Further Results for Systems with Repeated Scalar Nonlinearities

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Abstract

The class of nonlinear systems described by a discrete-time state-equation containing a repeated scalar nonlinearity is considered. The paper sharpens several previous results on the performance synthesis and model reduction for such systems. Extensions to the case that the nonlinearity is not an odd function are also discussed.

Keywords: Diagonally dominant matrices, Stieltjes matrices, M-matrices, Linear Matrix Inequalities, Integral Quadratic Constraints, recurrent neural networks, model reduction.

1 Introduction

In this paper we consider the discrete-time nonlinear system described by

$$x_i^+ = \sum_{j=1}^{n} a_{ij}\phi(x_j) + \sum_{j=1}^{m} b_{ij}u_j, \quad i = 1,2,\ldots,n \quad (1)$$

$$y_i = \sum_{j=1}^{n} c_{ij}\phi(x_j) + \sum_{j=1}^{m} d_{ij}u_j, \quad i = 1,2,\ldots,p \quad (2)$$

The superscript + denotes the time-shift, i.e. $x_{i[k]}^+ = x_{i[k+1]}$ for any time instant k, and $\phi : \mathbb{R} \to \mathbb{R}$ is an odd and 1-Lipschitz nonlinear function:

$$\forall s \in \mathbb{R}, \qquad \phi(-s) = -\phi(s) \quad (3)$$

$$\forall s,t \in \mathbb{R}, \qquad |\phi(s) - \phi(t)| \leq |s - t| \quad (4)$$

For convenience, we call such a system a ϕ-system and abbreviate (1), (2) to

$$x^+ = A\phi(x) + Bu \quad (5)$$

$$y = C\phi(x) + Du \quad (6)$$

or in matrix form

$$\begin{bmatrix} x^+ \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \phi(x) \\ u \end{bmatrix} \quad (7)$$

where $u \in \mathbb{R}^m$ is the input, $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output and $\phi(x)$ actually means

$$\begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_n) \end{bmatrix}$$

Note that the same nonlinearity acts on every component of the state. A ϕ-system may be regarded as a discrete-time recurrent neural network and can approximate a general class of nonlinear system in a certain sense. In [1, 2], various properties of the ϕ-system, including its stability, induced norm, model reduction and controller synthesis, have been extensively studied using the method of positive definite diagonally dominant matrices and Linear Matrix Inequalities. The objective of this paper is to further sharpen and generalize some of those results.

2 Basics

For a matrix

$$M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

where P, S are square and S is invertible, the Schur complement of S in M, denoted by M/S, is defined as

$$P - QS^{-1}R$$

(The notation M/S is standard in the matrix theory and it is known that $M/S = (M/\bullet)/(S/\bullet)$ if $\bullet$ is a principal submatrix of S[3].)

Definition 1 A matrix P is said to be positive diagonally dominant (pdd for short) if P is (symmetric) positive definite (denoted by $P > 0$) and (row) diagonally dominant:

$$\forall i, \quad |p_{ii}| \geq \sum_{j \neq i} |p_{ij}| \tag{8}$$

◁

It is known that if P is pdd, then all its principal submatrices and their Schur complements are pdd. (A proof of the latter can also be found in [2].) However, the inverse of P need not be pdd.

Definition 2 The ϕ-system (7) is said to be dd-stable and have a dd-gain $< \gamma$ if there exist a pdd P and a positive number γ satisfying the following LMI (Linear Matrix Inequality):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I_m \end{bmatrix} < 0 \tag{9}$$

(where the superscript $*$ denotes the transpose). ◁

According to [1], a ϕ-system that satisfies the condition in Definition 2 is stable and its ℓ_2-ℓ_2 induced gain $< \gamma$.

3 A New Necessary Condition for the Performance Synthesis Problem

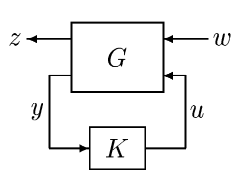
The performance synthesis problem in [2] may be stated as follows. Given a ϕ-plant G

$$G \; : \; \begin{bmatrix} x^+ \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \phi(x) \\ w \\ u \end{bmatrix} \tag{10}$$

which is not necessarily stable, we want to design a ϕ-controller K

$$K \; : \; \begin{bmatrix} \xi^+ \\ u \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} \phi(\xi) \\ y \end{bmatrix} \tag{11}$$

such that the closed-loop



(12)

is dd-stable and has a dd-gain $< \gamma$. $w \in \mathbb{R}^{m_1}$, $u \in \mathbb{R}^{m_2}$, $z \in \mathbb{R}^{p_1}$, $y \in \mathbb{R}^{p_2}$ are the external input, control input, regulated output and measured output of the plant respectively. As in the standard $\mathcal{H}_\infty$-control problem, we may assume $D_{22} = 0$ without loss of generality.

