_{j=0}^{N-1} E\left\{A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}}\mathbf{1}_{\{\theta(i+1)=k\}}\,\middle|\, A\right\} \\ &= \sum_{j=0}^{N-1} E\Big\{A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}} \\ &\qquad \cdot E\{\mathbf{1}_{\{\theta(i+1)=k\}}|\,\boldsymbol{x}(i), \boldsymbol{\theta}(i)\}\,\Big|\, A\Big\}. \end{aligned}$$

Using the fact that all statistics involving $\boldsymbol{\theta}(i+1)$ are completely known if only $\boldsymbol{\theta}(i)$ is given, the following simplifications apply:

$$\begin{aligned} Q_k(i+1|A) &= \sum_{j=0}^{N-1} E\Big\{A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}} \\ &\qquad \cdot\ E\{\mathbf{1}_{\{\theta(i+1)=k\}}|\,\boldsymbol{\theta}(i)\}\,\Big|\, A\Big\} \\ &= \sum_{j=0}^{N-1} E\Big\{A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}} \\ &\qquad \cdot P\left\{\boldsymbol{\theta}(i+1) = k \mid \boldsymbol{\theta}(i)\right\} \Big|\, A\Big\} \\ &= \sum_{j=0}^{N-1} E\left\{A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}}\Pi_{\theta(i),k}\,\middle|\, A\right\} \\ &= \sum_{j=0}^{N-1} E\left\{A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}}\,\middle|\, A\right\}\Pi_{j,k}. \end{aligned}$$

Now apply the column stacking operator $vec(\cdot)$ to both sides of the equation above.

$$\begin{aligned} \tilde{q}_k(i+1|A) &:= vec(Q_k(i+1|A)) \\ &= vec\left(\sum_{j=0}^{N-1} E\left\{A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\mathbf{1}_{\{\theta(i)=j\}}\,\middle|\, A\right\}\right)\Pi_{j,k} \\ &= \sum_{j=0}^{N-1} E\left\{vec\left(A_j\boldsymbol{x}(i)\boldsymbol{x}^T(i)A_j^T\right)\mathbf{1}_{\{\theta(i)=j\}}\,\middle|\, A\right\}\Pi_{j,k} \\ &= \sum_{j=0}^{N-1}(A_j \otimes A_j)E\left\{vec(\boldsymbol{x}(i)\boldsymbol{x}^T(i))\mathbf{1}_{\{\theta(i)=j\}}|\, A\right\}\Pi_{j,k} \\ &= \sum_{j=0}^{N-1}(A_j \otimes A_j)\,\tilde{q}_j(i|A)\Pi_{j,k}. \end{aligned}$$

Next, stacking the vectors $\tilde{q}_k(i+1|A)$ into a column vector and allowing A to be random, the following *time-invariant* stochastic difference equation results:

$$\begin{aligned} \tilde{q}(i+1|\boldsymbol{A}) &:= \left[\tilde{q}_0^T(i+1|\boldsymbol{A}), \ldots, \tilde{q}_{N-1}^T(i+1|\boldsymbol{A})\right]^T \\ &= \mathcal{A}_1(\boldsymbol{A})\,\tilde{q}(i|\boldsymbol{A}). \end{aligned}$$

Under the stated conditions, it is elementary to show that such a system is *stable in the mean* if and only if

$\rho(\mathcal{A}_1(\boldsymbol{A})) \leq 1$ a.e. (a proof is outlined in an appendix for completeness). In which case, the latter condition is equivalent to having

$$\lim_{i\to\infty} E\{\tilde{q}(i|\boldsymbol{A})\} = \lim_{i\to\infty} \tilde{q}(i) = 0,$$

which in turn is equivalent to the condition

$$\begin{aligned}\lim_{i\to\infty} Q(i) &= \lim_{i\to\infty} vec^{-1}(\tilde{q}(i)) \\ &= vec^{-1}\left(\lim_{i\to\infty} \tilde{q}(i)\right) = 0.\end{aligned}$$

Hence, the system is mean square stable if and only if $\rho(\mathcal{A}_1(\boldsymbol{A})) \leq 1$ a.e. ∎

In order to better understand the consequences of a system *not* being mean square stable, consider the following theorem adapted from [11]. (Its converse is known to be false.)

