

# Robust Global Stabilization with Input Unmodeled Dynamics: An ISS Small-Gain Approach\*

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*Abstract:* This paper addresses the global asymptotic stabilization of nonlinear systems in the presence of unmodeled dynamics appearing at the input. The unmodeled dynamics are restricted to be minimum-phase and relative degree zero. On the basis of nonlinear small-gain arguments, we design a static feedback control law to achieve global asymptotic stabilization. In the absence of full state information an observer-based control design is developed.

## 1 Introduction

The robust stabilization problem for nonlinear systems with input unmodeled dynamics has been studied in the last decade. Krstić *et al.* [9], and Jiang *et al.* [6] have studied linear unmodeled dynamics, and proposed redesigns that guarantee boundedness of the closed-loop signals. These results have been extended to nonlinear unmodeled dynamics by Praly and Wang [11], and Jiang and Mareels [5]. All these studies impose a small-gain condition on the unmodeled dynamics. Alternative passivation redesigns by Janković *et al.* [4], Krstić [8] and Hamzi and Praly [3] do not require the small-gain condition, but, instead, restrict the unmodeled dynamics to be strictly passive. Arcak and Kokotović [1] replaced small-gain and passivity assumptions with the less restrictive requirement that the unmodeled dynamics subsystem be stable, relative degree zero, and minimum-phase.

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For nonlinear unmodeled dynamics, studied in [2], the minimum-phase requirement is replaced by a robust stability property for the zero dynamics. The redesigns of [1, 2] are based on a dynamic control law which requires *a priori* knowledge about the stability margin of the unmodeled dynamics.

This paper presents a static redesign which further relaxes the restrictions on the unmodeled dynamics subsystem. The key idea of the redesign is to use small-gain assignment tools developed by Jiang *et al.* [7] in combination with a change of variables which transforms the input unmodeled dynamics to state-driven unmodeled dynamics. The resulting control law guarantees boundedness of closed-loop solutions, and their convergence to a compact set around the origin which can be rendered arbitrarily small. If, in addition, the zero dynamics are locally exponentially stable (LES), then we guarantee global asymptotic stability (GAS) and LES for the closed-loop system. In the absence of full state information an observer-based variant of the design is developed for a subclass of the systems considered. An appealing feature of our redesign is that no *a priori* knowledge of stability margin is assumed about the underlying unmodeled dynamics. While, as in previous redesigns, the unmodeled dynamics are assumed to be relative degree zero and minimum phase, in this paper they are not restricted to be stable. This means that the result can be employed to simplify control designs for systems in which a certain subsystem can be treated as unmodeled dynamics. Thus, the redesign serves also to enlarge the classes of systems for which standard design methods such as backstepping and forwarding are applicable.

In Section 2 we present our redesign via state-feedback. In Section 3, we present a dynamic output-feedback design with input unmodeled dy-

namics. Conclusions are given in Section 4.

## 2 Redesign by state feedback

We refer the reader to Sontag [12, 13] for the definitions of class  $\mathcal{K}$ ,  $\mathcal{K}_\infty$  and  $\mathcal{KL}$  functions and an input-to-state stable (for short, ISS) system. A “perturbed” version of ISS is given in [7] and is defined as follows. We use  $|\cdot|$  to mean the Euclidean norm and  $\|\cdot\|$  to mean the  $\mathcal{L}_\infty$  norm.

**Definition 1** *A nonlinear control system  $\dot{x} = f(x, u)$ , with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , is said to be input-to-state practically stable (ISpS) if, for any bounded input  $u$  and any initial condition  $x(0)$ , the solution  $x(t)$  exists for every  $t \geq 0$  and satisfies*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|) + d \quad (1)$$

where  $\beta$  is of class  $\mathcal{KL}$ ,  $\gamma$  is of class  $\mathcal{K}$  and  $d \geq 0$ . When  $d = 0$  in (1), ISpS becomes ISS.