The solution to the performance synthesis problem given in [2] is essentially the following:

Proposition 1 Given the ϕ-plant G in (10) where $D_{22} = 0$, $\check{U} := \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}$ has full column rank and $\check{V} := \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ has full row rank. Let $\check{U}_\perp$, $\check{V}_\perp$ be such that $\begin{bmatrix} \check{U} & \check{U}_\perp \end{bmatrix}$, $\begin{bmatrix} \check{V} \\ \check{V}_\perp \end{bmatrix}$ are invertible and $\check{U}^* \check{U}_\perp = 0$, $\check{V} \check{V}_\perp^* = 0$. Then there exists a ϕ-controller K of the form (11) and order r such that the closed-loop (12) is dd-stable and has a dd-gain $< \gamma$ if and only if there exist X, X_{12}, X_{22}, Y, Y_{12}, Y_{22} such that X_{22} is $r \times r$,

$$\hat{X} := \begin{bmatrix} X & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \text{ is pdd} \tag{13}$$

$$\hat{X}^{-1} = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix} \tag{14}$$

and

$$\check{V}_\perp \left(M^* \begin{bmatrix} X & 0 \\ 0 & \gamma^{-1} I_{p_1} \end{bmatrix} M - \begin{bmatrix} X & 0 \\ 0 & \gamma I_{m_1} \end{bmatrix} \right) \check{V}_\perp^* < 0 \tag{15}$$

$$\check{U}_\perp^* \left(M \begin{bmatrix} Y & 0 \\ 0 & \gamma^{-1} I_{m_1} \end{bmatrix} M^* - \begin{bmatrix} Y & 0 \\ 0 & \gamma I_{p_1} \end{bmatrix} \right) \check{U}_\perp < 0 \tag{16}$$

where $M = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}$. ◁

It is noted that the key to the solution is the so-called "dilation problem of pdd matrices", i.e. given a pdd matrix X, how to dilate it to a larger pdd matrix $\hat{X}$ such that the upper-left corner of the inverse of $\hat{X}$ is fixed. This problem is difficult. Necessary and sufficient conditions are not available, and only sufficient conditions and necessary conditions were given in [2]. The best necessary condition obtained in [2] is $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$, X pdd and Y^{-1} pdd, which follows from the fact that $Y^{-1} = \hat{X}/X_{22}$. A tighter necessary condition is now given below.

Lemma 2 Given X, Y. If there exist X_{12}, X_{22}, Y_{12},

Y_{22} such that (13), (14) hold, then $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$ and

$$\begin{bmatrix} Y^{-1}+X & Y^{-1}-X \\ Y^{-1}-X & Y^{-1}+X \end{bmatrix} \text{ is pdd} \qquad (17)$$

Furthermore, if X_{22} is $r \times r$,

$$\text{rank}(X - Y^{-1}) \leq r \qquad (18)$$

◁

This is indeed a tighter condition since (17) essentially means that not only X and Y^{-1}, but the matrix composed of the smaller diagonal entries and larger off-diagonal entries of X and Y^{-1} is still pdd.

Proof. It suffices to check that $Y^{-1} = \hat{X}/X_{22}$ and the matrix in (17) is 2 times the Schur complement of $2X_{22}$ in

$$\begin{bmatrix} X & 0 & X_{12} \\ 0 & X & X_{12} \\ X_{12}^* & X_{12}^* & 2X_{22} \end{bmatrix}$$

which is pdd. ■

4 Two New Sufficient Conditions for the Model Reduction Problem

The model reduction problem in [1] may be stated as follows. Given a stable ϕ-system G

$$G \;:\; \begin{bmatrix} x^+ \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \phi(x) \\ u \end{bmatrix} \qquad (19)$$

of order n, we want to find a ϕ-model G_r

$$G_r \;:\; \begin{bmatrix} \xi^+ \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} \phi(\xi) \\ u \end{bmatrix} \qquad (20)$$

of order r, $r < n$, such that the error system $E := G - G_r$ is dd-stable and has a dd-gain $< \gamma$.