Theorem 2.2 *If the system (1) is mean square stable then for any given* $\mathbf{x}_o$ *and initial distribution* ν*, it is almost surely convergent to* 0*, i.e.,* $\mathbf{x}(i) \to 0$ *w.p. 1 as* $i \to \infty$*.*

When this result is combined with Theorem 2.1, the following lemma results.

Lemma 2.1 *In the context of Theorem 2.1,*

$$P\{\mathbf{x}(i) \not\to 0\} \leq P\{\rho(\mathcal{A}_1(\boldsymbol{A})) \geq 1\}.$$

Thus, if a system is not mean square stable, then the probability mass of $\rho(\mathcal{A}_1(\boldsymbol{A}))$ exceeding unity provides an upper bound on the probability that any given state trajectory will not converge to zero. The utility of this result is exploited in the next section.

3. EM Disturbance Modeling

In general, there are two basic parts in any disturbance model for a control problem: a model for the *exosystem* (any part of the system which is not the plant, sensors, controller or actuators) and an *interference model* which describes exactly how the exosystem interferes with the normal operation of the closed-loop system. The EM disturbance model in Figure 1 consists of a signal injection system and/or a control law perturbation for the interference model driven by a Markovian exosystem. A complete characterization of this model appears in [7]-[9]. Below a brief overview the relevant features are presented.

<ins>**The Exosystem**</ins>: The typical flight control computer operates in a complex electromagnetic environment consisting of radiation at many different frequencies, powers, and angles of incidence. A drastically simplified model for this environment is to quantize the nature of the EM disturbances in a suitable way. At any

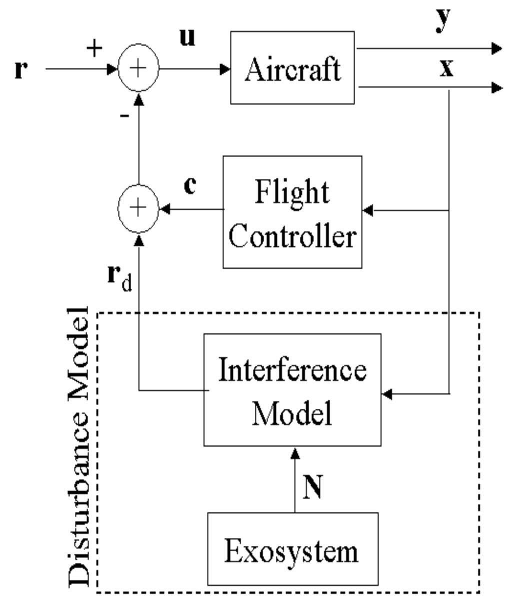

Figure 1: A closed-loop flight control system with the EM disturbance model

specific time instant $t \in \mathbb{R}$, let $\mathbf{N}(t)$ denote the number of active EM disturbances. In this simplified model, the i-th disturbance is characterized by its arrival time, $\mathbf{t}_i$, and its total duration, $\mathbf{d}_i$. We assume the random variables $\mathbf{t}_i$ constitute a Poisson process with constant parameter λ; the $\mathbf{d}_i$ have an exponential distribution with parameter μ, and $\mathbf{N}(t)$ is a memoryless Markovian continuous-time random process with transition probability rates

$$\begin{aligned}P\{\mathbf{N}(t+\Delta t) = k+1|\mathbf{N}(t) = k\} &\approx \beta_k \Delta t \\ P\{\mathbf{N}(t+\Delta t) = k-1|\mathbf{N}(t) = k\} &\approx \delta_k \Delta t.\end{aligned}$$

These so called *birth* and *death rates* determine the probability of adding or removing a disturbance in the near future given the current number of disturbances. The remaining transition probability rates are taken to be zero so that the set of all transition rates can be represented by a tridiagonal matrix Λ. It follows directly that $\beta_k = \lambda$ for all $k \geq 0$. But the death rate, for example, could either be fixed and independent of k ($\delta_k = \delta$), in which case the number of current disturbances does not affect the probability that another will be removed in the near future, or the death rate might be proportional to k ($\delta_k = k\ \delta$), then a disturbance is more likely to be removed in the near future when k is large. The first scenario may seem more heuristically appealing for our application, but in [7, 9] it is asserted that the latter case is a better model. As a consequence, the statistics of $\mathbf{N}(t)$ are assumed equivalent to those of a $(M|M|\infty)$ queue. The $(M|M|\infty)$ equilibrium state probabilities, p_k, and the transition probability rates, $\Lambda_{i,j}$, are summarized in Table 1.