To make the main features of our redesign more apparent we first consider a scalar system

$$\dot{x} = f(x) + g(x)v \quad (2)$$

$$\dot{\xi} = A(\xi) + bu \quad (3)$$

$$v = c(\xi) + du, \quad (4)$$

where  $\xi \in \mathbb{R}^p$  is the state of unmodeled dynamics driven by the input  $u \in \mathbb{R}$ . It is assumed that  $f$ ,  $g$  are known smooth functions with  $f(0) = 0$ , and

$$g(x) \geq g_0 > 0, \quad \forall x \in \mathbb{R}. \quad (5)$$

$A(\cdot)$  and  $c(\cdot)$  are unknown functions vanishing at the origin,  $b$  is an unknown vector and  $d$  is an unknown constant. The stability properties to be analyzed are with respect to the origin  $(x, \xi) = (0, 0)$ , which is an equilibrium for the open-loop system (2)-(4).

The nominal system  $\dot{x} = f(x) + g(x)u$  is globally asymptotically stabilizable (GAS) by a smooth control law  $u = \alpha(x)$ . Our task is to redesign this nominal control law to make it robust against the destabilizing effect of the unmodeled dynamics. Thus, a new control law  $u = \vartheta(x)$  is to be designed to guarantee GAS for the system (2)-(4).

The admissible unmodeled dynamics are characterized by the following assumptions:

**(A1)** The unmodeled dynamics subsystem has relative degree zero, that is,  $d \neq 0$ . Furthermore, the sign of  $d$  is known, say,  $d > 0$ .

**(A2)** There exists a known smooth function  $\hat{c}(\cdot)$  of class  $\mathcal{K}$  such that

$$|c(\xi)| \leq \hat{c}(|\xi|), \quad \forall \xi \in \mathbb{R}^p. \quad (6)$$

Our final assumption requires a robust stability property for the zero dynamics of the  $\xi$ -subsystem (3)-(4), that is

$$\dot{z} = A(z) - \frac{1}{d}bc(z) =: A_0(z). \quad (7)$$

**(A3)** The zero-dynamics subsystem (7) disturbed by  $(w_1, w_2)$

$$\dot{z} = A_0(z + w_1) + w_2 \quad (8)$$

is ISS with respect to the input  $(w_1, w_2)$ , that is, there exist class  $\mathcal{K}$  functions  $\gamma_{01}(\cdot)$  and  $\gamma_{02}(\cdot)$  and a class  $\mathcal{KL}$  function  $\beta_0(\cdot, \cdot)$  such that the solutions of (8) satisfy

$$|z(t)| \leq \beta_0(|z(0)|, t) + \gamma_{01}(\|w_1\|) + \gamma_{02}(\|w_2\|) \quad (9)$$

**Remark 1** Clearly, (A3) is satisfied for linear unmodeled dynamics  $A(\xi) = A\xi$ ,  $c(\xi) = c\xi$  that are minimum-phase and relative degree zero.

**Proposition 1** *Under Assumptions (A1), (A2) and (A3), there exists a smooth function  $\vartheta(x)$  such that the system (2) in closed-loop with  $u = \vartheta(x) + \tilde{u}$  is ISpS with  $\tilde{u}$  as input. Furthermore, if the first-order approximation of  $\dot{z} = A_0(z)$  is asymptotically stable at  $z = 0$ , then we can render the closed-loop system ISS, and at the same time, LES when  $\tilde{u} = 0$ .*

**Proof.** Let  $b_0 := \frac{1}{d}b$ , and introduce the variable

$$\bar{\xi} = \xi - b_0 \int_0^x \frac{1}{g(s)} ds \quad (10)$$

which, from (2)-(3), is governed by

$$\dot{\bar{\xi}} = A_0(\bar{\xi} + b_0 \int_0^x \frac{1}{g(s)} ds) - b_0 \frac{f(x)}{g(x)}. \quad (11)$$

Substituting (10) into the  $x$ -subsystem (2) yields

$$\dot{x} = f(x) + g(x) \left( du + c(\bar{\xi} + b_0 \int_0^x \frac{1}{g(s)} ds) \right)$$

Thus, we have converted the *input* unmodeled dynamics  $\xi$ -subsystem into system (11) driven only by  $x$ . This important observation allows us to use gain assignment theorems [7] to design a robust controller  $u = \vartheta(x)$ .