The following notation is introduced for convenience. For a $n \times n$ matrix M, let $(M)_{11}$, $(M)_{22}$, $(M)_1$, $(M)_2$ denote the top-left $r \times r$ submatrix, the bottom-right $(n-r) \times (n-r)$ submatrix, the first r columns, and the last $n-r$ columns of M respectively. Then according to [1], we have the following sufficient condition for the model reduction problem:

Proposition 3 Given the ϕ-system G in (19) of order n. Suppose there exist Q, P satisfying

$$A^*QA - Q + C^*C/\sigma \;<\; 0 \qquad (21)$$
$$APA^* - P + BB^*/\sigma \;<\; 0 \qquad (22)$$

such that

$$\begin{bmatrix} P^{-1}+Q & P^{-1}-Q \\ P^{-1}-Q & P^{-1}+Q \end{bmatrix} \text{ is pdd} \qquad (23)$$

and

$$(Q - P^{-1})_2 = 0 \qquad (24)$$

Then there exists a ϕ-model G_r of the form (20) and order r such that the error system $E := G - G_r$ is dd-stable and has a dd-gain $< 2\sigma$. ◁

The condition in Proposition 3 is a standard LMI problem but it depends on the particular ordering of the state-components. To be general, we may replace (24) by the condition that the null space of $Q - P^{-1}$ contains a $(n-r)$-dimensional coordinate subspace as in [1], which is then not convex, however. For ease of comparison with the new results, we leave the condition to be as shown in Proposition 3. Methods to obtain a suitable ordering of the state-components can be found in [1].

Two new sufficient conditions for the model reduction problem will be presented here. The first one says that if $Q - P^{-1}$ is furthermore non-negative diagonal, then we can actually find a reduced-order ϕ-model such that the error is bounded by σ instead of 2σ. The second one relaxes the requirement that $(Q - P^{-1})_2 = 0$ in Proposition 3.

Proposition 4 Given the ϕ-system G in (19) of order n. Suppose there exist Q, P satisfying (21), (22) such that

$$P^{-1} \text{ is pdd} \qquad (25)$$
$$(Q - P^{-1})_{11} \text{ is positive diagonal} \qquad (26)$$

and (24) holds. Then there exists a ϕ-model G_r of the form (20) and order r such that the error system $E := G - G_r$ is dd-stable and has a dd-gain $< \sigma$. ◁

Note that (23) is implied by (25), (26), (24) and therefore the condition in Proposition 4 is stronger than the condition in Proposition 3, but in return the error bound is reduced by half.

Proof. The model reduction problem may be regarded as a special performance synthesis problem with the plant

$$\hat{G} \;:\; \begin{bmatrix} x^+ \\ y - \tilde{y} \\ u \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ C & D & -I_p \\ 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} \phi(x) \\ u \\ \tilde{y} \end{bmatrix} \qquad (27)$$

As demonstrated by Beck *et al.* in the linear case[4], we may choose $\check{U}_\perp$, $\check{V}_\perp$ in Proposition 1 to be $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$,

$\begin{bmatrix} I_n & 0 \end{bmatrix}$ respectively due to the special forms of B_2, C_2, D_{12}, D_{21}, and then (15), (16) reduce to

$$A^*XA - X + C^*C/\gamma < 0 \quad (28)$$
$$AYA^* - Y + BB^*/\gamma < 0 \quad (29)$$

which are essentially (21), (22) with $X = Q$, $Y = P$ and $\gamma = \sigma$. Let

$$\hat{X} := \begin{bmatrix} Q & -(Q - P^{-1})_1 \\ -(Q - P^{-1})_1^* & (Q - P^{-1})_{11} \end{bmatrix} \quad (30)$$

It follows from (25), (26) that $\hat{X}$ is diagonally dominant. Both $(Q - P^{-1})_{11}$ and $\hat{X}/(Q - P^{-1})_{11} = P^{-1}$ invertible implies $\hat{X}$ invertible and hence pdd, and

$$\hat{X}^{-1} = \begin{bmatrix} P & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix} \quad (31)$$

for some Y_{12}, Y_{22}. The proof is complete by noting that $(Q - P^{-1})_{11}$ is $r \times r$. ■