Another useful exosystem characterization follows from defining the (state) events *disturbance absent* and *disturbance exists*, respectively, as

$$A := \{\mathbf{N}(t) = 0\}, \quad E := \{\mathbf{N}(t) > 0\}$$

with the corresponding transition events

$$\begin{aligned} A \mapsto E &:= \{\mathbf{N}(t+\Delta t) > 0 | \mathbf{N}(t) = 0\} \\ E \mapsto A &:= \{\mathbf{N}(t+\Delta t) = 0 | \mathbf{N}(t) > 0\}. \end{aligned}$$

The equilibrium state probabilities and the transition probability rates are also given in Table 1. Now in the event that disturbances are *rare*, the average duration of a disturbance, $1/\mu$, is much less than the average interarrival spacing of disturbance, $1/\lambda$. Thus, it follows that $\lambda \ll \mu$, and the *traffic parameter* $\rho := \lambda/\mu \ll 1$ and $p_k \approx 0$ for $k > 1$. We denote the corresponding state and transition probability rates with either a superscript or subscript r.

Equilibrium State Probabilities		Transition Probability Rates	
ρ	$\frac{\lambda}{\mu}$	$\Lambda_{k,k+1}$	λ
p_k	$e^{-\rho}\frac{\rho^k}{k!}$	$\Lambda_{k,k-1}$	$k\,\mu$
$P(A)$	$e^{-\rho}$	Λ_{AE}	λ
$P(E)$	$1-e^{-\rho}$	Λ_{EA}	$\lambda\,\frac{e^{-\rho}}{1-e^{-\rho}}$
$P(A)_r$	$1-\rho$	Λ^r_{AE}	λ
$P(E)_r$	ρ	Λ^r_{EA}	$\mu-\lambda$

Table 1: Equilibrium state probabilities and transition probability rates associated with the $(M|M|\infty)$ Markovian exosystem.

General Interference Model: It is assumed that the plant is modeled by the sampled-data system

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i) \\ y(i) &= Cx(i), \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. When no radiation is present, the nominal closed-loop system with $u(i) = r(i) - Fx(i)$ will be denoted by (A_0, B, C). The interference mapping is defined as

$$\begin{aligned} \mathcal{I} &: Z^+ \mapsto \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m} \times [0,1] \\ &: k \mapsto (\Delta F_k, G_k, p_k^u), \end{aligned}$$

where $Z^+ := \{0,1,2,\ldots\}$ and ΔF_k denotes the perturbation to the nominal state space gain matrix applied with upset probability p_k^u when $\mathbf{N}(t) = k$. The matrix G_k is a weighting for a noise sequence $\{\mathbf{w}(i) : i \in Z^+\}$ that is additively injected into the closed-loop also with probability p_k^u. Let $\boldsymbol{\theta}(i)$ denote the state of the sampled exosystem $\mathbf{N}(t)$, i.e., $\boldsymbol{\theta}(i) = \mathbf{N}(iT)$, where T is the sampling period. For sufficiently small T, $\boldsymbol{\theta}(i)$ is a discrete-time Markov process characterized by the transition probability matrix $\Pi = e^{\Lambda T}$. Given that $\mathbf{N}(t) = k$, the probability of an upset condition resulting is p_k^u. Thus, define a second discrete-time Markov process $\{\tilde{\boldsymbol{\theta}}(i) : i \in Z^+\}$ consisting of two states for each state of $\{\boldsymbol{\theta}(i) : i \in Z^+\}$, one for the upset condition and one for the no upset condition. The corresponding probability transition matrix is denoted by $\tilde{\Pi}$ (see [7, 9]). With this setup, the closed-loop input assumes the form

$$\begin{aligned} \mathbf{u}(i) &= r(i) - \left(F + \Delta F_{\tilde{\theta}(i)}\right)\mathbf{x}(i) + G_{\tilde{\theta}(i)}\mathbf{w}(i) \\ &= r(i) - (\underbrace{F\mathbf{x}(i)}_{c(i)} + \underbrace{\Delta F_{\tilde{\theta}(i)}\mathbf{x}(i) - G_{\tilde{\theta}(i)}\mathbf{w}(i)}_{\mathbf{r}_d(i)}) \end{aligned}$$

yielding a family of closed-loop systems

$$\begin{aligned} \mathbf{x}(i+1) &= A_{\tilde{\theta}(i)}\mathbf{x}(i) + B_{\tilde{\theta}(i)}\mathbf{w}(i) + Br(i) \\ \mathbf{y}(i) &= C\mathbf{x}(i), \end{aligned}$$

where for $\ell \in Z^+$:

$$A_\ell = \begin{cases} A - BF & : \ \ell \text{ even} \\ A - B(F + \Delta F_\ell) & : \ \ell \text{ odd} \end{cases}$$

$$B_\ell = \begin{cases} 0 & : \ \ell \text{ even} \\ BG_\ell & : \ \ell \text{ odd.} \end{cases}$$