We first consider the system (11), with  $x$  regarded as its input. Because of (A3), the  $\bar{\xi}$ -system is ISS with respect to the inputs  $w_1 = b_0 \int_0^x \frac{1}{g(s)} ds$  and  $w_2 = -b_0 \frac{f(x)}{g(x)}$ . From (5), we have

$$\left| b_0 \int_0^x \frac{1}{g(s)} ds \right| \leq \frac{1}{g_0} |b_0 x| \quad (12)$$

$$\left| b_0 \frac{f(x)}{g(x)} \right| \leq |b_0| \hat{f}(|x|) \quad (13)$$

for some smooth, class  $\mathcal{K}$  function  $\hat{f}(\cdot)$ . The inequalities (12) and (13) together with (A3) imply that the solutions  $\bar{\xi}(t)$  of (11) satisfy

$$|\bar{\xi}(t)| \leq \beta_0(|\bar{\xi}(0)|, t) + \gamma_0(\|x\|) \quad (14)$$

where  $\gamma_0(\|x\|) = \gamma_{01}(\frac{|b_0|}{g_0} \|x\|) + \gamma_{02}(|b_0| \hat{f}(\|x\|))$ . From (14) and (12), it follows that the  $\bar{\xi}$ -system is also input-to-output stable (IOS) from the input  $x$  to the output  $y_0 = c(\bar{\xi} + b_0 \int_0^x \frac{1}{g(s)} ds)$ . More precisely,

$$|y_0(t)| \leq \beta_{\text{ios}}(|\bar{\xi}(0)|, t) + \gamma_{\text{ios}}(\|x\|) \quad (15)$$

where  $\beta_{\text{ios}}(|\bar{\xi}(0)|, t) = \hat{c}(2\beta_0(|\bar{\xi}(0)|, t))$  and  $\gamma_{\text{ios}}(\|x\|) = \hat{c}(2\gamma_0(\|x\|) + \frac{2|b_0|}{g_0} \|x\|)$ . Now, consider the  $x$ -system (2), rewritten as

$$\dot{x} = f(x) + g(x)(du + y_0). \quad (16)$$

The control law  $u = \vartheta(x) + \tilde{u}$  will be chosen in such a way that the  $x$ -system is ISS with respect to the new inputs  $y_0$  and  $\tilde{u}$ , that is, there exists a class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$  and a class  $\mathcal{K}$  function  $\gamma_x(\cdot)$  such that

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_x(\|y_0\|) + \gamma_x(d\|\tilde{u}\|). \quad (17)$$

The ISS gain  $\gamma_x(\cdot)$  that will be assigned by the control law  $u = \vartheta(x) + \tilde{u}$  is to satisfy the small-gain condition  $2\gamma_x \circ 2\gamma_{\text{ios}}(s) \leq s$  for all  $s \geq s_0 \geq 0$ , or, equivalently,

$$2\gamma_{\text{ios}}(2s) \leq \gamma_x^{-1}(s), \quad \forall s \geq 0.5s_0. \quad (18)$$

When  $s_0 > 0$  in (18),  $\gamma_x(\cdot)$  can be taken as a smooth, class  $\mathcal{K}_\infty$  function that is linear around zero.