Proposition 5 Given the ϕ-system G in (19) of order n. Suppose there exist Q, P satisfying

$$A^*(P^{-1} + Q)A - (P^{-1} + Q) + C^*C/\sigma + (P^{-1} - Q)_2(P^{-1} + Q)_{22}^{-1}(P^{-1} - Q)_2^* < 0 \quad (32)$$
$$A^*QA - Q < 0 \quad (33)$$
$$APA^* - P + BB^*/\sigma < 0 \quad (34)$$

such that (23) holds. Then there exists a ϕ-model G_r of the form (20) and order r such that the error system $E := G - G_r$ is dd-stable and has a dd-gain $< 2\sigma$. ◁

Proof. (23) implies that

$$\hat{X} := \frac{1}{2}\left(\begin{bmatrix} P^{-1} + Q & P^{-1} - Q \\ P^{-1} - Q & P^{-1} + Q \end{bmatrix} / (P^{-1} + Q)_{22}\right) \quad (35)$$

is pdd. Also

$$\hat{X}^{-1} = \frac{1}{2}\begin{bmatrix} Q^{-1} + P & -(Q^{-1} - P)_1 \\ -(Q^{-1} - P)_1^* & (Q^{-1} + P)_{11} \end{bmatrix} \quad (36)$$

Hence, letting X, Y be the top-left $n \times n$ submatrices of $\hat{X}$, $\hat{X}^{-1}$ respectively:

$$X = \frac{1}{2}(P^{-1} + Q) - \frac{1}{2}(P^{-1} - Q)_2(P^{-1} + Q)_{22}^{-1}(P^{-1} - Q)_2^* \quad (37)$$
$$Y = \frac{1}{2}(Q^{-1} + P) \quad (38)$$

and putting them and $\gamma = 2\sigma$ into (28), (29) gives

$$A^*(P^{-1} + Q)A - (P^{-1} + Q) + C^*C/\sigma - A^*(P^{-1} - Q)_2(P^{-1} + Q)_{22}^{-1}(P^{-1} - Q)_2^*A + (P^{-1} - Q)_2(P^{-1} + Q)_{22}^{-1}(P^{-1} - Q)_2^* < 0 \quad (39)$$
$$A\left(Q^{-1} + P\right)A^* - \left(Q^{-1} + P\right) + BB^*/\sigma < 0 \quad (40)$$

It is easy to verify that (32), (33), (34) are sufficient for (39), (40). As $\hat{X}$ in (35) is $(n + r) \times (n + r)$, the proposition follows. ■

The reason of using (32), (33), (34) instead of (39), (40) is that the former can be written as a LMI in Q, P^{-1} and σ while the latter cannot.

Corollary 6 Given the ϕ-system G in (19) of order n. Suppose there exist Q, P satisfying

$$A^*QA - Q + C^*C/\sigma + (Q - P^{-1})_2(Q + P^{-1})_{22}^{-1}(Q - P^{-1})_2^* < 0 \quad (41)$$
$$APA^* - P + BB^*/\sigma < 0 \quad (42)$$

such that (23) holds. Then there exists a ϕ-model G_r of the form (20) and order r such that the error system $E := G - G_r$ is dd-stable and has a dd-gain $< 2\sigma$. ◁

The corollary is obvious if we note that (41), (42) are sufficient for (32), (33), (34). The significance is to see that Corollary 6 has relaxed the requirement in Proposition 3 since $(Q - P^{-1})_2$ is no longer restricted to 0, but if it were, the condition in Corollary 6 would simply reduce to the original one in Proposition 3.

Having solved the LMI, one may follow the controller synthesis procedure in [2] to compute the reduced-order ϕ-model G_r. Alternatively, note that (9) is a LMI in A, B, C, D, γ^2 once P is fixed. Therefore, once $\hat{X}$ in the above propositions is obtained, the reduced-order ϕ-model G_r can be computed readily via a convex optimization. The former procedure is computationally cheaper, while the latter might allow the incorporation of additional requirements of A_r, B_r, C_r, D_r into the reduced-order ϕ-model.

To have a feeling of how the new sufficient conditions perform, some statistics are collected and summarized in Table 1, which are the average values over the 22 numerical examples studied in [5]. The dimension of the model is originally $(m, n, p) = (1, 4, 2)$, which is then reduced to (m, r, p). For each sufficient condition and each $r \in \{1, 2, 3\}$, all possible orderings of the state-components have been considered and the data in the table reflect the one that gives the best error bound (2σ in Proposition 3, 5 or Corollary 6, or σ in Proposition 4). The error bound is denoted by $\bar{e}$ in the table, and the entries are normalized with respect to the original error bound $\bar{e}_0$ of Proposition 3. This explains why the last column gives an 1. Mathematically, it is guaranteed that the error bound of Proposition 5 is smaller than that of Corollary 6, which in turn is smaller than the original error bound of Proposition 3. However, we are not certain about the bound of Proposition 4. Table 1 shows that it seems to be the best in average, but in theory it need not always be smaller than the bound of Proposition 5, as confirmed by some of the 22 numerical examples.