The perturbations $\{\Delta F_\ell\}$ need to be characterized experimentally. In the stability analysis presented in [7]-[9], the perturbations are assumed to be fixed, i.e., deterministic. Thus, the stability boundaries for radiation parameters usually represent a worst-case analysis. For example, one might assume that all EM disturbances always introduce the worst-case bit flip error in memory. A more reasonable approach, of course, is to allow bit flips errors at random memory addresses. But this implies that the corresponding perturbation, say $\boldsymbol{\Delta F}_k$, will have some random components, and likewise for $\boldsymbol{A}_k$.

Stability Analysis of Digital Memory Upsets: Consider the situation where bit flips are randomly introduced into the memory module containing the feedback gain, F. The specific effect of such bit flips depends on the binary representation of its components. If single precision (32-bit) IEEE floating-point representation is used, then a number n is represented by $n = s \times m \times 2^e$, where s is the sign bit, m is the mantissa with binary representation $1.\underbrace{bbb\ldots b}_{23}$, and e is the 8-bit biased exponent [10]. In practice, single parity, block parity and parity check codes (linear codes) are widely used to detect and correct memory errors in fault tolerant systems. Let $n_F = [F]_{\nu,\omega}$ be the IEEE floating-point representation of the (ν,ω) component in the nominal state feedback gain matrix. Assume that a single parity bit is used for error detection, and it is not subject to bit corruption. Thus, we have a 33-bit word = 1 sign bit + 23 mantissa bits + 8 exponent bits + 1 parity bit. Also assume that only the 23 mantissa bits and the sign bit are subject to corruption, otherwise the resulting closed-loop control signal is likely to saturate. Such nonlinear effects are not handled by the present theory. For a given radiation level $k \geq 1$, the corresponding $\Delta\mathbf{F}_k$ is a random matrix where each component, $\boldsymbol{\alpha} = [\Delta\mathbf{F}_k]_{\nu\omega}$, is a random variable with a probability density function

$$f_{\nu\omega}(\alpha) = \sum_{j=0}^{47} \phi_j \delta(\alpha - \alpha_j), \tag{4}$$

where

$$\alpha_j = \begin{cases} 0 & : \ j = 0 \text{ (no bit flips)} \\ \pm 2^{e-l} & : \ j = 1 \text{ to } 23 \ (l = j) \\ & \quad \text{(single bit flip, no sign change)} \\ -2n_F & : \ j = 24 \text{ (sign bit flip)} \\ -(2n_F \pm 2^{e-l}) & : \ j = 25 \text{ to } 47 \ (l = j - 24) \\ & \quad \text{(single bit flip, sign bit flip),} \end{cases}$$

with e the exponent of n_F, $l = 1, ..., 23$ denotes the bit flip position in the mantissa, and ϕ_j is the probability associated with the j^{th} bit flip event. For equally likely, independent bit errors with probability p_e, it is easily shown that the probabilities associated with the j^{th} bit flip event are $\bar{\phi}_0 = (1 - p_e)^{24}$, $\bar{\phi}_j = p_e(1 - p_e)^{23}$ for $1 \leq j \leq 23$, $\bar{\phi}_{24} = p_e(1-p_e)^{23}$, and $\bar{\phi}_j = (p_e)^2(1-p_e)^{22}$ for $25 \leq j \leq 47$. The desired probabilities need to be normalized (conditioned) so that only this specific set of bit corruptions can occur. Thus, they are given by $\phi_j = \bar{\phi}_j / \phi_{tot}$, where $\phi_{tot} = \sum_{j=0}^{47} \bar{\phi}_j$. To apply the stability criterion in Theorem 2.1, it is only necessary to numerically generate the density function for the random variable $\rho(\mathcal{A}_1(\boldsymbol{A}))$. Since the density function for each component of $\boldsymbol{\Delta F}_k$ is discrete, so is the corresponding density function for each component of $\boldsymbol{A}_k = A - B(F + \boldsymbol{\Delta F}_k)$. The resulting necessary and sufficient condition for mean square stability is the density function for $\rho(\mathcal{A}_1(\boldsymbol{A}))$ having no probability mass exceeding one. This yields a feasible computation method for determining the stability thresholds for all the relevant radiation parameters. This idea is illustrated in the next section by example.