Let  $\alpha(x) = (-kx - f(x))/g(x)$  with  $k > 0$ . The time derivative of  $V(x) = \frac{1}{2}x^2$  along solutions of (16) satisfies

$$\dot{V} = -kx^2 + xg(x)(du + y_0 - \alpha(x)). \quad (19)$$

Let  $\hat{\alpha}$  be a smooth function satisfying that  $|\alpha(x)| \leq |x|\hat{\alpha}(x)$  and let  $\kappa$  be a positive constant such that  $\kappa d \geq 1$ . Choose the following smooth control law

$$\vartheta(x) = -\kappa x \hat{\alpha}(x) - 2\gamma_x^{-1}(|x|)\text{sign}(x). \quad (20)$$

Then, substituting  $u = \vartheta(x) + \tilde{u}$  in (19) gives

$$\dot{V} \leq -kx^2 + |x|g(x) \left( |y_0 + d\tilde{u}| - 2\gamma_x^{-1}(|x|) \right). \quad (21)$$

Because of Sontag's ISS algorithm [12], the ISS property (17) follows from (21).

Thanks to the small-gain condition (18) between  $\gamma_{\text{ios}}$  and  $\gamma_x$ , the proof of the first statement in Proposition 1 is completed with the help of Theorem 2.1 in [7].

The second statement can be proved as in [7, Corollary 2.3] by taking advantage of the LES condition on the first-order approximation of  $\dot{z} = A_0(z)$ . Indeed, using the LES property of the linearization of  $\dot{z} = A_0(z)$ , a direct application of [7, Lemma A.2] yields that  $\gamma_{01}$  and  $\gamma_{02}$  can be picked as  $\mathcal{K}$  functions being linearly bounded near the origin. This implies that  $\gamma_0$  and  $\gamma_{\text{ios}}$  are linearly bounded near the origin. Hence, (18) holds with  $s_0 = 0$ , leading to the desired ISS property. Finally, the LES property of the entire closed-loop system follows from the fact that the linearization of the entire closed-loop system is asymptotically stable for sufficiently large  $k$  and  $\kappa$ .  $\square$

**Corollary 1** *If the unmodeled dynamics are linear, minimum-phase and relative degree zero, then there exists a robust controller  $u = \vartheta(x)$  that achieves GAS and LES for the system (2).*

We now proceed with higher-dimensional systems of the form

$$\begin{aligned} \dot{X} &= F(X, x) \\ \dot{x} &= f(X, x) + g(X, x)v \\ \dot{\xi} &= A(\xi) + bu \\ v &= c(\xi) + du \end{aligned} \quad (22)$$

where  $X \in \mathbb{R}^{n-1}$ ,  $x \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^p$  is the state of the unmodeled dynamics system,  $g(X, x) \geq g_0 > 0$  for all  $(X, x)$ , and  $A(\cdot)$ ,  $b$ ,  $c(\cdot)$  and  $d$  satisfy Assumptions (A1)-(A3).

**Proposition 2** *Consider system (22). If there exists a smooth function  $\alpha(\cdot)$ , with  $\alpha(0) = 0$ , such that  $\dot{X} = F(X, \alpha(X))$  is GAS at  $X = 0$ , then there exists a robust controller  $\vartheta(X, x)$  such that the entire system in closed-loop with  $u = \vartheta(X, x) + \tilde{u}$  is ISpS with respect to the input  $\tilde{u}$ . Furthermore, if the first-order approximation of both  $\dot{X} = F(X, \alpha(X))$  and  $\dot{z} = A_0(z)$  are LES at the origin, then we can render the closed-loop system ISS, and at the same time, LES when  $\tilde{u} = 0$ .*

**Proof.** From Theorem 1 in [13], the GAS property for the  $X$ -system implies the existence of a smooth real-valued function  $K(\cdot)$ ,  $K(0) = 0$ , and a smooth globally invertible function  $G(X)$ , with  $0 < |G(x)| \leq 1$ , such that  $\dot{X} = F(X, K(X) + G(X)u)$  is ISS with input  $u$ .