Note that $\bar{e}$ is only a bound of the dd-gain of the error system. In general, it is not the smallest upper bound. The "actual error" e in Table 1 means the infimum of γ that satisfies Definition 2 for $E := G - G_r$, which is the tightest upper bound of the ℓ_2-ℓ_2 induced gain we can derive so far for the error system, so it might be of interest. The values are summarized in Table 1. It should be understood that although the new sufficient conditions in Proposition 5 and Corollary 6 always improve $\bar{e}$ over $\bar{e}_0$, it does not follow that the "actual error" must be smaller. In average it is, according to Table 1. It is also observed that although Proposition 4 gives the smallest bound, the "actual error" is in average the highest. In other words, if we consider the bounds given by the propositions as upper bounds of the "actual errors", the one given by Proposition 4 is usually the tightest. The "tightness" in each case, reflected as $\bar{e}/e$, is recorded in the table.

Table 1 shows that in average, the new sufficient conditions can reduce the error bound by about 40%, and the "actual error" by about 20%. Another important indicator is the "success rate" in Table 1. If the sufficient condition gives a reduced-order ϕ-model G_r such that $e \leq 10\%$ of the gain of $G - D$, it is regarded as successful. It is seen that the original success rate is low: only 5 out of 22 examples are considered successful, but the new conditions can increase the number of success up to 9.

Table 1: Comparison of model reduction conditions

		Cor. 6	Prop. 5	Prop. 4	Prop. 3
$r=3$	$\bar{e}/\bar{e}_0$	0.6841	0.5674	0.5225	1.0000
	e/e_0	0.8125	0.8096	0.8613	1.0000
	$\bar{e}/e$	1.4523	1.2196	1.0493	1.7335
$r=2$	$\bar{e}/\bar{e}_0$	0.6972	0.5652	0.5072	1.0000
	e/e_0	0.7631	0.7517	0.8079	1.0000
	$\bar{e}/e$	1.5219	1.2547	1.0411	1.6610
$r=1$	$\bar{e}/\bar{e}_0$	0.7170	0.5722	0.5060	1.0000
	e/e_0	0.8305	0.7923	0.8388	1.0000
	$\bar{e}/e$	1.4644	1.2317	1.0318	1.7146
success rate		0.4091	0.4091	0.3636	0.2273

$\bar{e}$: error bounds given by the conditions

e: actual errors

$\bar{e}_0$: original error bound given by Proposition 3

e_0: original error after applying Proposition 3

5 Relaxing the Requirement that ϕ should be an Odd Function

Sometimes the requirement that ϕ should be odd might be restrictive. In this section, we discuss how the results in [1, 2] and this paper can be extended to the case that ϕ is not odd.

All the results in [1, 2] were built upon the following quadratic inequality: If ϕ is odd and 1-Lipschitz, P is pdd, then

$$\forall \zeta \in \mathbb{R}^n, \quad \phi(\zeta)^* P \phi(\zeta) \leq \zeta^* P \zeta \tag{43}$$

Note that a pdd matrix can be written into 3 components:

$$P = P^{\setminus} + P^{-} + P^{+}$$

where $P^{\setminus}$ is non-negative diagonal, P^- and P^+ are symmetric such that

$$\begin{aligned} p^-_{ij} &\leq 0 \text{ if } i \neq j \\ p^-_{ii} &= -\sum_{j\neq i} p^-_{ij} \geq 0 \\ p^+_{ij} &\geq 0 \\ p^+_{ii} &= \sum_{j\neq i} p^+_{ij} \end{aligned}$$

If ϕ is odd and 1-Lipschitz, then

$$\forall \zeta \in \mathbb{R}^n, \qquad \phi(\zeta)^* P^{\setminus} \phi(\zeta) \leq \zeta^* P^{\setminus} \zeta \tag{44}$$
$$\forall \zeta \in \mathbb{R}^n, \qquad \phi(\zeta)^* P^{-} \phi(\zeta) \leq \zeta^* P^{-} \zeta \tag{45}$$
$$\forall \zeta \in \mathbb{R}^n, \qquad \phi(\zeta)^* P^{+} \phi(\zeta) \leq \zeta^* P^{+} \zeta \tag{46}$$

However, if ϕ is 1-Lipschitz but not odd, it only satisfies (45). Since P^- is only positive semidefinite but not definite, we should further require that $\phi(0) = 0$ so that (44) also holds. Then if $P^{\setminus} \neq 0$, $P^{\setminus} + P^-$ will be positive definite.