4. Simulation Studies

In this section a simple example is considered under the rare event scenario. The system and simulation parameters are given in Table 2. The radiation parameters are $D_\lambda := 1/(\lambda T)$ and $D_\mu := 1/(\mu T)$, which correspond respectively to the average interarrival spacing of an EM disturbance in sample periods and the average duration of a EM disturbance in sample periods. In the rare event case then $D_\mu \ll D_\lambda$.

Parameter	Value
D_λ	80
D_μ	$[3, 4]$
$\{p_0^u, p_1^u\}$	$\{0, 1\}$
p_e	0.05
T	0.001 sec
$\{A_0, A_1, A_2\}$	$\{0.999, 0.999, 0.999\}$
$\boldsymbol{A}_3$	random variable with density function in equation (4)
$\boldsymbol{x}_o$	uniform on $[-10, 10]$
Monte Carlo runs	1000
total time	100 sec

Table 2: The system and simulation parameters for a simple test example.

The simulation program uses the Monte Carlo method to generate estimates of the second order statistics $Q(i)$ as a function of time. Keeping all parameters fixed and sweeping only D_μ, it appears from Figure 2 that the radiation stability threshold is between $D_\mu = 3$ and $D_\mu = 4$. This claim is validated by simulation in Figure 3 where the second order moments are plotted over time. Finally in Figure 4, the *steady-state* values of sample paths of $\boldsymbol{x}(t)$ are plotted for each Monte Carlo run. In light of Corollary 2.1, it is expected than no more than 4.6 percent of the runs will diverge when $D_\mu = 4$, and this appears to be the case here. In the more realistic example of a stabilizing controller for the longitudinal dynamics of the AFTI/F-16 aircraft appearing in [6], this type of statistic helps quantify the probability of a system instability given a specific probability that controller memory errors can occur.

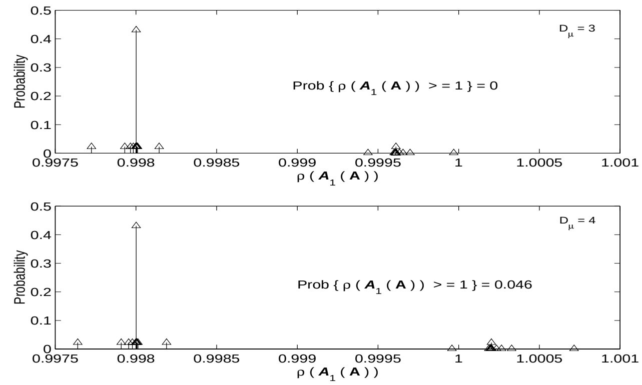

Figure 2: The density functions for $\rho(\mathcal{A}(\boldsymbol{A})_1)$ when $D_\mu = 3$ (stable) and when $D_\mu = 4$ (unstable).

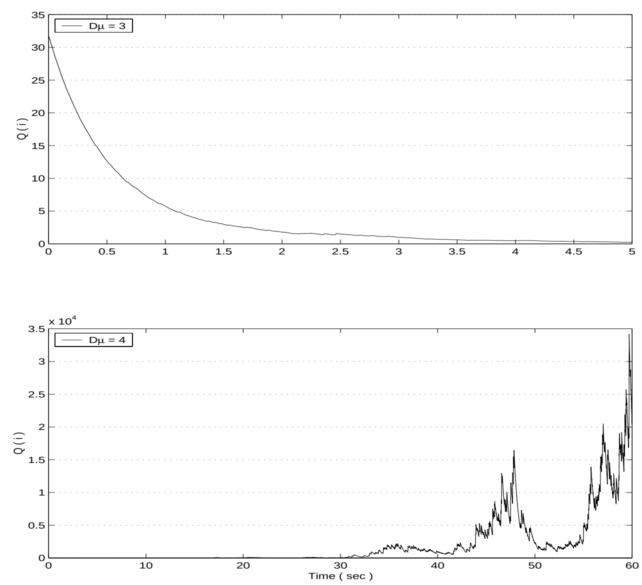

Figure 3: Second order moment estimates via Monte Carlo simulation as a function of time when $D_\mu = 3$ (stable) and when $D_\mu = 4$ (unstable).