As in the proof of Proposition 1, we introduce a change of variables

$$\bar{x} = \frac{x - K(X)}{G(X)} \quad (23)$$

$$\bar{\xi} = \xi - b_0 \int_0^x \frac{1}{g(X, s)} ds. \quad (24)$$

The mapping  $(X, x, \xi) \mapsto (X, \bar{x}, \bar{\xi})$  is a global diffeomorphism preserving the origin. In the new coordinates  $(X, \bar{x}, \bar{\xi})$ , the system (21) is rewritten as

$$\begin{aligned} \dot{X} &= F(X, K(X) + G(X)\bar{x}) \\ \dot{\bar{x}} &= \frac{f(X, K(X) + G(X)\bar{x})}{G(X)} + \left[ (K(X) + G(X)\bar{x}) \right. \\ &\quad \left. \frac{\partial}{\partial X} \left( \frac{1}{G(X)} \right) - \frac{\partial}{\partial X} \left( \frac{K(X)}{G(X)} \right) \right] \\ &\quad F(X, K(X) + G(X)\bar{x}) + \frac{g(X, K(X) + G(X)\bar{x})}{G(X)} \\ &\quad \left( du + c(\bar{\xi} + b_0 \int_0^{K(X) + G(X)\bar{x}} \frac{ds}{g(X, s)}) \right) \\ \dot{\bar{\xi}} &= A_0 \left( \bar{\xi} + b_0 \int_0^{K(X) + G(X)\bar{x}} \frac{ds}{g(X, s)} \right) \\ &\quad - b_0 \frac{f(X, K(X) + G(X)\bar{x})}{g(X, K(X) + G(X)\bar{x})} \\ &:= A_0(\bar{\xi} + w_1) + w_2 \end{aligned}$$

Thanks to (A3), the above  $\bar{\xi}$ -system is ISS with respect to the inputs  $(w_1, w_2)$ .

Then, it follows that the  $\bar{\xi}$ -subsystem is ISS with  $X$  and  $\bar{x}$  considered as inputs. For notational simplicity, denote  $z = (X, \bar{\xi})$ . Thus, as a cascade-interconnection of two ISS systems, the  $z$ -system is ISS with respect to the input  $\bar{x}$ . That is, there exist a class  $\mathcal{KL}$  function  $\beta_z(\cdot, \cdot)$  and a class  $\mathcal{K}$  function  $\gamma_z(\cdot)$  such that

$$|z(t)| \leq \beta_z(|z(0)|, t) + \gamma_z(\|\bar{x}\|). \quad (25)$$

The rest of the proof is similar to the proof of Proposition 1.  $\square$

### 3 Redesign by output-feedback

In this section, we deal with a class of dynamically input-perturbed *output feedback* systems which are described by

$$\begin{aligned} \dot{z} &= q(z, y) \\ \dot{x}_1 &= x_2 + \phi_1(z, y) \\ &\vdots \\ \dot{x}_n &= v + \phi_n(z, y) \\ \dot{\xi} &= A(\xi) + b\sigma(y)u \\ v &= c(\xi) + d\sigma(y)u \\ y &= x_1 \end{aligned} \quad (26)$$

where  $y = x_1 \in \mathbb{R}$  is the output,  $z \in \mathbb{R}^{p_1}$ ,  $x \in \mathbb{R}^n$ ,  $\xi \in \mathbb{R}^{p_2}$ , and  $A(\cdot)$ ,  $b$ ,  $c(\cdot)$  and  $d$  are as in (2), and fulfill Assumptions (A2) and (A3). We assume that  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is a globally invertible function. It is further assumed that

**(A7)** The  $z$ -subsystem of (26) is ISS when  $y$  is considered as its input.

**(A8)** For each  $1 \leq i \leq n$ , there exist two known smooth, class  $\mathcal{K}$  functions  $\varphi_{i1}(\cdot)$  and  $\varphi_{i2}(\cdot)$  such that

$$|\phi_i(z, y)| \leq \varphi_{i1}(|z|) + \varphi_{i2}(|y|). \quad (27)$$

Our control task is to find a dynamic output-feedback law of the type

$$u = \vartheta(y, \chi), \quad \dot{\chi} = \varpi(y, \chi) \quad (28)$$

which globally asymptotically stabilizes (26) at the origin. We treat the cases  $n = 1$  and  $n > 1$  separately. When  $n = 1$ , a slight modification of the design in the previous section yields a static

output-feedback control law  $u = \vartheta(y)$ . This result is stated below, whose proof is omitted due to the page limitation.