In summary, if ϕ is not odd but $\phi(0) = 0$, and ϕ is 1-Lipschitz, then the pdd matrix in the results of [1, 2] should be replaced by:

$$P^{\setminus} + P^{-} \text{ such that } P^{\setminus} \neq 0 \tag{47}$$

which is called a diagonally dominant Stieltjes matrix in the literature (i.e. a diagonally dominant, symmetric non-singular M-matrix[6]). Following the lines of [2], it is obvious that the inverses of such matrices will play an important role in the synthesis problem. We remark here that the inverses of a diagonally dominant Stieltjes matrix always has non-negative entries such that any off-diagonal entry $\leq$ the corresponding diagonal entry. On the other hand, if a matrix is strictly ultrametric, then its inverse is a strictly diagonally dominant Stieltjes matrix[7, 8, 9, 10].

To prove the claim that (44), (45) hold for a ϕ which is not necessarily odd, one may modify the proofs of [1, Lemmas 1, 2] accordingly. Another way to look at it is via the $\mathcal{S}$-procedure[11]. ϕ being 1-Lipschitz means that

$$\forall \zeta_i, \zeta_j \in \mathbb{R}, \;\; (\phi(\zeta_i) - \phi(\zeta_j))^2 \leq (\zeta_i - \zeta_j)^2 \tag{48}$$

which may be written as

$$\forall \zeta \in \mathbb{R}^n, \;\; \phi(\zeta)^* \Pi_{ij} \phi(\zeta) \leq \zeta^* \Pi_{ij} \zeta \tag{49}$$

where Π_{ij} is a $n \times n$ matrix such that its entries are all 0 except the (i,i), (j,j)-th entries which are 1 and the (i,j), (j,i)-th entries which are -1. Now it suffices to note that P^- is just a non-negative linear combination of Π_{ij}, $i,j = 1,2,\ldots,n$.

In a similar way, we can prove that

$$\forall \zeta \in \mathbb{R}^n, \;\; \zeta^* P^- \phi(\zeta) \leq \zeta^* P^- \zeta \tag{50}$$

by noting that

$$\forall \zeta_i, \zeta_j \in \mathbb{R}, \;\; (\zeta_i - \zeta_j)(\phi(\zeta_i) - \phi(\zeta_j)) \leq (\zeta_i - \zeta_j)^2 \tag{51}$$

Also, if ϕ is in addition monotonically increasing, we have

$$\forall \zeta_i, \zeta_j \in \mathbb{R}, \;\; (\phi(\zeta_i) - \phi(\zeta_j))^2 \leq (\zeta_i - \zeta_j)(\phi(\zeta_i) - \phi(\zeta_j)) \tag{52}$$

and hence

$$\forall \zeta \in \mathbb{R}^n, \;\; \phi(\zeta)^* P^- \phi(\zeta) \leq \zeta^* P^- \phi(\zeta) \tag{53}$$

It is easily seen that inequalities (50), (53) hold for a pdd P as well if ϕ is odd. A useful consequence is that we can replace (9) by

$$\begin{bmatrix} -R & P & 0 \\ P & A^*RA - 2P + C^*C & A^*RB + C^*D \\ 0 & B^*RA + D^*C & B^*RB + D^*D - \gamma^2 I_m \end{bmatrix} < 0 \tag{54}$$

where $R > 0$ need not be diagonally dominant, to obtain a tighter upper bound of the ℓ_2-ℓ_2 induced gain of the ϕ-system when ϕ is also monotonically increasing. More generally, quadratic inequalities (45), (50), (53) can all be incorporated into the IQC (Integral Quadratic Constraint) framework of Megretski and Rantzer[12] to provide analysis results for any systems involving repeated appearance of such a nonlinearity ϕ. However, how the latter two can be useful to synthesis and model reduction is still under investigation.

6 Conclusions

In this paper, a new necessary condition for the performance synthesis problem and two new sufficient conditions for the model reduction problem for systems with repeated scalar nonlinearities have been presented. How to relax the requirement that the nonlinearity should be an odd function has also been discussed.

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