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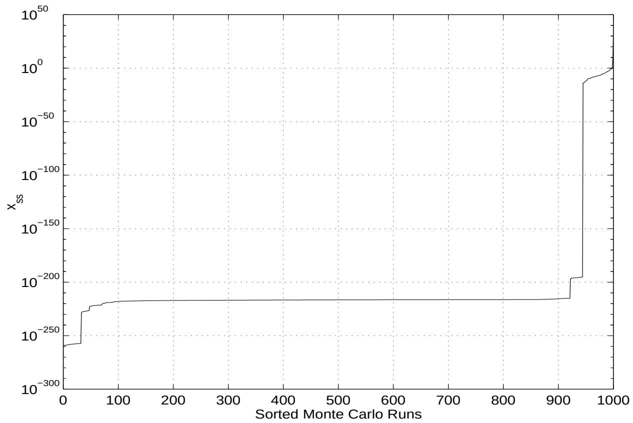

Figure 4: The convergence statistic x_{ss} for the 1000 Monte Carlo runs when $D_\mu = 4$.

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Appendix

Lemma A.1 *Let $\boldsymbol{A}$ be a random $n \times n$ matrix with finite joint moments. Then the dynamical system*

$$\boldsymbol{x}(i+1) = \boldsymbol{A}\boldsymbol{x}(i), \quad \boldsymbol{x}(0) = \boldsymbol{x}_o, \tag{5}$$

for arbitrary $\boldsymbol{x}_o$ with finite mean and with $\boldsymbol{A}$ and $\boldsymbol{x}_o$ independent is stable in the mean if and only if $P\{\rho(\boldsymbol{A}) < 1\} = 1$.

Proof:
Sufficiency: Suppose $\rho(\boldsymbol{A}) < 1$ a.e. Then $\lim_{i\to\infty} \boldsymbol{x}(i) = 0$ a.e. (componentwise). Also, for any $j = 1, 2, \ldots, n$

$$|\boldsymbol{x}_j(i)| \leq \|\boldsymbol{x}(i)\| \leq \|\boldsymbol{x}(0)\| \quad \forall i \geq 0 \;\; a.e.$$

since $\boldsymbol{A}$ acts as a contraction a.e. Now applying the Dominated Convergence Theorem componentwise [1], it follows that $\lim_{i\to\infty} E\{\boldsymbol{x}(i)\} = 0$, again in a componentwise sense. So the system is stable in the mean

Necessity: First consider the scalar case, i.e., $n = 1$. Suppose that $\lim_{i\to\infty} E\{\boldsymbol{x}(i)\} = 0$. Clearly the solution to (5) with $\boldsymbol{A} = \boldsymbol{a}$ is

$$\boldsymbol{x}(i) = \boldsymbol{a}^i \boldsymbol{x}_o.$$

Using the independence assumption and the fact that all moments of $\boldsymbol{a}$ are finite gives

$$E\{\boldsymbol{x}(i)\} = E\{\boldsymbol{a}^i\}\, E\{\boldsymbol{x}_o\}.$$

Since $E\{\boldsymbol{x}_o\}$ is arbitrary then in general

$$\lim_{i\to\infty} E\{\boldsymbol{x}(i)\} = 0 \Leftrightarrow \lim_{i\to\infty} E\{\boldsymbol{a}^i\} = 0.$$

But all subsequences of a convergent sequence converge, hence

$$\lim_{i\to\infty} E\{(\boldsymbol{a}^2)^i\} = 0.$$

Next apply Bienaymé's inequality [12] to the random variable $\boldsymbol{y} = \boldsymbol{a}^2$ such that

$$P\{\boldsymbol{a}^2 \geq 1\} \leq E\{\boldsymbol{a}^{2i}\}, \quad \forall i \geq 0$$

Thus, $P\{|\boldsymbol{a}| < 1\} = 1$, or equivalently, $\rho(\boldsymbol{a}) < 1$ a.e. This proves the scalar case. To extend this result to the general case, first transform the system to Jordan normal form and then apply the scalar results to each Jordan block. ∎