**Theorem 1** *Under Assumptions (A2), (A3), (A7) and (A8), if there is a known constant  $\sigma_0 > 0$  such that  $\sigma(y) \geq \sigma_0$  for all  $y$ , then we can find a smooth static output-feedback law  $u = \vartheta(y)$  that drives the output  $y$  to an arbitrarily small neighborhood of the origin while guaranteeing the boundedness of all closed-loop signals. Furthermore, if the first-order approximation of both  $\dot{z} = q(z, 0)$  and  $\dot{\bar{\xi}} = A_0(\bar{\xi})$  are asymptotically stable, then the origin of the closed-loop system is GAS.*

When  $n > 1$ , output-feedback global stabilization is achieved by restricting the unmodeled dynamics subsystem to be linear.

(A9) There exist a constant matrix  $\bar{A}$  and a constant vector  $\bar{c}$  such that  $A(\xi) = \bar{A}\xi$  and  $c(\xi) = \bar{c}\xi$ . In addition,  $d > 0$  and  $\bar{A} - b\bar{c}/d$  is an asymptotically stable matrix.

We further assume that  $\sigma(y)$  is known, and, without loss of generality,  $\sigma(y) = 1$ .

**Theorem 2** *Under Assumptions (A7), (A8) and (A9), for the system (26) with  $n > 1$  and  $\sigma(y) = 1$ , we can find a smooth dynamic output-feedback law (28) that drives the output  $y$  to an arbitrarily small neighborhood of the origin while guaranteeing the boundedness of all closed-loop signals. Furthermore, if the first-order approximation of  $\dot{z} = q(z, 0)$  is asymptotically stable, then the origin of the closed loop system (26) and (28) is GAS.*

**Proof.** Our design starts with the change of variables  $\bar{\xi} = \xi - \sum_{i=1}^n \theta_i x_i$  with  $\theta_n = \frac{b}{d}$  and  $\theta_{i-1} = \bar{A}\theta_i \forall i = 2, \dots, n$ . Then, direct computation yields

$$\dot{\bar{\xi}} = (\bar{A} - b\bar{c}/d)\bar{\xi} + \left( \bar{A}\theta_1 y - \sum_{i=1}^n (\theta_n \bar{c} + I_p)\theta_i \phi_i(z, y) \right)$$

where  $I_p$  is the  $p \times p$  identity matrix. Hence, the system (26) can be rewritten as

$$\begin{aligned} \dot{z} &= q(z, y) \\ \dot{x} &= Fx + dG(u + \bar{c}\bar{\xi}/d) + \phi(z, y) \\ \dot{\bar{\xi}} &= (\bar{A} - b\bar{c}/d)\bar{\xi} + (\bar{A}\theta_1 y - \sum_{i=1}^n (\theta_n \bar{c} + I_p)\theta_i \phi_i(z, y)) \\ y &= x_1 \end{aligned} \quad (29)$$

where  $\phi = (\phi_1, \dots, \phi_n)^T$ ,  $F$  and  $G$  are constant matrices defined as

$$F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{c}\theta_1 & \bar{c}\theta_2 \cdots \bar{c}\theta_n \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (30)$$

Defining  $\tilde{x} = \frac{1}{d}Px$  with  $P = (p_{ij})$  a lower-triangular matrix of order  $n$  satisfying that  $p_{ii} = 1$  and  $p_{ij} = 0$  for each  $1 \leq i \leq n$  and  $j > i$ , it is not difficult to show that the values of  $p_{ij}$ ,  $i > j$ , can be chosen so that

$$PFP^{-1} = \begin{bmatrix} -p_{21} & & & \\ \vdots & I_{n-1} & & \\ -p_{n1} & & & \\ \bar{c}\theta_1 & 0 \cdots 0 & & \end{bmatrix}. \quad (31)$$

Letting

$$\psi_i(z, y) = -p_{i+1,1}y + \sum_{j=1}^i p_{ij}\phi_j(z, y) \quad (32)$$

$$\psi_n(z, y) = \bar{c}\theta_1 y + \sum_{j=1}^n p_{nj}\phi_j(z, y) \quad (33)$$

$\forall 1 \leq i \leq n-1$ , it follows that

$$\dot{\tilde{x}} = A_c \tilde{x} + G(u + \bar{c}\bar{\xi}) + \psi(z, y)/d \quad (34)$$

where  $A_c$  is the  $n \times n$  matrix with ones on the upper-diagonal and zeros elsewhere. Note that the nominal part of the  $\tilde{x}$ -system (34) is in the controllable canonical form. Also note that  $\tilde{x}_1 = y/d$ . We follow the global stabilization algorithm in [10] to introduce the observer-like dynamic system

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \ell_i(y - \hat{x}_1), \quad 1 \leq i \leq n-1 \\ \dot{\hat{x}}_n &= u + \ell_n(y - \hat{x}_1) \end{aligned} \quad (35)$$

where the design parameters  $\ell_i$  are chosen in such a way that the observation error  $e = \tilde{x} - \hat{\tilde{x}}$  (i.e. each component is  $e_i = \tilde{x}_i - \hat{\tilde{x}}_i$ ) satisfies

$$\dot{e} = Me - L(1 - \frac{1}{d})y + G\bar{c}\bar{\xi} + \psi(z, y)/d \quad (36)$$

with  $M$  an asymptotically stable matrix and  $L = (\ell_1, \dots, \ell_n)^T$ .

In view of (29), (34), (35) and (36), we obtain the following system, which is in a suitable form for our robust control redesign

$$\begin{aligned}
\dot{z} &= q(z, y) \\
\dot{\bar{\xi}} &= (\bar{A} - b\bar{c}/d)\bar{\xi} \\
&\quad + (\bar{A}\theta_1 y - \sum_{i=1}^n (\theta_n \bar{c} + I)\theta_i \phi_i(z, y)) \\
\dot{e} &= Me - L(1 - \frac{1}{d})y + G\bar{c}\bar{\xi} + \psi(z, y)/d \\
\dot{y} &= d\hat{x}_2 + de_2 + \psi_1(z, y) \\
\dot{\hat{x}}_2 &= \hat{x}_3 + \ell_2(1 - \frac{1}{d})y + \ell_2 e_1 \\
&\vdots \\
\dot{\hat{x}}_n &= u + \ell_n(1 - \frac{1}{d})y + \ell_n e_1. \tag{37}
\end{aligned}$$

To complete our robust redesign using gain assignment theorems of [7], it is important to note that the information of partial state  $(y, \hat{x}_2, \dots, \hat{x}_n)$  is available to the designer and that the cascaded  $(z, \bar{\xi}, e)$ -system is ISS when  $y$  is considered as the input. With these remarks in mind, a repeated application of Proposition 1 gives a robust nonlinear controller  $\vartheta(y, \hat{x}_2, \dots, \hat{x}_n)$  that solves the global output feedback stabilization problem.  $\square$

## 4 Concluding remarks

We have presented a redesign that achieves robust stabilization in the presence of input unmodeled dynamics. The class of unmodeled dynamics is restricted to be relative degree zero, and to have ISS zero dynamics. An interesting feature of our redesign is that, in contrast to previous results, it does not require any *a priori* knowledge of stability margin on the unmodeled dynamics. (In fact, unmodeled dynamics are not required to be stable). The use of our small-gain redesign in conjunction with the observer design in [10] led to an output-feedback scheme that ensures robustness against input unmodeled dynamics. A new research task would be to compare the closed-loop performance of our static redesign with that of the dynamic redesign in [2].

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